Multivariable Calculus – Lecture #4 Notes

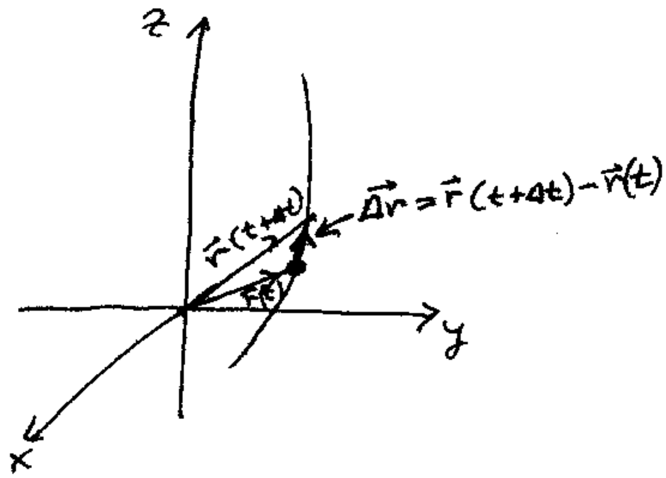
This lecture completes the discussion of parameterized curves the cross product in $\mathbf{R}^2$, $\mathbf{R}^3$, (and $\mathbf{R}^n$) with applications to finding tangent vectors to parameterized surfaces. Using this, we look at graphs of functions of two variables and begin to address the matter of rates of change of functions – leading to the definition of partial derivative. In the next lecture we'll address any necessary technical details regarding limits, continuity, and differentiability of functions.

Definition: A **parametrized curve** in $\mathbf{R}^n$ is a vector-valued function $\mathbf{r}(t) = \langle x_1(t), x_2(t), \cdots, x_n(t) \rangle$ in which the component functions vary (at least) continuously in *t*.

We think of $\mathbf{r}(t)$ as tracing out the position vector of points along a curve as we vary the parameter *t*.

Velocity vectors and tangent vectors

Perhaps the greatest advantage of using parameterized curves is how we can borrow basic physics concepts to assist in the geometry. The idea of a **velocity vector** is at the top of our list. If we imagine a parameterized curve where the component functions vary as *differentiable* functions of the parameter *t*, then by comparing the position vector $\mathbf{r}(t)$ at "time" *t* and the position vector $\mathbf{r}(t+\Delta t)$ a moment later, the difference of these vectors $\Delta\mathbf{r} = \mathbf{r}(t+\Delta t) - \mathbf{r}(t)$ will line up essentially tangent to the curve traced out by the position vector. Using some physics ideas, we can then divide this by the brief change in time to get $\dfrac{\Delta\mathbf{r}}{\Delta t} = \dfrac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$ which might be identified for an actual physical motion as the "average velocity vector". Taking the limit as $\Delta t \to 0$ gives the "instantaneous velocity vector" $\lim\limits_{\Delta t \to 0}\left(\dfrac{\Delta\mathbf{r}}{\Delta t}\right) = \lim\limits_{\Delta t \to 0}\left(\dfrac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}\right) \equiv \dfrac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \mathbf{v}(t)$. This **velocity vector** yields a vector that will be tangent to the path traced out by the position vector at every point along the path. This is very useful for finding tangent lines, for example. In terms of the component functions, if $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for a parameterized curve in $\mathbf{R}^3$, then $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.



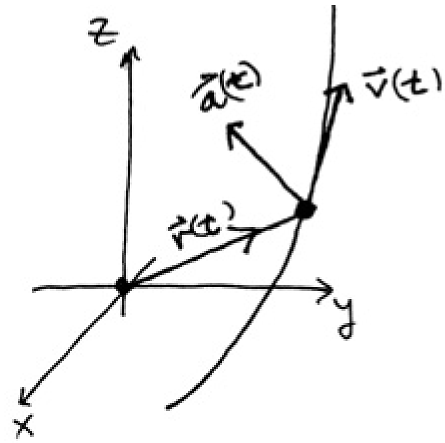
Definition: The **speed** is the magnitude of the velocity vector, i.e. $\|\mathbf{v}(t)\|$.

Basic Rules of Differentiation for Vector-Valued Functions:

If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are vector-valued functions, $f(t)$ is any scalar-valued function, and *c* is any constant, then:

1) $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$ [The derivative of a sum is the sum of the derivatives.]
2) $\frac{d}{dt}[c\,\mathbf{u}(t)] = c\,\mathbf{u}'(t)$ [Scalars pass through derivatives just as with ordinary differentiation.]
3) $\frac{d}{dt}[f(t)\,\mathbf{u}(t)] = f(t)\,\mathbf{u}'(t) + f'(t)\cdot\mathbf{u}(t)$ [Product Rule for scalar multiplication.]
4) $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}(t)\cdot\mathbf{v}'(t) + \mathbf{u}'(t)\cdot\mathbf{v}(t)$ [Product Rule for dot products.]
5) $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}(t)\times\mathbf{v}'(t) + \mathbf{u}'(t)\times\mathbf{v}(t)$ [Product Rule for cross products.]
6) $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\,\mathbf{u}'(f(t))$ [Chain Rule for compositions.]

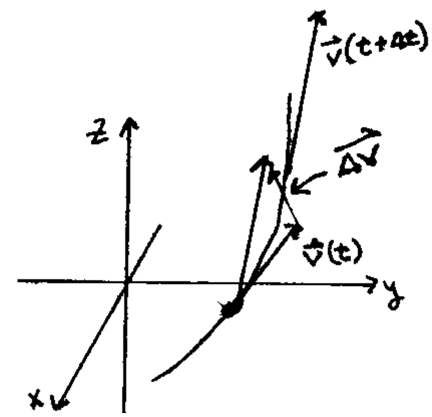
Each of these follows by applying the familiar rules of differentiation in the components. In particular, note that since the cross product is not commutative (it's anti-commutative), the order of the factors matters.



Vector-valued functions abound in physics and in the study of the geometry of curves and surfaces. This includes such physically important examples as **force**, the **angular momentum** vector, **torque**, and the most basic notions of **position**, **velocity**, and **acceleration**.

If we focus on parameterized curves (or paths) where the parameter is the time t (and even when this is not the case it is often helpful to *think* of t as representing time in order to have a notion of motion), we can describe motion (in $\mathbf{R}^3$, but the ideas are the same in $\mathbf{R}^2$) by specifying the position at any time t by its coordinates $(x(t), y(t), z(t))$. The **position vector** (relative to the origin) is the vector $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ from the origin to this point. The tip of the position vector will then trace out the path. The geometric argument above showed that the derivative $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle = \mathbf{v}(t)$ (the **velocity vector**) will be tangent to the path in the direction of motion (or the zero vector when the motion stops) and its units are in terms of distance per time.

Acceleration



We can similarly define the **acceleration vector** $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$ as the derivative of the velocity vector. As the figure illustrates, if we compare the velocity vector $\mathbf{v}(t)$ and the velocity vector $\mathbf{v}(t+\Delta t)$ a moment later, the difference vector $\Delta\mathbf{v} = \mathbf{v}(t+\Delta t) - \mathbf{v}(t)$ will generally be in the direction of the turn (unless the path is a straight line), with a positive tangential component if the velocity is increasing and a negative tangential component if the velocity is decreasing. This same observation will then hold for the difference quotient $\dfrac{\Delta\mathbf{v}}{\Delta t} = \dfrac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}$. If we pass to the limit, we define the acceleration vector as $\mathbf{a}(t) = \dfrac{d\mathbf{v}}{dt} = \mathbf{v}'(t) = \lim\limits_{\Delta t \to 0}\left(\dfrac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}\right) = \mathbf{r}''(t)$. This acceleration vector will generally have a tangential component (in the direction of the curve) if the motion is speeding up or slowing down, and a normal component perpendicular to the curve if the motion is turning.

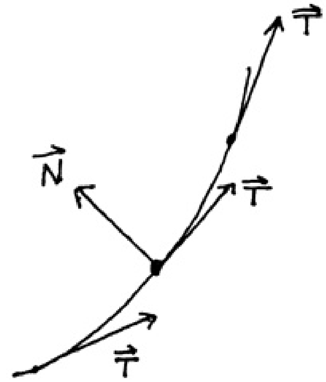
Arclength

If we think of a parameterized curve as defined starting with time $t = a$, we can let s denote the arclength of the curve to any other point on the path. (For simplicity, we presume that the speed is never zero and that the arclength is a strictly increasing function of t, i.e. $s = s(t)$.) We can then, alternatively, also think of the speed simply as $\frac{ds}{dt}$, the time rate of change of distance traveled, and we can conclude that $\|\mathbf{v}(t)\| = \frac{ds}{dt}$. In terms of differentials, we can then write $ds = \|\mathbf{v}(t)\|\,dt$ and, using a little bit of basic integral calculus, conclude that the **total arclength** traveled along a curve for $a \le t \le b$ must be $\boxed{L = \int_a^b \|\mathbf{v}(t)\|\,dt}$.

It's worth noting that since $\|\mathbf{v}(t)\|$ almost always involves square roots, this arclength integral tends to be one of the more difficult integrals to calculate using integration techniques. It is often best to use numerical integration in its calculation.

Velocity, unit tangent vector, unit normal vector, and curvature

For a parameterized curve described by $\mathbf{r}(t)$, as long as its velocity vector $\mathbf{v}(t)$ is nonzero, this velocity vector will be tangent to the path for all t in the direction of the path. We can divide out its magnitude (the speed) to define the **unit tangent vector** $\boxed{\mathbf{T}(t)=\dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}}$. This is a unit vector in the direction of the path. [It's worth noting that we can use the position vector at a given time t to determine a point on the path and either the velocity vector or the unit tangent vector at that time t to determine a tangent vector to the path at the given point. These can then be used to write down a parameterization of the tangent line to the path at that point.]



Since $\mathbf{T}$ is a unit vector, it follows that $\mathbf{T}\cdot\mathbf{T}=\|\mathbf{T}\|^2=1$. We can then differentiate both sides (with respect to t) to get that $\mathbf{T}\cdot\mathbf{T}'+\mathbf{T}'\cdot\mathbf{T}=2\mathbf{T}\cdot\mathbf{T}'=0$. So $\mathbf{T}\cdot\mathbf{T}'=0$. This means that either $\mathbf{T}'=\mathbf{0}$ (which would be the case for a straight line path) or, more generally, $\mathbf{T}'\perp\mathbf{T}$. That is, the rate of change of the (unit) direction vector is perpendicular to the path for all t. [Note: We would draw the same conclusion regardless of the parameter – a useful thing in the case where we may choose to parameterize the curve by arclength s rather than by time t.]

If the path is parameterized by (time) t, we can use the previous observation to define a **unit normal vector** to the path by $\mathbf{N}(t)=\dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$, a unit vector perpendicular to the path that points in the direction of the turn.

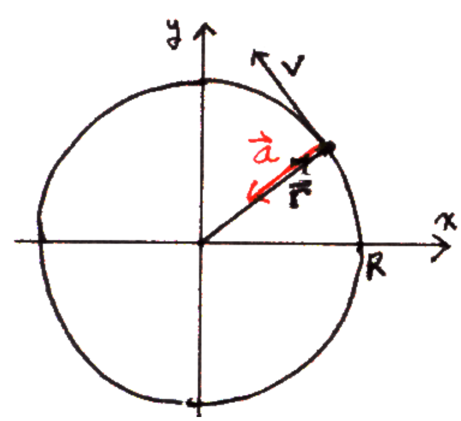
It is often more convenient to think of a path being parameterized by the arclength traveled along the path from the start (assuming, of course, that you always move in the same direction along the path and don't retrace the path). In this case we can think of the arclength s as a function of time t, i.e. $s=s(t)$. This is especially useful when thinking about the intrinsic shape of the path independent of how quickly you move along it. It then becomes natural to define the curvature in terms of the rate of change of direction per distance traveled, i.e. how fast you are turning per distance traveled. We thus define the **curvature** κ by $\boxed{\kappa=\left\|\dfrac{d\mathbf{T}}{ds}\right\|}$.

Though we can conceptually think about parametrizing a path by arclength, it is rarely simple to actually do this. Nonetheless, we can use the idea in our definitions and then find alternatives for calculation. We do this via the Chain Rule.

For example, if we think of $\mathbf{T}(s(t))$ as the unit tangent vector parameterized by arclength and the arclength, in turn, as a function of time, then we can compute $\dfrac{d\mathbf{T}}{dt}=\dfrac{d}{dt}\left[\mathbf{T}(s(t))\right]=s'(t)\mathbf{T}'(s(t))=\dfrac{d\mathbf{T}}{ds}\dfrac{ds}{dt}=\|\mathbf{v}(t)\|\dfrac{d\mathbf{T}}{ds}$ where we've conveniently used the fact that $\|\mathbf{v}(t)\|=\frac{ds}{dt}$.

So $\dfrac{d\mathbf{T}}{ds}=\dfrac{\mathbf{T}'(t)}{\|\mathbf{v}(t)\|}$ and, therefore, the curvature may also be written as $\boxed{\kappa=\left\|\dfrac{d\mathbf{T}}{ds}\right\|=\dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|}}$.

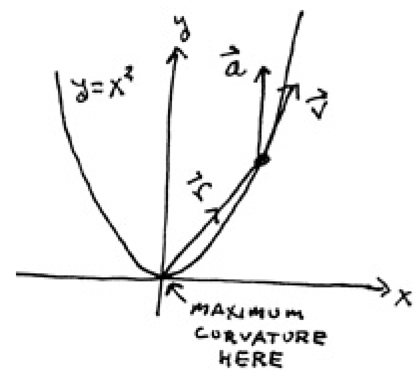
Example (motion around a circle): For a circle parameterized as $\mathbf{r}(t)=\langle R\cos\omega t, R\sin\omega t\rangle=R\langle\cos\omega t,\sin\omega t\rangle$, we calculate that $\mathbf{v}(t)=\mathbf{r}'(t)=\langle -R\omega\sin\omega t, R\omega\cos\omega t\rangle=R\omega\langle-\sin\omega t,\cos\omega t\rangle$. The speed is $\|\mathbf{v}(t)\|=R\omega$ which in physics might be described by saying that the "tangential velocity" is the product of the radius of the circle and the angular velocity. Note that for all t, $\mathbf{v}(t)\perp\mathbf{r}(t)$ as is expected for a circle about the origin. We can also calculate the acceleration $\mathbf{a}(t)=\mathbf{r}''(t)=R\omega^2\langle-\cos\omega t,-\sin\omega t\rangle$. Note that the acceleration is a centripetal acceleration (directed back toward the center of the circle. This



is because the speed is constant for this parameterization. If we calculate the curvature, we first note that $\mathbf{T}(t)=\dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}=\langle -\sin\omega t,\cos\omega t\rangle$, so $\mathbf{T}'(t)=\omega\langle -\cos\omega t,-\sin\omega t\rangle$ and $\|\mathbf{T}'(t)\|=\omega$. Therefore the curvature is given by $\kappa=\dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|}=\dfrac{\omega}{\omega R}=\dfrac{1}{R}$. That is, the curvature and the radius are inversely proportional. This makes sense geometrically for a circle but, perhaps more importantly, this allows us to define the radius of curvature generally for any curve – not just for circles.

Definition: The **radius of curvature** for any (parameterized) curve is $\boxed{R=\dfrac{1}{\kappa}}$ where κ is the curvature at a given point. The radius of curvature represents the radius of an ***osculating circle*** that is tangent to the curve and matches the curvature at the given point.

Example: Let's look at the familiar parabola $y=x^2$ in the *xy*-plane. This is a static, algebraic equation for the parabola, but we can also describe it parametrically by letting $x=t$ (and therefore $y=t^2$). We thus get the position vector and parameterization $\mathbf{r}(t)=\langle t,t^2\rangle$. We can then easily calculate that $\mathbf{v}(t)=\langle 1,2t\rangle$ and $\mathbf{a}(t)=\langle 0,2\rangle$. These are all shown in the accompanying sketch. We can also quickly calculate the speed $\|\mathbf{v}(t)\|=\sqrt{1+4t^2}$ as well as the unit tangent vector $\mathbf{T}(t)=\dfrac{\mathbf{v}}{\|\mathbf{v}\|}=\dfrac{\langle 1,2t\rangle}{\sqrt{1+4t^2}}=\left\langle \dfrac{1}{\sqrt{1+4t^2}},\dfrac{2t}{\sqrt{1+4t^2}}\right\rangle$. If we exercise our differentiation skills (especially the quotient rule), we can then calculate



$\mathbf{T}'(t)=\left\langle \dfrac{-4t}{(1+4t^2)^{3/2}},\dfrac{2}{(1+4t^2)^{3/2}}\right\rangle$. Its magnitude is then $\|\mathbf{T}'(t)\|=\dfrac{\sqrt{16t^2+4}}{(1+4t^2)^{3/2}}=\dfrac{2\sqrt{4t^2+1}}{(1+4t^2)^{3/2}}=\dfrac{2}{1+4t^2}$. The curvature is therefore given by $\kappa=\dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|}=\dfrac{2}{(1+4t^2)^{3/2}}$. Note that $t=0$ corresponds to the origin $(0,0)$ and at this point the curvature is $\kappa=2$. Further note that for $t\neq 0$, the curvature decreases. This is consistent with the shape of the parabola. The radius of curvature at the vertex is $\frac{1}{2}$ which represents the radius of the circle that can be snugly fitted to the curve at this point.

It's worth noting that had we simply focused on the function $f(x)=x^2$ in analyzing this parabola, the first derivative $f'(x)=2x$ would accurately give the slope of a tangent line for any given *x*, but the second derivative $f''(x)=2$, generally referred to as the *concavity*, is not an accurate measure of the *curvature* for the graph of this function. The sign of the second derivative allows us to conclude that the graph is "concave up" everywhere, but we can conclude nothing further just from the second derivative. The concavity, as calculated above, gives the accurate measure of just how *curved* the curve is at any given point.

A little more physics – tangential and normal components of acceleration

If we rewrite $\mathbf{T}(t)=\dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$ as $\mathbf{v}(t)=\|\mathbf{v}(t)\|\mathbf{T}(t)=\dfrac{ds}{dt}\mathbf{T}(t)$, we can differentiate to get

$\mathbf{a}(t)=\dfrac{d^2s}{dt^2}\mathbf{T}(t)+\dfrac{ds}{dt}\dfrac{d\mathbf{T}}{dt}=\dfrac{d^2s}{dt^2}\mathbf{T}(t)+\dfrac{ds}{dt}\left(\dfrac{d\mathbf{T}}{ds}\dfrac{ds}{dt}\right)=\dfrac{d^2s}{dt^2}\mathbf{T}+\left(\dfrac{ds}{dt}\right)^2\kappa\mathbf{N}$. If we further identify the speed as the "tangential velocity" v_T and note that $\kappa=\dfrac{1}{R}$ where *R* is the radius of curvature, we get that $\boxed{\mathbf{a}=\dfrac{dv_T}{dt}\mathbf{T}+\dfrac{{v_T}^2}{R}\mathbf{N}}$.

This shows the **tangential and normal components of acceleration**. The tangential component is just the rate of change of the speed. The normal component has the same value as the centripetal acceleration for circular motion around a circle whose radius is the radius of curvature at any given point.

If we imagine this curve to be the motion of a particle of mass m moving subject to a force, then Newton's 2nd Law gives that $\mathbf{F} = m\mathbf{a} = m\frac{dv_T}{dt}\mathbf{T} + \frac{m{v_T}^2}{R}\mathbf{N}$. If we identify $\frac{dv_T}{dt} = a_T$ as the tangential component of acceleration, then we see that the tangential component of force is just ma_T and the normal component of the force is given by $\frac{m{v_T}^2}{R}$. In the case of uniform circular motion this corresponds to the **centripetal force**.

Unit tangent, unit normal, and unit binormal vectors for a space curve

If we parameterize a curve in $\mathbf{R}^3$ using arclength as the parameter, i.e. $\mathbf{r}(s) = \langle x(s), y(s), z(s) \rangle$, then as long as the curve is not straight (except possibly at isolated points along the curve) then we can define a "moving frame" as follows:

First, if we think of $s = t$ we can observe that $\frac{ds}{dt} = 1$ (unit speed) and $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{st} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$. That is, if we travel at unit speed, the unit tangent vector coincides with the velocity vector. Since $\mathbf{T} \cdot \mathbf{T} = 1$, we calculate as before to get that $\frac{d\mathbf{T}}{ds} \perp \mathbf{T}$ and since $\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$ we conclude that $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$ where $\mathbf{N}$ is the unit normal vector.

[*Note*: At any point where the curve "straightened out" the curvature would be 0 and the unit normal vector would not be defined.] We can then take the unit tangent vector **T** and the unit normal vector **N** and define a third perpendicular unit vector called the **binormal vector** using the cross product, $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. These three mutually perpendicular unit vectors $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ then constitute a "moving frame" at each point along the curve.

Using our basic differentiation rules and previously defined quantities, we calculate $\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \mathbf{T} \times \frac{d\mathbf{N}}{ds} + \frac{d\mathbf{T}}{ds} \times \mathbf{N}$. But we know that $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$, so $\frac{d\mathbf{T}}{ds} \times \mathbf{N} = \kappa\mathbf{N} \times \mathbf{N} = \mathbf{0}$, so $\frac{d\mathbf{B}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$. But we also know that $\mathbf{B} \cdot \mathbf{B} = \|\mathbf{B}\|^2 = 1$, so in a calculation similar to those we have already done, we know that $\frac{d\mathbf{B}}{ds} \perp \mathbf{B}$. Therefore $\frac{d\mathbf{B}}{ds}$ must lie in the plane of **T** and **N** (formally we would say that it must be a linear combination of these two vectors). But we can also observe from the fact that $\frac{d\mathbf{B}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$ that $\frac{d\mathbf{B}}{ds}$ must be perpendicular to **T**, so $\frac{d\mathbf{B}}{ds}$ must be a simple multiple of the unit normal vector **N**. The magnitude of this vector is called the **torsion** τ, and we traditionally choose the sign so that $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$. [Note that $\tau = \left\| \frac{d\mathbf{B}}{ds} \right\|$.]

We can also observe that not only is $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ but also $\mathbf{T} = \mathbf{N} \times \mathbf{B}$ and $\mathbf{N} = \mathbf{B} \times \mathbf{T}$. Differentiating the last of these, we calculate that $\frac{d\mathbf{N}}{ds} = \frac{d}{ds}(\mathbf{B} \times \mathbf{T}) = \mathbf{B} \times \frac{d\mathbf{T}}{ds} + \frac{d\mathbf{B}}{ds} \times \mathbf{T} = \mathbf{B} \times \kappa\mathbf{N} - \tau\mathbf{N} \times \mathbf{T} = -\kappa\mathbf{T} + \tau\mathbf{B}$. If we assemble the above relations we get the Serret-Frenét formulas: $\left\{ \begin{aligned} \frac{d\mathbf{T}}{ds} &= \kappa\mathbf{N} \\ \frac{d\mathbf{N}}{ds} &= -\kappa\mathbf{T} + \tau\mathbf{B} \\ \frac{d\mathbf{B}}{ds} &= -\tau\mathbf{N} \end{aligned} \right\}$.

The combination of the curvature and torsion associated with a space curve provides important characteristic information about of a space curve.

For any curve that lies entirely in a plane (either the *xy*-plane or *any* plane), then the unit tangent **T** and unit normal vector **N** will necessary also lie in that plane. Consequently their cross product $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ will be a unit vector that is in a direction normal to the plane, i.e. it will be a constant vector. Therefore $\frac{d\mathbf{B}}{ds} = \mathbf{0}$ and since $\tau = \left\| \frac{d\mathbf{B}}{ds} \right\|$ we see that any plane curve will have zero torsion everywhere along the curve.

Example: Consider motion along a helix, e.g. $\mathbf{r}(t) = \langle a\cos\omega t, a\sin\omega t, bt \rangle$. We calculate the velocity $\mathbf{v}(t) = \langle -a\omega\sin\omega t, a\omega\cos\omega t, b \rangle$, the speed $\|\mathbf{v}(t)\| = \|\langle -a\omega\sin\omega t, a\omega\cos\omega t, b \rangle\| = \sqrt{a^2\omega^2 + b^2}$ (constant speed), and the unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\langle -a\omega\sin\omega t, a\omega\cos\omega t, b \rangle}{\sqrt{a^2\omega^2 + b^2}}$.

We then have $\mathbf{T}'(t) = \frac{\langle -a\omega^2\cos\omega t, -a\omega^2\sin\omega t, 0 \rangle}{\sqrt{a^2\omega^2 + b^2}}$, so $\|\mathbf{T}'(t)\| = \frac{a\omega^2}{\sqrt{a^2\omega^2 + b^2}}$ and $\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|} = \frac{a\omega^2}{a^2\omega^2 + b^2}$ (constant curvature).

We also get that $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \langle -\cos\omega t, -\sin\omega t, 0 \rangle$, so we can calculate that

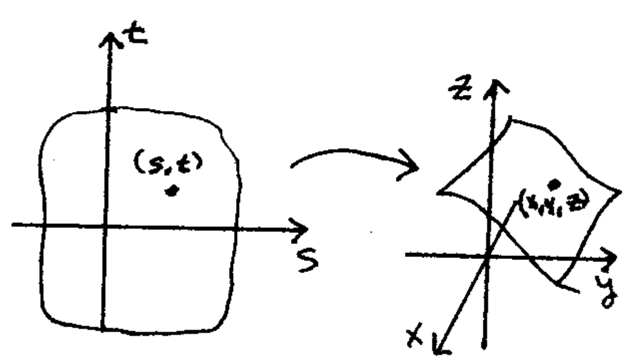
$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \left(\frac{\langle -a\omega\sin\omega t, a\omega\cos\omega t, b \rangle}{\sqrt{a^2\omega^2 + b^2}} \right) \times \langle -\cos\omega t, -\sin\omega t, 0 \rangle = \frac{\langle b\sin\omega t, -b\cos\omega t, a\omega \rangle}{\sqrt{a^2\omega^2 + b^2}}.$$

Now $\frac{d\mathbf{B}}{dt} = \frac{d\mathbf{B}}{ds}\frac{ds}{dt} = \|\mathbf{v}(t)\| \frac{d\mathbf{B}}{ds}$, so $\frac{d\mathbf{B}}{ds} = \frac{1}{\|\mathbf{v}(t)\|}\frac{d\mathbf{B}}{dt}$ and therefore $\tau = \left\| \frac{d\mathbf{B}}{ds} \right\| = \frac{1}{\|\mathbf{v}(t)\|}\left\| \frac{d\mathbf{B}}{dt} \right\|$. From the expression above we calculate $\frac{d\mathbf{B}}{dt} = \frac{b\omega\langle \cos\omega t, \sin\omega t, 0 \rangle}{\sqrt{a^2\omega^2 + b^2}}$, so $\left\| \frac{d\mathbf{B}}{dt} \right\| = \frac{b\omega}{\sqrt{a^2\omega^2 + b^2}}$ Therefore $\tau = \left\| \frac{d\mathbf{B}}{ds} \right\| = \frac{b\omega}{a^2\omega^2 + b^2}$.

If you consider movement around a helix with a fixed radius *a* and angular velocity ω, then you can consider the torsion as a function of the vertical velocity *b*, i.e. $\tau(b) = \frac{b\omega}{a^2\omega^2 + b^2}$. Critical point analysis then shows that the helix will have maximum torsion when $b = a\omega$, i.e. if the vertical component of the velocity matches the angular velocity associated only with the circular motion. In other words, if your ascent makes a constant angle with the horizontal of $45°$. Less than that brings you closer to a plane curve (with zero torsion) and more than that causes the helix to become more straightened out.

Parameterized surfaces and tangent vectors

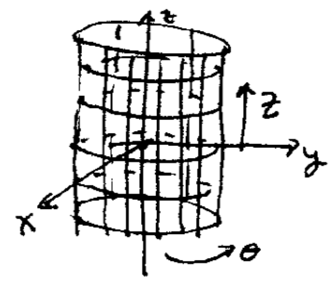
Just as we can parameterize a curve (one degree of freedom) using a single parameter as $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, we can do a similar construction for a surface (two degrees of freedom). The basic idea is to come up with a one-to-one correspondence between locations in a two-dimensional parameter space and the given surface. As with curves, there are many ways to do this for any given surface.



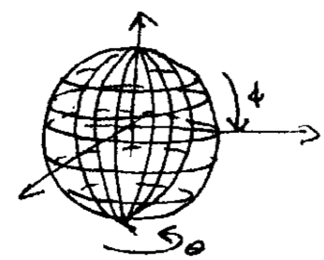
The basic idea is that we will express $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$ using two independent parameters. Varying each parameter independently will allow the freedom to move about on the given surface. Varying just one parameter at a time will produce parameterized curves embedded in the surface. We'll discuss surface in greater detail later in the course, but for now consider the following three examples:

Cylinder of radius *R*: Using polar coordinates to describe a circle for x and y and allowing the z coordinate to vary freely, we can describe the cylinder with equation $x^2 + y^2 = R^2$ parametrically using θ and z as independent parameters:

$\begin{Bmatrix} x = R\cos\theta \\ y = R\sin\theta \\ z = z \end{Bmatrix}$. Expressed in this way, z does double-duty as both a coordinate as a parameter. While this may make intuitive sense, it's generally best to distinguish the parameters as things you choose and the coordinates that then correspond to a given choice of parameters. We might therefore express the parametrization as $\mathbf{r}(\theta, t) = \langle R\cos\theta, R\sin\theta, t\rangle$, but often we'll simply write $\mathbf{r}(\theta, z) = \langle R\cos\theta, R\sin\theta, z\rangle$. Note that if we fix a value of z and vary θ only, we produce circular curves at the given *z* level. Similarly, if we fix θ and vary z only we'll get the vertical lines that are also embedded in the cylinder.



Sphere of radius *R*: If you recall the discussion of spherical coordinates from Lecture #1, we had the relations $\begin{Bmatrix} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{Bmatrix}$. Fixing the radius at $\rho = R$, we get a sphere described parametrically as $\begin{Bmatrix} x = R\cos\theta\sin\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\phi \end{Bmatrix}$. We can express this as $\mathbf{r}(\phi, \theta) = \langle R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi\rangle$. We get the entire sphere by varying $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. The one-to-one correspondence breaks down at the poles, but otherwise it's a one-to-one correspondence between pairs of parameters and points on the sphere. Note that if we fix the azimuth angle θ (longitude) and choose to think of the parameter ϕ as "time", then by varying this parameter we'll be heading south along a longitude. By treating θ as constant, we can differentiate to get the velocity vector $\mathbf{v}_\phi = \langle R\cos\theta\cos\phi, R\sin\theta\cos\phi, -R\sin\phi\rangle$ which basically says which way is "south". Similarly we could have fixed the latitude by fixing $\phi =$ constant and thinking of the parameter θ as "time". Differentiating with respect to this parameter would then yield the "east" velocity vector $\mathbf{v}_\theta = \langle -R\sin\theta\sin\phi, R\cos\theta\sin\phi, 0\rangle$. Note that the *z*-component would be zero which makes sense if you're heading east along a latitude.



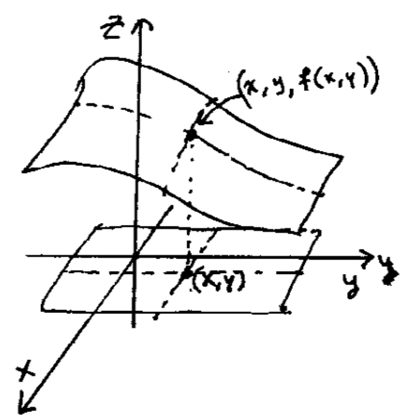
Graph of a function $z = f(x, y)$

Given a reasonably well-behaved function $f(x, y)$, it's graph will generally yield a surface with a natural one-to-one correspondence between any (x, y) and the point $(x, y, f(x, y))$ on its graph.

As parametric equations we might write: $\begin{Bmatrix} x = x \\ y = y \\ z = f(x, y) \end{Bmatrix}$.

As a vector-valued function we would write $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$.

Again, we probably should express this as $\mathbf{r}(s, t) = \langle s, t, f(s, t)\rangle$ to keep clear the distinction between parameters and coordinates, but as long as



you're clear about the context this isn't really necessary.

Looking at the sketch of the graph, it should be clear that each of the cross-section curves pictured will have well-defined slopes just as long as the graph surface is smooth. We'll formalize that in the next lecture when we discuss in greater the ideas of limits, continuity , and differentiability, but for now we can simply note that these slopes can be calculated by simply varying just one of the independent variables (parameters) at a time.

Partial derivatives

Referring to the previous example, we define the partial derivatives of $f(x,y)$ as follows:

$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}\right) = f_x(x,y)$ is the **partial derivative of *f* with respect to *x***

$\frac{\partial f}{\partial y} = \lim_{\Delta x \to 0}\left(\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}\right) = f_y(x,y)$ is the **partial derivative of *f* with respect to *y***

These represent, respectively, the slopes of tangent lines to the cross-section curves in the graph of $f(x,y)$ as we vary just x and just y holding the other variable fixed. The same idea extends to functions of any number of variables. To calculate the partial derivative with respect to one independent variable, you treat any other variables as though they were constant. In Latin, the expression is *ceteris paribus* meaning "with other things the same" or "all other things being equal".

Generally we will rarely need to use this formal definition to calculate partial derivatives. By simply understanding what the definitions is really saying, we can just use familiar rules for differentiation by literally "treating the other variables as though constant".

For example, if $f(x,y) = x^2 y + 7y^3$ we easily calculate that $\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial f}{\partial y} = x^2 + 21y^2$.

Finally, we can take a vector approach with the parameterization $\mathbf{r}(x,y) = \langle x, y, f(x,y)\rangle$ to produce

$\frac{\partial \mathbf{r}}{\partial x} = \langle 1, 0, f_x \rangle$ as a tangent vector to the cross-section where only x is varied;

$\frac{\partial \mathbf{r}}{\partial y} = \langle 0, 1, f_y \rangle$ as a tangent vector to the cross-section where only y is varied.

The cross product of these two tangent vectors to the graph surface will then give a normal vector to the graph at any point on the graph: $\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \langle 1, 0, f_x(x_0,y_0)\rangle \times \langle 0, 1, f_y(x_0,y_0)\rangle = \langle -f_x(x_0,y_0), -f_y(x_0,y_0), 1\rangle = \mathbf{n}$.

We can then use this normal vector and the point $(x_0, y_0, f(x_0,y_0))$ to get an equation for a tangent plane to the graph: $\langle -f_x(x_0,y_0), -f_y(x_0,y_0), 1\rangle \cdot \langle x - x_0, y - y_0, z - f(x_0,y_0)\rangle = 0$ or, after solving for z:

$$\boxed{z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$ (equation for tangent plane to graph)

Once we clear up a few ideas about what it means to be differentiable, we will then be able to say that for any (x,y) near (x_0,y_0) it should be the tangent plane provides a very good linear approximation to the actual graph.

That is:

$$\boxed{f(x,y) \cong f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$ (linear approximation for (x,y) near (x_0,y_0))

Notes by Robert Winters

Revised January 7, 2017