Multivariable Calculus – Lecture #9 Notes

In this lecture we continue the discussion of integration of functions of several variables. We'll develop a library of applications and also discuss methods for calculating multiple integrals using iterated single integrals.

Recall from the previous lecture:

1) If $f(x, y)$ is a function of two variables defined over a (closed and bounded) domain D and if this function has positive values in this domain, the ***volume under this graph*** (and above the xy-plane and within the vertical "curtain wall" lying above the boundary of the domain D) is given by $\iint_D f(x, y)dA$. In the case where the function $f(x, y)$ takes on positive and negative values in different parts of the domain, we can interpret $\iint_D f(x, y)dA$ as a ***signed volume***.

2) The ***total area of a region D*** can be calculated as: $\boxed{\text{Area}(D) = \iint_D dA}$.

3) If $\sigma(x, y)$ is a mass density function for an object that occupies a domain D in the xy-plane then the ***total mass*** of this object (often referred to as a *lamina*) is given by: $\boxed{\text{mass}(D) = \iint_D \sigma(x, y)dA}$.

   If the density measure electrical charge, the integral $\iint_D \sigma(x, y)dA$ would then give the ***total charge*** (summing both positive and negatively charged regions to produce the net charge). If $\sigma(x, y)$ measured population density in a region D, then $\iint_D \sigma(x, y)dA$ would give the ***total population*** in the region.

   Generally, $\iint_D \sigma(x, y)dA$ will give the ***total amount*** of any quantity with associated density function $\sigma(x, y)$.

Average value of a function over a domain

Most of us were first introduced to the idea of average value in elementary school. Given a discrete list of numerical scores $\{S_1, S_2, \cdots, S_n\}$, we define the (unweighted) average $\overline{S}$ by adding up these values and dividing by the number of scores, i.e. $\overline{S} = \dfrac{S_1 + \cdots + S_n}{n} = \dfrac{1}{n}\sum_{i=1}^{n} S_i$. That is, the sum of the scores should be the same as what we would get if we added up an equal number of scores each of which had the average score. This can be expressed as $n \cdot \overline{S} = S_1 + \cdots + S_n = \sum_{i=1}^{n} S_i$. If we think of this in terms of a bar graph with equal width of 1 unit per score, this says simply that the total area of the bar graph should be the same as a rectangle with the same total width (n) but with constant height equal to the average. This perspective allows us to extend the idea of average value to any integrable function $f(x)$ defined on an interval $[a, b]$. We define the average value $\overline{f}$ to be that value such that (length of $[a, b]$) $\cdot \overline{f} = (b - a) \cdot \overline{f} = \int_a^b f(x)dx$. So $\boxed{\overline{f} = \dfrac{\int_a^b f(x)dx}{\text{length of }[a, b]} = \dfrac{\int_a^b f(x)dx}{b - a}}$. A good way to think of this (unweighted) average is that it's the integral of the function over its domain divided by the "geometric content" of its domain, i.e. its length. This can be completely generalized.

4) In the case of a function of two variables $f(x, y)$ defined over a domain D, we can (at least in the case of a function with positive values) think of the average value as an average height $\overline{f}$ such that

$$(\text{Area of } D) \cdot \overline{f} = \iint_D f(x, y)\, dA = [\text{Volume under graph}]. \text{ So } \boxed{\overline{f} = \dfrac{\iint_D f(x, y)\, dA}{\text{Area of } D}}.$$

For example, if a region D in the xy-plane has an associated mass density function $\sigma(x, y)$, we would calculate the **average density** as $\overline{\sigma} = \dfrac{\iint_D \sigma(x, y)\, dA}{\text{Area of } D} = \dfrac{\text{Mass of } D}{\text{Area of } D}$, and this coincides with the more familiar way of thinking about average density.

Weighted averages
There are situations where it is completely appropriate to compute an average value where different "weights" are assigned to the values, e.g. the way a course grade might be calculated by counting the Final Exam more than the individual midterm exams. This is relatively simple to quantify. If the scores are $\{S_1, S_2, \cdots, S_n\}$ and we assign to these scores the respective weights $\{w_1, w_2, \cdots, w_n\}$, then we would define the weighted average implicitly by the condition that $(\text{total weight}) \cdot \overline{S}_{wtd} = S_1 w_1 + \cdots + S_n w_n = \sum_{i=1}^{n} S_i w_i$, so $\overline{S}_{wtd} = \dfrac{\sum_{i=1}^{n} S_i w_i}{\text{total weight}}$.

To define a weighted average of a function of more than one variable in a region D, it's helpful to use the fact that for a small piece D_i its mass will be approximately $\Delta m_i \cong \sigma(x_i, y_i)\Delta A_i$ where $\sigma(x, y)$ is an associated density function. We can use differential notation to capture this idea as it enters into an integral expression as a limit by expressing this as $\boxed{dm = \sigma\, dA}$. This will be very useful notation for succinctly expressing some of the quantities that follow.

5) In the case of a function of two variables $f(x, y)$ defined over a domain D, if we have an associated density function $\sigma(x, y)$ for the region D that permits us to "weigh" some parts more than others, we can define the **weighted average of the function** implicitly by the condition that $(\text{Mass of } D) \cdot \overline{f}_{wtd} = \iint_D f(x, y)\, dm$. This

gives the definition $\boxed{\overline{f}_{wtd} = \dfrac{\iint_D f(x, y)\, dm}{\text{Mass of } D} = \dfrac{\iint_D f(x, y)\sigma(x, y)\, dA}{\iint_D \sigma(x, y)\, dA}}.$

Given any region D in the xy-plane, we can use the idea of averaging to define the geometric center (**centroid**) of the region as well as the **center of mass** of the region. We just have to average the coordinate functions in the region.

6) The **centroid** or geometric center of a region D in the xy-plane is the point $(\overline{x}, \overline{y})$ where

$$\boxed{\overline{x} = \dfrac{\iint_D x\, dA}{\text{Area of } D}} \text{ and } \boxed{\overline{y} = \dfrac{\iint_D y\, dA}{\text{Area of } D}}.$$

7) The **center of mass** of a region D in the xy-plane with mass density function $\sigma(x, y)$ is the point $(\overline{x}_{cm}, \overline{y}_{cm})$

where $\boxed{\overline{x}_{cm} = \dfrac{\iint_D x\,dm}{\text{Mass of } D} = \dfrac{\iint_D x\,\sigma(x, y)dA}{\iint_D \sigma(x, y)dA}}$ and $\boxed{\overline{y}_{cm} = \dfrac{\iint_D y\,dm}{\text{Mass of } D} = \dfrac{\iint_D y\,\sigma(x, y)dA}{\iint_D \sigma(x, y)dA}}$.

It is worth noting that in the case of constant density the centroid and center of mass will coincide.

We now have a pretty respectable library of applications involving double integrals. We now turn to the issue of how to most effectively compute them.

Calculation of double integrals via iterated single integrals (Method of Successive Slicing)

For the sake of simplicity, let's start by assuming that a function $f(x, y)$ is positive everywhere in a region D in

the xy-plane. In this case, we can interpret the integral $\iint_D f(x, y)dA$ as a volume. However, in single-variable

Calculus we learned another way to find such a volume by integrating the areas of cross-sections. For example, if the solid region was such that we could find an expression for the cross-sectional area $A(x)$ for any given x,

and if the region was such that $a \le x \le b$, then the volume would be given by $\int_a^b A(x)dx$.

This was the basis for finding volumes of solids of revolution for figures with known cross-sections such as disks and washers, but the method is more general than just those specific applications. [Some call it the **Stacking Principle**, but it is known historically as **Cavalieri's principle** (after Bonaventura Cavalieri, 1598-1647) and grew out of what was known as the "method of indivisibles" going back to the time of Archimedes (287-212 BC).]
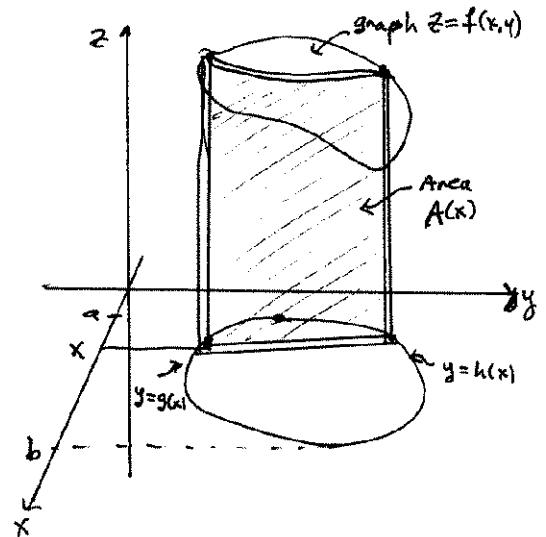
In the current context, suppose the region D is bounded in such a way that for each x we can uniquely describe the lower bound of the region algebraically by $y = g(x)$ and we can uniquely describe the upper bound of the region algebraically by $y = h(x)$. If this is the case, we can recognize the area $A(x)$ of a cross-sectional planar slice as the area under a curve $z = f(x, y)$ where x is held constant and y varies between the limits $y = g(x)$ and $y = h(x)$. Thus, we

can calculate this as $A(x) = \int_{y=g(x)}^{y=h(x)} f(x, y)\,dy$. It's important to

emphasize that the variable x is treated as a constant in this integration.

If we insert this into $\int_a^b A(x)dx$, we get that this volume can be

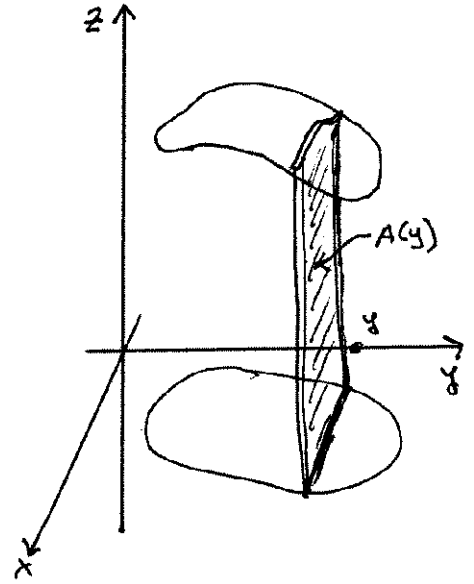calculated via iterated single integrals as $\int_{x=a}^{x=b}\left[\int_{y=g(x)}^{y=h(x)} f(x, y)\,dy\right]dx$. So

$\boxed{\iint_D f(x, y)dA = \int_{x=a}^{x=b}\left[\int_{y=g(x)}^{y=h(x)} f(x, y)\,dy\right]dx}$.

Though it is better perhaps to include the parentheses and the specific references in the integral limits, this is

often expressed more succinctly as $\iint_D f(x, y)dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\,dy\,dx$. Note that this has an prescribed order

of integration – x for the outside integral and y for the inside integral – and this is often captured by saying that the "area element" is $\boxed{dA = dy\,dx}$.

Before proceeding with any calculations, it's important to note that we could have sliced up this solid region in a completely different way. That is, suppose the region D is bounded in such a way that for each y (where $c \le y \le d$) we can uniquely describe the lower (left-hand) bound of the region algebraically by $x = k(y)$ and we can uniquely describe the upper (right-hand) bound of the region algebraically by $x = l(y)$. If this is the case, we can recognize the area $A(y)$ of a cross-sectional planar slice as the area under a curve $z = f(x,y)$ where y is held constant and x varies between the limits $x = k(y)$ and $x = l(y)$. Thus, we can calculate this as $A(y) = \int_{x=k(y)}^{x=l(y)} f(x,y)\,dx$. It's important to emphasize that the variable y is treated as a constant in this integration. If we then insert this into $\int_c^d A(y)\,dy$, we get that this volume can be

calculated via iterated single integrals as $\int_{y=c}^{y=d}\left[\int_{x=k(y)}^{x=l(y)} f(x,y)\,dx\right]dy$. So

$\boxed{\iint_D f(x,y)\,dA = \int_{y=c}^{y=d}\left[\int_{x=k(y)}^{x=l(y)} f(x,y)\,dx\right]dy}$. Once again, this is often expressed more succinctly as

$\iint_D f(x,y)\,dA = \int_c^d \int_{k(y)}^{l(y)} f(x,y)\,dx\,dy$. Note that this has an prescribed order of integration – y for the outside integral and x for the inside integral – and this is often captured by saying that the "area element" is $\boxed{dA = dx\,dy}$.

Fubini Theorem: If the function $f(x,y)$ is integrable over the region D, and if we can calculate both of the iterated single integral expressions $\int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$ and $\int_c^d \int_{k(y)}^{l(y)} f(x,y)\,dx\,dy$, then these will yield the same value, i.e. $\boxed{\iint_D f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx = \int_c^d \int_{k(y)}^{l(y)} f(x,y)\,dx\,dy}$. We can think of this theorem more simply as "no matter how you slice it, the volume is still the volume."

It's important to now emphasize that although we used the idea of volume to develop this method of calculating double integrals, the method works just as well to calculate any double integral (assuming you can actually find the necessary antiderivates using the Fundamental Theorem of Calculus). It is sometimes the case that we will be unable to do these integrations using one order of integration, so the Fubini Theorem gives us the flexibility to try doing it by *reversing the order of integration*. It is generally necessary to draw the region D and determine the algebraic expressions for the boundaries when doing this. It is not as simple as switching the integral limits.

Now, on to some examples.

Example 1: Calculate the volume under the graph of the function $f(x,y) = 9 - x^2 - y^2$ over the rectangular region D with $0 \le x \le 1$ and $0 \le y \le 2$.

Solution: (a) If we use the order of integration associated with the area element $dA = dy\,dx$, we first note that $0 \le x \le 1$ and that for each x in between 0 and 1, y will vary between the constant limits $y = 0$ and $y = 2$ (because the region is a simple rectangle). The integral limits are determined completely by the region. What the graph of $f(x,y)$ looks like isn't relevant as long as the function is positive in this domain and we're calculating the volume under the graph. The iterated integrals corresponding to this method of slicing is therefore $\int_0^1 \int_{y=0}^{y=2} (9 - x^2 - y^2)\,dy\,dx$. We then calculate:

Inner Integral: $\int_{y=0}^{y=2}(9-x^2-y^2)\,dy = \left[(9-x^2)y - \frac{y^3}{3}\right]_{y=0}^{y=2} = (9-x^2)2 - \frac{8}{3} = \frac{46}{3} - 2x^2$

Outer Integral: $\int_{0}^{1}\left(\frac{46}{3}-2x^2\right)dx = \left[\frac{46}{3}x - \frac{2}{3}x^3\right]_{0}^{1} = \frac{46}{3} - \frac{2}{3} = \boxed{\frac{44}{3}}$

(b) If we had instead used the order of integration associated with the area element $dA = dx\,dy$, we would first note that $0 \le y \le 2$ and that for each y in between 0 and 2, x will vary between the constant limits $x = 0$ and $x = 2$. The iterated integrals corresponding to this method of slicing is therefore $\int_{0}^{2}\int_{x=0}^{x=1}(9-x^2-y^2)\,dx\,dy$.

We then calculate:

Inner Integral: $\int_{x=0}^{x=1}(9-x^2-y^2)\,dx = \left[(9-y^2)x - \frac{x^3}{3}\right]_{x=0}^{x=1} = 9 - y^2 - \frac{1}{3} = \frac{26}{3} - y^2$

Outer Integral: $\int_{0}^{2}\left(\frac{26}{3}-y^2\right)dy = \left[\frac{26}{3}y - \frac{1}{3}y^3\right]_{0}^{2} = \frac{52}{3} - \frac{8}{3} = \boxed{\frac{44}{3}}$

As you can see, the Fubini Theorem guaranteed that these results had to be the same.

Things are a bit more complicated for non-rectangular domains.

Example 2: Find the centroid of the quarter-circular region D by the x-axis, the y-axis, and the circle of radius R (centered at the origin), with $x \ge 0$ and $y \ge 0$, i.e. in the first quadrant.

Solution: It is a good protocol to always first write down what it is you wish to calculate. In this case, the centroid is the point $(\bar{x}, \bar{y})$ where $\bar{x} = \dfrac{\iint_D x\,dA}{\text{Area of } D}$ and $\bar{y} = \dfrac{\iint_D y\,dA}{\text{Area of } D}$. We can simplify our calculations considerably by noting that for this particular region we must have $\bar{x} = \bar{y}$ by symmetry. Furthermore, we can calculate the denominator without Calculus as $Area(D) = \frac{1}{4}\pi R^2$. If we choose to calculate $\bar{y}$, we might choose the order of integration so that $\iint_D y\,dA = \int_{x=0}^{x=R}\left[\int_{y=0}^{y=\sqrt{R^2-x^2}} y\,dy\right]dx$. Note that the upper limit on the inner integral is $y = \sqrt{R^2 - x^2}$ since the (upper) circular boundary is described by the equation $x^2 + y^2 = R^2$. When using Cartesian coordinates, non-rectangular regions will yield non-constant integral limits. We calculate:

Inner: $\int_{y=0}^{y=\sqrt{R^2-x^2}} y\,dy = \left[\frac{1}{2}y^2\right]_{y=0}^{y=\sqrt{R^2-x^2}} = \frac{1}{2}(R^2 - x^2)$

Outer: $\int_{x=0}^{x=R}\frac{1}{2}(R^2-x^2)\,dx = \frac{1}{2}\left[(R^2 x - \frac{1}{3}x^3)\right]_{0}^{R} = \boxed{\frac{1}{3}R^3}$

This calculation could also have been done with the order of integration reversed, though the integration would be marginally more difficult.

Therefore $\bar{y} = \dfrac{\iint_D y\,dA}{\text{Area of } D} = \dfrac{\frac{1}{3}R^3}{\frac{1}{4}\pi R^2} = \boxed{\frac{4}{3\pi}R \cong .4244R}$ and the centroid is therefore at $(\bar{x}, \bar{y}) = \left(\frac{4}{3\pi}R, \frac{4}{3\pi}R\right)$.

Integration over regions in $\mathbf{R}^2$ using polar coordinates
There is nothing in the original definition of the double integral that says we must use Cartesian coordinates. The only requirement is that we should be able to partition the given region into small pieces so that the integral may be defined when we take the limit as the mesh of the partition goes to 0. We can do this relatively easily using polar coordinates to divide things up. If in the above example we were to carve up this region using concentric circles and rays emanating from the origin with the circles spaced by very small distance Δr and the
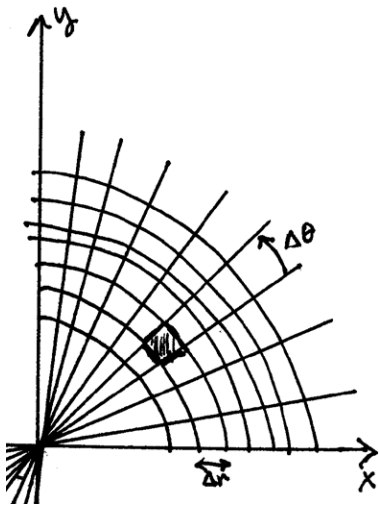
rays spaced by a very small $\Delta\theta$ (using radian measure, of course), then the resulting pieces of the region lying between circles and between rays will be approximately rectangular with side lengths Δr along the rays and $r\Delta\theta$ along the circular arcs. The area of each piece will then be approximately $\Delta A \cong (\Delta r)(r\Delta\theta) = r\Delta r\Delta\theta$. Note that these areas grow as the radius grows. In terms of differential quantities for the purpose of integration, we say that the area element in polar coordinates is $\boxed{dA = r\,dr\,d\theta}$. The r factor in this area element is a fundamentally important part of setting up any integral in polar coordinates.

It is very important to note that the shape of the region is what usually drives the decision about what coordinates to use in the integration. In the above example, the region is clearly better described in terms of polar coordinates. If we were to redo the integral $\iint_D y\,dA$ in polar coordinates, we have to address

three things – the integrand, the area element, and the integral limits. For the integrand, we note that in polar coordinates we have $y = r\sin\theta$. The area element will be $dA = r\,dr\,d\theta$. The region is such that θ varies between $\theta = 0$ and $\theta = \frac{\pi}{2}$, and for each θ in between, the radius will vary between $r = 0$ (the origin) and $r = R$. The

integral then becomes $\iint_D y\,dA = \int_0^{\pi/2}\int_0^R r^2\sin\theta\,dr\,d\theta$

Inner: $\displaystyle\int_0^R r^2\sin\theta\,dr = \left[\frac{r^3}{3}\right]_0^R \sin\theta = \tfrac{1}{3}R^3\sin\theta$

Outer: $\displaystyle\int_0^{\pi/2}\tfrac{1}{3}R^3\sin\theta\,d\theta = \tfrac{1}{3}R^3\left[-\cos\theta\right]_0^{\pi/2} = \tfrac{1}{3}R^3(0+1) = \tfrac{1}{3}R^3$

This agrees with the previous result.

Example 3: Calculate the area of a circular disk of radius R using a double integral.

Solution: We already know the answer, of course, but it's remarkably simple to get the result using polar coordinates. If we call this disk D, then $\text{Area}(D) = \iint_D dA = \int_0^{2\pi}\int_0^R r\,dr\,d\theta$. The Inner integral is

$\displaystyle\int_0^R r\,dr = \left[\tfrac{1}{2}r^2\right]_0^R = \tfrac{1}{2}R^2$ and the Outer integral is $\displaystyle\int_0^{2\pi}\left(\tfrac{1}{2}R^2\right)d\theta = \left(\tfrac{1}{2}R^2\right)(2\pi) = \boxed{\pi R^2}$.

For regions not bounded entirely by circles and rays, polar coordinates will likely produce some non-constant integral limits (just as was the case for non-rectangular regions in Cartesian coordinates).

Example 4: Calculate the average distance from the origin for points in the triangular region D bounded by the x-axis, the line $x = 4$, and the line $y = x$.

Solution: First, the distance from the origin to any arbitrary point (x, y) in this region is given by

$$f(x, y) = \sqrt{x^2 + y^2} = r\text{, so we want to calculate } \overline{f} = \frac{\iint_D f\,dA}{\text{Area}(D)} = \frac{\iint_D \sqrt{x^2 + y^2}\,dA}{\tfrac{1}{2}(4)(4)} = \tfrac{1}{8}\iint_D \sqrt{x^2 + y^2}\,dA.$$

The region suggests either Cartesian coordinates or polar coordinates might be a good choice, but the integration will likely be much simpler in polar coordinates. The only difficulty is that although the polar angle will vary from $\theta = 0$ to $\theta = \frac{\pi}{4}$, for each θ in between the radius will vary from $r = 0$ out to the vertical line

where $x = 4$. That's not expressed in polar coordinates, but we can rewrite it as $r\cos\theta = 4$ or $r = 4\sec\theta$. The integral therefore becomes:

$$\tfrac{1}{8}\iint_D \sqrt{x^2 + y^2}\, dA = \tfrac{1}{8}\int_{\theta=0}^{\theta=\pi/4}\int_{r=0}^{r=4\sec\theta}(r)r\,dr\,d\theta = \tfrac{1}{8}\int_{\theta=0}^{\theta=\pi/4}\int_{r=0}^{r=4\sec\theta} r^2\,dr\,d\theta$$

Inner: $\displaystyle\int_{r=0}^{r=4\sec\theta} r^2\,dr = \tfrac{1}{3}\Big[r^3\Big]_{r=0}^{r=4\sec\theta} = \tfrac{64}{3}\sec^3\theta$

Outer: $\displaystyle\tfrac{1}{8}\int_{\theta=0}^{\theta=\pi/4}\tfrac{64}{3}\sec^3\theta\,d\theta = \tfrac{8}{3}\int_{\theta=0}^{\theta=\pi/4}\sec^3\theta\,d\theta = \tfrac{4}{3}\Big[\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)\Big]_0^{\pi/4} = \tfrac{4}{3}\Big[\sqrt{2} + \ln\big(1+\sqrt{2}\big)\Big] \cong 3.06$.

This is a very plausible result for this region.

One class of multiple integrals for which polar coordinates are especially suitable are those that involve polar graphs such as roses, cardioids, lemniscates, etc.

Example 5: Find the area of one leaf of the rose determined by the polar equation $r = 4\cos 3\theta$.
Solution: The radius will reach its maximum when $\theta = 0$ and close in to the origin when $3\theta = \pm\frac{\pi}{2}$, i.e. when $\theta = \pm\frac{\pi}{6}$. To calculate the area of one leaf of this rose, we calculate:

$$\text{Area}(D) = \iint_D dA = \int_{-\pi/6}^{+\pi/6}\int_{r=0}^{r=4\cos 3\theta} r\,dr\,d\theta$$

Inner: $\displaystyle\int_{r=0}^{r=4\cos 3\theta} r\,dr = \tfrac{1}{2}\Big[r^2\Big]_{r=0}^{r=4\cos 3\theta} = 8\cos^2 3\theta$

Outer: $\displaystyle\int_{-\pi/6}^{+\pi/6} 8\cos^2 3\theta\,d\theta = 4\int_{-\pi/6}^{+\pi/6}(1+\cos 6\theta)\,d\theta = 4\Big[\theta + \tfrac{1}{6}\sin 6\theta\Big]_{-\pi/6}^{+\pi/6} = \boxed{\tfrac{4\pi}{3}}$.

In the next lecture we'll look at integration over regions in space, i.e. triple integrals, using Cartesian coordinates, cylindrical coordinates, and spherical coordinates. We'll also show how to change coordinates more generally and also see some examples of integration on surfaces (though that's a topic we will address more fully later).

Notes by Robert Winters and Renée Chipman