Multivariable Calculus – Lecture #3 Notes

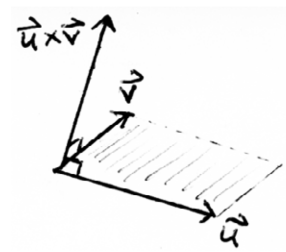
This lecture completes the discussion of the cross product in $\mathbf{R}^3$ and addresses the variety of different ways that mathematical objects can be described via equations, functions, graphs, parameterization in $\mathbf{R}^2$, $\mathbf{R}^3$, and $\mathbf{R}^n$.

Geometric definition of the Cross Product: Suppose **u** and **v** are vectors in $\mathbf{R}^3$. Then the cross product $\mathbf{u}\times\mathbf{v}$ is the unique vector in $\mathbf{R}^3$ such that:

(1) $\mathbf{u}\times\mathbf{v}$ is orthogonal to both **u** and **v**;

(2) the magnitude of the cross product $\|\mathbf{u}\times\mathbf{v}\|$ is equal to the area of the parallelogram determined by **u** and **v**;

(3) $\mathbf{u}\times\mathbf{v}$ is oriented according to the Right-Hand Rule (as explained in class and elsewhere).



Algebraic definition of the Cross Product

If we use the above geometric criteria and some desirable properties such as the distributive law, we can derive that the only plausible algebraic definition for the **cross product** of two vectors (defined only in $\mathbf{R}^3$) is that given two vectors $\mathbf{u}=\langle u_1,u_2,u_3\rangle$ and $\mathbf{v}=\langle v_1,v_2,v_3\rangle$ in $\mathbf{R}^3$, we define the **cross product** $\mathbf{u}\times\mathbf{v}$ as follows:

$$\mathbf{u}\times\mathbf{v}=\langle u_1,u_2,u_3\rangle\times\langle v_1,v_2,v_3\rangle=\langle u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1\rangle$$

This may also be expressed in terms of 2×2 determinants as:

$$\mathbf{u}\times\mathbf{v}=\left\langle \begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix},\begin{vmatrix}u_3 & u_1\\ v_3 & v_1\end{vmatrix},\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\right\rangle=\left\langle \begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix},-\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix},\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\right\rangle$$

This representation allows us to perform the calculation by creating a 2×3 array from the given two vectors $\begin{bmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{bmatrix}$ and then respectively covering the 1st, 2nd, and 3rd columns and calculating the determinant of the resulting 2×2 determinants (with appropriate sign switch of the middle component.

Some people prefer to express this procedure using $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ notation by formally calculating the 3×3 determinant $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}=\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\mathbf{k}$.

Using only this algebraic definition for the cross product, we can derive the following properties:

Algebraic Properties of the Cross Product: Suppose **u**, **v**, and **w** are vectors in $\mathbf{R}^3$ and that t is any scalar.

1) $\mathbf{v}\times\mathbf{u}=-\mathbf{u}\times\mathbf{v}$ (anticommutative) [Corollary: $\mathbf{u}\times\mathbf{u}=\mathbf{0}$ for any vector $\mathbf{u}$]

2) $\left\{\begin{matrix}\mathbf{u}\times(\mathbf{v}+\mathbf{w})=\mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}\\ (\mathbf{u}+\mathbf{v})\times\mathbf{w}=\mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}\end{matrix}\right\}$ (left and right distributive laws)

3) $(t\,\mathbf{u})\times\mathbf{v}=t(\mathbf{u}\times\mathbf{v})=\mathbf{u}\times(t\,\mathbf{v})$ (how the dot product behaves relative to scaling of vectors)

4) $\mathbf{u}\times\mathbf{0}=\mathbf{0}$

5) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$ (triple scalar product)

6) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ (triple vector product)

All of the above algebraic properties of the cross product except for the last one are straightforward. You can prove the last one by noting that the first component would be:

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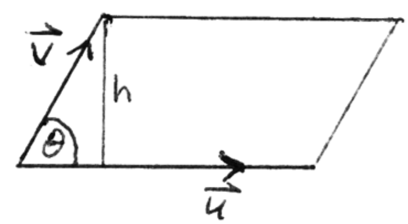
$$\begin{vmatrix} u_2 & u_3 \\ v_3w_1 - v_1w_3 & v_1w_2 - v_2w_1 \end{vmatrix} = u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3) = u_2v_1w_2 - u_2v_2w_1 - u_3v_3w_1 + u_3v_1w_3$$
$$= u_2v_1w_2 - u_2v_2w_1 - u_3v_3w_1 + u_3v_1w_3 + u_1v_1w_1 - u_1v_1w_1 = (u_1w_1 + u_2w_2 + u_3w_3)v_1 - (u_1v_1 + u_2v_2 + u_3v_3)w_1$$
$$= (\mathbf{u}\cdot\mathbf{w})v_1 - (\mathbf{u}\cdot\mathbf{v})w_1$$

Similarly, we can show that the 2nd and 3rd components are $(\mathbf{u}\cdot\mathbf{w})v_2 - (\mathbf{u}\cdot\mathbf{v})w_2$ and $(\mathbf{u}\cdot\mathbf{w})v_3 - (\mathbf{u}\cdot\mathbf{v})w_3$.

Together these give that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$. Physicists (and others) often refer to this property as the "BAC-CAB Rule" and express it as $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$.

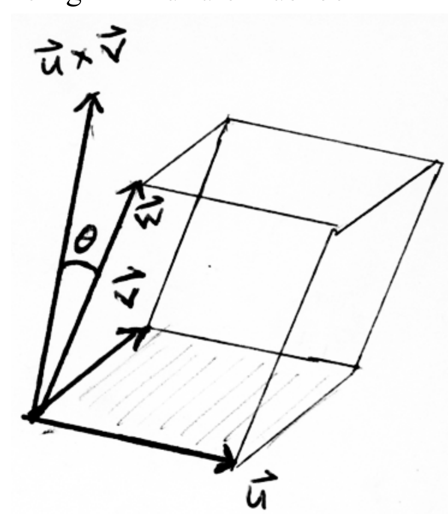
To complete the demonstration that the algebraic and geometric definitions of the cross product are equivalent, we can now show that the algebraic definition and the algebraic properties above imply all of the geometric properties that we used to motivate the algebraic definition. Specifically:

(1) $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = (\mathbf{u}\times\mathbf{u})\cdot\mathbf{v} = \mathbf{0}\cdot\mathbf{v} = 0$, so $\mathbf{u}\times\mathbf{v}$ is orthogonal to the vector **u**;
$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{v}) = \mathbf{u}\cdot\mathbf{0} = 0$, so $\mathbf{u}\times\mathbf{v}$ is orthogonal to the vector **v**.



(2) If we consider the parallelogram determined by **u** and **v** and let θ be the angle between these vectors (drawing a picture is advisable), then the area of the parallelogram will be given by
(length of base)($\perp$ height) $= \|\mathbf{u}\|\|\mathbf{v}\||\sin\theta|$. Squaring both sides gives
$(\text{Area})^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1-\cos^2\theta) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2\cos^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$.
On the other hand, $\|\mathbf{u}\times\mathbf{v}\|^2 = (\mathbf{u}\times\mathbf{v})\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\cdot[\mathbf{v}\times(\mathbf{u}\times\mathbf{v})] = \mathbf{u}\cdot[(\mathbf{v}\cdot\mathbf{v})\mathbf{u} - (\mathbf{v}\cdot\mathbf{u})\mathbf{v}] = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$.
Therefore $(\text{Area})^2 = \|\mathbf{u}\times\mathbf{v}\|^2$, so $\text{Area} = \|\mathbf{u}\times\mathbf{v}\|$.

(3) You can easily calculate using the algebraic definition that $\mathbf{i}\times\mathbf{j} = \mathbf{k}$ which satisfies the Right-Hand Rule. Then argue using a continuity argument that if this is true for these two vectors than by continuously varying these vectors in $\mathbf{R}^3$ to align them with the given two vectors, the right-hand rule must be preserved.



Volume and the Triple Scalar Product:
Property (5) of the Algebraic Properties of the Cross Product (the triple scalar product) has an interesting geometric interpretation. Note that $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \|\mathbf{u}\times\mathbf{v}\|\|\mathbf{w}\|\cos\theta$ where θ is the angle between the vectors $\mathbf{u}\times\mathbf{v}$ and **w**. [For simplicity, we're considering the case where θ is an acute angle. If it is obtuse, everything's the same except for a change in sign.] Note that $\|\mathbf{u}\times\mathbf{v}\|$ gives the area of the parallelogram that forms the base of the ***parallelepiped*** determined by the three vectors **u**, **v**, and **w** in $\mathbf{R}^3$; and $\|\mathbf{w}\|\cos\theta$ corresponds to the perpendicular height of this parallelepiped, i.e. the scalar projection of the vector w in the direction of the vector $\mathbf{u}\times\mathbf{v}$. Thus $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \|\mathbf{u}\times\mathbf{v}\|(\|\mathbf{w}\|\cos\theta) = (\text{area of base})(\perp\text{ height}) = \text{Volume}$ of the parallelepiped (up to sign). So the volume is $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$ or $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|$.

The triple scalar product may also be calculated as $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$, a 3×3 determinant.

Using the tools of the dot product and cross product, we can easily calculate lengths, angles, areas, and volumes. We can also use these tools to find most or all of the necessary ingredients for working with lines, planes, and their intersections and similar constructions.

Example 1: Find an equation for the plane that contains the three points $P(1,1,1)$, $Q(3,1,2)$, and $R(4,5,-2)$.

Solution: First calculate the difference vectors $\overrightarrow{PQ} = \langle 2,0,1 \rangle$ and $\overrightarrow{PR} = \langle 3,4,-3 \rangle$. A normal vector to the plane is then given by their cross product $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2,0,1 \rangle \times \langle 3,4,-3 \rangle = \langle -4,9,8 \rangle = \mathbf{n}$. We can then use this normal vector and any one of the three given points to get the equation of the plane to be $\boxed{-4x+9y+8z=13}$.

Example 2: Find the area of the triangle PQR for the points given in the previous example.

Solution: The area of the triangle is just half the area of the parallelogram determined by the vectors $\overrightarrow{PQ} = \langle 2,0,1 \rangle$ and $\overrightarrow{PR} = \langle 3,4,-3 \rangle$, and that area is $\left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\| = \left\| \langle -4,9,8 \rangle \right\| = \sqrt{16+91+64} = \sqrt{171}$. So the area of the triangle is $\boxed{\tfrac{1}{2}\sqrt{171}}$.

Example 3: Find the distance from the point $S(2,5,7)$ to the plane passing through the three points $P(1,1,1)$, $Q(3,1,2)$, and $R(4,5,-2)$.

Solution: We determined a normal vector to this plane above, namely $\mathbf{n} = \langle -4,9,8 \rangle$. Observe that (up to sign) the distance from the point to the plane is just the scalar projection onto this normal vector of the vector from any point on the plane to the point S. For example, we can calculate $\overrightarrow{PS} = \langle 1,4,6 \rangle$ and then find the scalar projection by $\overrightarrow{PS} \cdot \dfrac{\mathbf{n}}{\|\mathbf{n}\|} = \langle 1,4,6 \rangle . \dfrac{\langle -4,9,8 \rangle}{\sqrt{171}} = \dfrac{80}{\sqrt{171}}$. In general, we would find this distance by $\left| \overrightarrow{PS} \cdot \dfrac{\mathbf{n}}{\|\mathbf{n}\|} \right|$.

Example 4: Find the volume of the tetrahedron $PQRS$ using the above points.

Solution: Note that the area of base triangle is given by $\frac{1}{2}\left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\|$ and the perpendicular height (over this base) is given by the (absolute value of) the scalar projection of the vector $\overrightarrow{PS} = \langle 1,4,6 \rangle$ onto the normal vector $\overrightarrow{PQ} \times \overrightarrow{PR}$. The volume of the tetrahedron will then be given by

$$\frac{1}{3}(\text{area base})(\perp \text{height}) = \frac{1}{3}\left(\frac{1}{2}\left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\| \right) \left| \overrightarrow{PS} \cdot \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\|} \right| = \frac{1}{6}\left| (\overrightarrow{PQ} \times \overrightarrow{PR}) \cdot \overrightarrow{PS} \right|.$$

Note that this is one-sixth the volume of the parallelepiped (triple scalar product) determined by the vectors $\overrightarrow{PQ} = \langle 2,0,1 \rangle$, $\overrightarrow{PR} = \langle 3,4,-3 \rangle$, and $\overrightarrow{PS} = \langle 1,4,6 \rangle$. So $(\overrightarrow{PQ} \times \overrightarrow{PR}) \cdot \overrightarrow{PS} = \langle -4,9,8 \rangle \cdot \langle 1,4,6 \rangle = 80$ and the volume is therefore $\frac{80}{6} = \frac{40}{3} = \boxed{13\tfrac{1}{3}}$.

Equations, Functions, Graphs, Parameterizations

Many familiar objects can be described algebraically in a variety of different but equivalent ways, e.g. as the set of points satisfying one or more algebraic equations (and there are many ways in which this can be done), as the graph of a function, or via parametric equations. We'll also soon use the idea of level sets of a function as yet another alternative.

Example: A line in $\mathbf{R}^2$

We are all familiar with the **slope intercept form** $\boxed{y = mx + b}$ and the **point slope form** $\boxed{y - y_0 = m(x - x_0)}$ of a line. If we define $f(x) = mx + b$, we can also consider this line to be the **graph of the function** $\boxed{\{(x,y) : y = f(x), x \in \mathbf{R} = \text{domain of } f\}}$. Many of you may also be familiar with the **double intercept form** of a line (assuming both a and b are nonzero) given by $\boxed{\frac{x}{a}+\frac{y}{b}=1}$. There is also the standard linear form(s) of the line given as $\boxed{cx+dy=e}$. Given a point (x_0, y_0) on the line and a direction vector $\mathbf{v}=\langle A, B\rangle$, we have now also seen the **vector form** $\boxed{\mathbf{x}=\mathbf{x}_0+t\mathbf{v}}$ where $\mathbf{x}_0=\langle x_0, y_0\rangle$ is the position vector of the given point and $\mathbf{x}=\langle x, y\rangle$ is the position vector of any other point on this line. This yields the corresponding **parametric equations** $\boxed{\begin{Bmatrix} x=x_0+At \\ y=y_0+Bt \end{Bmatrix}_{t\in\mathbf{R}}}$ for the line. We could also (in the case where *A* and *B* are nonzero) eliminate the parameter by solving for $t=\boxed{\frac{x-x_0}{A}=\frac{y-y_0}{B}}$, called **symmetric equations** for this line. Shortly we'll see that if we define the function of two variables $F(x,y)=cx+dy$, then the line is just a **level set** for this function, i.e. a set of the form $\boxed{\{(x,y): F(x,y)=\text{constant}\}}$. The point is that there are many different ways to represent this line.

Example: A line in $\mathbf{R}^3$

Given a point (x_0, y_0, z_0) on the line and a direction vector $\mathbf{v}=\langle A, B, C\rangle$, we derived the **vector form** $\boxed{\mathbf{x}=\mathbf{x}_0+t\mathbf{v}}$ and the corresponding **parametric equations** $\boxed{\begin{Bmatrix} x=x_0+At \\ y=y_0+Bt \\ z=z_0+Ct \end{Bmatrix}_{t\in\mathbf{R}}}$ for the line. We could also (in the case where *A*, *B*, and *C* are nonzero) eliminate the parameter by solving for $t=\boxed{\frac{x-x_0}{A}=\frac{y-y_0}{B}=\frac{z-z_0}{C}}$, called **symmetric equations** for this line.

Example: A plane in $\mathbf{R}^3$

We have seen that given a point (x_0, y_0, z_0) and a normal vector $\mathbf{n}=\langle A, B, C\rangle$, we can express the equation of the plane as $\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)=0$ where $\mathbf{x}_0=\langle x_0, y_0, z_0\rangle$ is the position vector of the given point and $\mathbf{x}=\langle x, y, z\rangle$ is the position vector of any other point on the plane. This yields the **single algebraic equation** $\boxed{A(x-x_0)+B(y-y_0)+C(z-z_0)=0}$ which can also be written as $\boxed{Ax+By+Cz=D}$ for appropriate choice of *D*. Soon we will consider functions of several variables such as $F(x,y,z)=Ax+By+Cz$ and we will then be able to consider the plane as a **level set** $\{(x,y,z): F(x,y,z)=D\}$.

We can also describe a plane <u>parametrically</u> by choosing a single point (x_0, y_0, z_0) and two nonparallel (independent) vectors <u>lying in the plane</u> $\mathbf{u}=\langle u_1, u_2, u_3\rangle$ and $\mathbf{v}=\langle v_1, v_2, v_3\rangle$. Then, in a construction analogous to that for a line, we can say that $\boxed{\mathbf{x}=\mathbf{x}_0+s\mathbf{u}+t\mathbf{v}}$ where *s* and *t* are independent parameters. By ranging over all possible choices of the parameters (s,t) we can realize all possible points on the plane. We can express this explicitly as $\langle x,y,z\rangle=\langle x_0,y_0,z_0\rangle+s\langle u_1,u_2,u_3\rangle+t\langle v_1,v_2,v_3\rangle$ or $\boxed{\begin{Bmatrix} x=x_0+u_1s+v_1t \\ y=y_0+u_2s+v_2t \\ z=z_0+u_3s+v_3t \end{Bmatrix}_{s,t\in\mathbf{R}}}$, call **parametric equations for the plane**.

Example: Give parametric equations for the plane containing the three points $P(1,1,1)$, $Q(3,1,2)$, and $R(4,5,-2)$.

Solution: Earlier we calculated $\overrightarrow{PQ} = \langle 2,0,1 \rangle$ and $\overrightarrow{PR} = \langle 3,4,-3 \rangle$. Both of these vectors lie in the plane, so we can use them and the point $P(1,1,1)$ to get the parametric equations $\left\{ \begin{array}{l} x = 1 + 2s + 3t \\ y = 1 \qquad + 4t \\ z = 1 + 1s - 3t \end{array} \right\}_{s,t \in \mathbf{R}}$. It's interesting to not that if we take the algebraic equation derived earlier for this plane, namely $-4x + 9y + 8z = 13$, we could arbitrarily (and independently) choose $x = s$ and $y = t$ and then solve for $z = \frac{13}{8} + \frac{1}{2}x - \frac{9}{8}y = \frac{13}{8} + \frac{1}{2}s - \frac{9}{8}t$.

These can be expressed as the parametric equations $\left\{ \begin{array}{l} x = s \\ y = t \\ z = \frac{13}{8} + \frac{1}{2}s - \frac{9}{8}t \end{array} \right\}_{s,t \in \mathbf{R}}$. It's algebraically a lot simpler to instead express the arbitrary choices as $x = 2s$ and $y = 8t$ and then solve for $z = \frac{13}{8} + s - 9t$. This yields the simpler parametric equations $\left\{ \begin{array}{l} x = 2s \\ y = 8t \\ z = \frac{13}{8} + s - 9t \end{array} \right\}_{s,t \in \mathbf{R}}$. These can also be expressed in vector form as

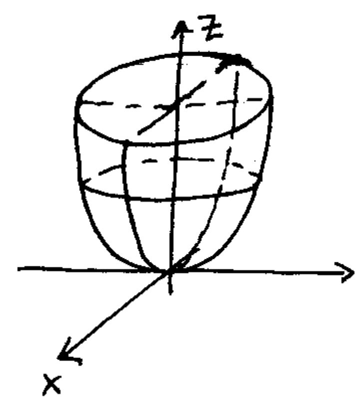
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{13}{8} \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$. From this we can see that the point $(0,0,\frac{13}{8})$ is a point on the plane and the two vectors $\mathbf{u} = \langle 2,0,1 \rangle$ and $\mathbf{v} = \langle 0,8,-9 \rangle$ are two independent vectors lying in the plane. You may wish to check that these vectors are, in fact, both orthogonal to the normal vector we found earlier. This gives a different, but equivalent parametrization than the one we constructed above from the given points in this plane.

Functions of two variables

In the most basic sense, a function of two variables is just a function $f(x, y)$ that assigns to any pair of independent inputs (x, y) an output value denoted by $f(x, y)$. This could be something defined in nature such as the latitude and longitude of a location having a particular temperature (at a given time). As such we might be interested in better understanding how these temperatures are distributed. We often specify a function simply by giving a formula saying how the output value is to be calculated from the inputs. For example, we might say that $f(x, y) = x^2 + y^2$. That would likely be sufficient to satisfy your calculator's quest for knowledge, but as human beings we often prefer to have a more geometric realization of the functions we define. There are various competing ways in which this can be accomplished.

The **graph of a function** $f(x, y)$ is the point set in $\mathbf{R}^3$ defined as $\{(x, y, z) : z = f(x, y), (x, y) \in \text{Domain}(f)\}$. The domain of a function is the same familiar idea as you will have seen when dealing with functions of one variable, i.e. the set of all points (x, y) where the function is defined.

Generically, the graph of a function of two variables will be a surface in $\mathbf{R}^3$.



For example, the function $f(x, y) = x^2 + y^2$ is defined for all (x, y) and its graph will be defined by the equation $z = x^2 + y^2$. One very effective tool for understanding and sketching graphs is to use cross-sections. This simply means that you try various choices of $x = \text{constant}$, $y = \text{constant}$, and $z = \text{constant}$ and see how these intersect the surface defined by the given equation. In the case of $z = x^2 + y^2$, choosing $x = \text{constant}$ will give an upward parabola. The same will be the case for cross-sections with $y = \text{constant}$. Choosing horizontal cross sections with $z = \text{constant}$ will give no intersection for negative choices

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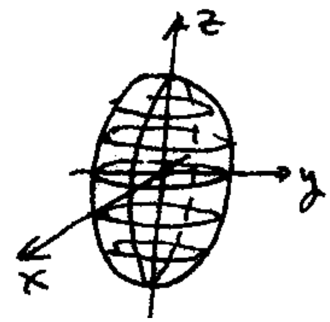
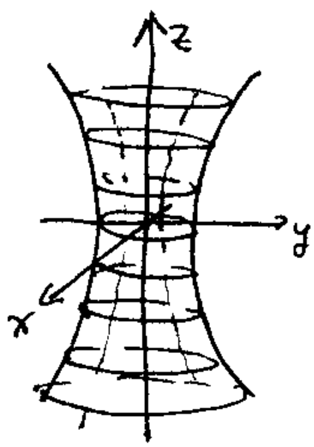
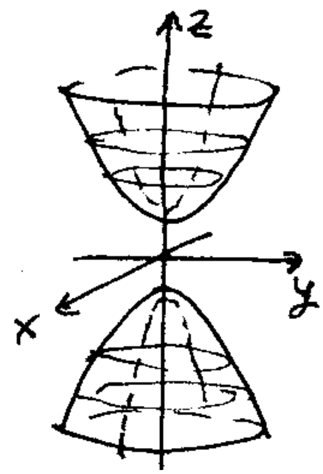
of this constant, will give a single point (the origin) when the constant is 0, and will give circular cross sections of the form $x^2 + y^2 = c$ for positive choices of c. The resulting surface is called a paraboloid.

This **method of cross-sections** can be used to sketch surfaces defined by equations other than just graphs of functions. For example, any equation of the form $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ will have ellipses as cross-section for any choice of $x = \text{constant}$ where the constant lies between $-a$ and $+a$ and similarly with $y = \text{constant}$ and $z = \text{constant}$ cross-sections. The resulting figure is called an ellipsoid.



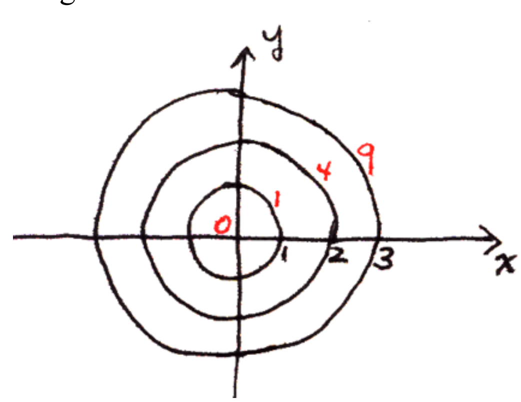
If we consider a surface defined by an equation of the form $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1$, we'll get a surface where the horizontal cross-sections are circular and the vertical cross sections are all hyperboloids. The resulting figure is called a hyperboloid of one sheet (left figure). Note, in particular, the narrowest cross section corresponding to the $z = 0$ slice. This is fundamentally different than a surface with equation of the form $-\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ which defines a surface known as a hyperboloid of two sheets (right figure).





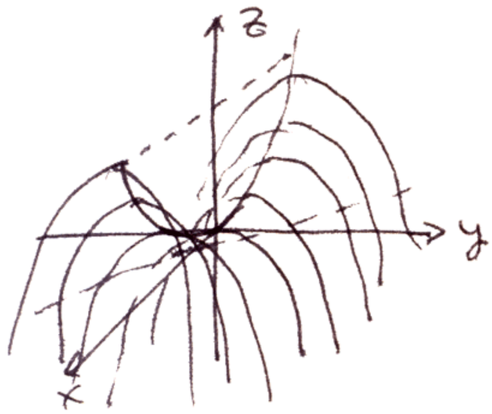
Graphing a function of two variables can sometimes be a challenge. A great alternative is to instead focus on the **level sets** (contours) of the function. For a function of two variables, the level sets will be curves in $\mathbf{R}^2$ of the form $\{(x, y) : f(x, y) = c\}$ for a sampler of values of the constant c. This is exactly the same idea used in topographic maps as a means of indicating elevations without actually having to visit or reconstruct the terrain.

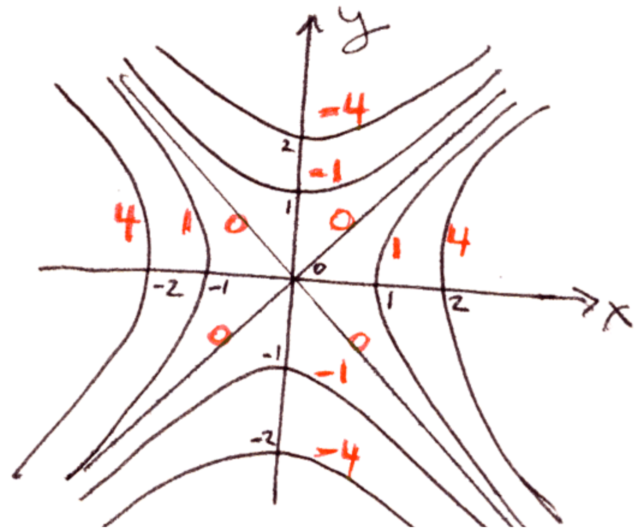
For example, the function $f(x, y) = x^2 + y^2$ will have level sets of the form $x^2 + y^2 = c$. The idea is to draw a sampler of level sets (contours) and to mark the value associated with each contour on the respective level set. It is generally best to use a different color to indicate these values. With a little practice you should be able to easily "see" the graph by imagining the "**contour diagram**" to be something like a "**bird's eye view**" of the graph with the elevations indicated by the values marked on the contours. It's important to note that the function may represent something like temperature or pressure rather elevations and this case we often refer to the level sets as either isotherms or isobars. In economics, the set of all points on which some economically defined quantity is constant are generally referred to simply as "**isoquants**".



Note, in particular, that level sets corresponding to different levels can never intersect because if they did the function $f(x,y)$ would have to assume different values at a single point. This is why level sets tend to exhibit a sort of parallelism where contours corresponding to nearby distinct values "line up" without intersecting each other. Observe also that if we choose to draw contours with equally spaced values, the contours will be much more closely spaced in regions where the graph of the function is steep (rapidly increasing or decreasing). These observations can greatly help in visualizing a graph from a corresponding contour diagram.

Example: Contour diagram vs. graph of the function $f(x,y) = x^2 - y^2$
If we were to rely solely on the graph to visualize this function, the graph would be defined by the equation $z = x^2 - y^2$. The $x = \text{constant}$ cross-sections will give downward parabolas. The $y = \text{constant}$ cross-sections will give upward parabolas. Choosing horizontal cross sections with $z = \text{constant}$ will hyperbolas. Drawing this can be a bit of a challenge, but the resulting graph is known as a **saddle surface**.





If you look at the contour diagram (right) and image the positive values rising up toward you in this bird's eye view and the negative values receding away from you, you should be able to "see" the saddle surface.

Parameterized curves
We have already seen parameterization in the context of lines (one parameter) and planes (two parameters). The idea of parameterization is very general and provides a much more *dynamic* way of thinking about curves and surfaces than just relying on algebraic equations.

If, for example, we wanted to consider a circle of radius 2 in the *xy*-plane centered around the origin, we could describe it by the single algebraic equation $x^2 + y^2 = 4$. An equation is something like a True/False question. If you would like to determine if a given point lies on the curve defined by this equation, you plug the values into the equation to see if it's satisfied. If it is, then you're on the curve. If not, then the point is not on the curve. This is fundamentally a static point of view. A parametric view is more dynamic in that it can provide "a notion of motion" if you imagine the parameter to be something like time. As you vary the parameter you "move" along the curve. We can then borrow some basic ideas from physics to assist in the description.

Definition: A **parametrized curve** in $\mathbf{R}^n$ is a vector-valued function $\mathbf{r}(t) = \langle x_1(t), x_2(t), \cdots, x_n(t) \rangle$ in which the component functions vary (at least) continuously in *t*.

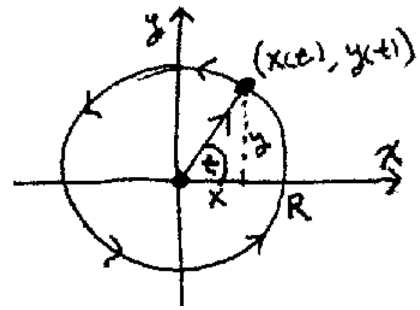
In particular, a parametrized curve in $\mathbf{R}^2$ is a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ where we think of $\mathbf{r}(t)$ as tracing out the position vector of points along a curve as we vary the parameter *t*. Similarly, a parametrized curve in $\mathbf{R}^3$ (also called a space curve) is a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ where we think of $\mathbf{r}(t)$ as tracing out the position vector of points along a curve in space as we vary the parameter *t*.

We have already encountered parameterized curves when we considered lines. For example, the line in $\mathbf{R}^3$ passing through the points $P(1,1,1)$ and $Q(3,-2,5)$ can be described parametrically by finding the direction vector $\mathbf{v} = \overrightarrow{PQ} = \langle 2,-3,4 \rangle$ and obtaining the parametric equations $\left\{ \begin{array}{l} x = 1+2t \\ y = 1-3t \\ z = 1+4t \end{array} \right\}_{t\in\mathbf{R}}$. The vector form of the line actually describes the parameterized curve by the position vector $\mathbf{r}(t) = \langle 1+2t, 1-3t, 1+4t \rangle$. If we of the parameter is time, we "move along the line".

We can describe all sorts of other curves parametrically, but there's something of an art to finding the parametrization. There are also many different ways to parameterize the same curve considered as a point set.

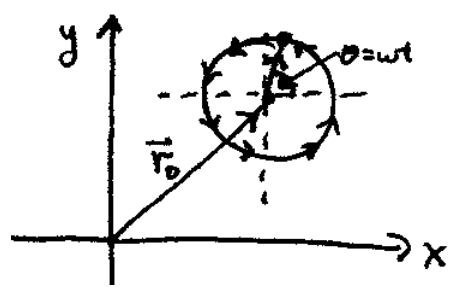
Parametrizing a circle

Perhaps the simplest way to parameterize a circle is to use the central angle as the parameter. For a circle of radius R centered at the origin in $\mathbf{R}^2$, basic trigonometry gives that $\left\{ \begin{array}{l} x = R\cos t \\ y = R\sin t \end{array} \right\}$ which can also be expressed as $\mathbf{r}(t) = \langle R\cos t, R\sin t \rangle = R\langle \cos t, \sin t \rangle$. We can trace one circuit about the circle by varying $0 \le t < 2\pi$.

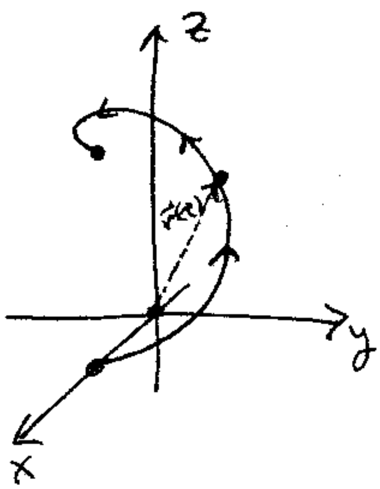


If we want to "move more quickly" about the circle, we might instead think of the central angle as $\theta = \omega t$ where ω is a prescribed "angular velocity". We would then have $\left\{ \begin{array}{l} x = R\cos \omega t \\ y = R\sin \omega t \end{array} \right\}$ or $\mathbf{r}(t) = \langle R\cos \omega t, R\sin \omega t \rangle = R\langle \cos \omega t, \sin \omega t \rangle$.

To parameterize a **circle with center translated** to the point (x_0, y_0) we can add the position vector of the center plus a vector from the center to the edge to get $\mathbf{r}(t) = \langle x_0, y_0 \rangle + R\langle \cos \omega t, \sin \omega t \rangle = \langle x_0 + R\cos \omega t, y_0 + R\sin \omega t \rangle$.



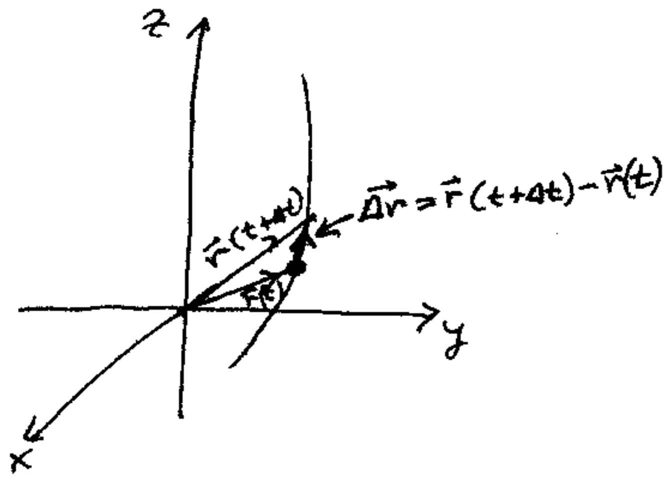
Parameterizing a helix



We can parameterize a vertical **helix** in $\mathbf{R}^3$ by thinking of the vertical "lift" independently of the circular motion around the axis. Only the "time" t connects the two. For example, if a helix is described as maintaining a fixed distance of 2 units from the *z*-axis and rises 10 units over the course of one turn of the helix, we can reason that if we use the parameter t as the angle we've rotated around the axis, then one turn will have $0 \le t \le 2\pi$ and if we think of the parameter t as time, then the fact that we must rise 10 units in 2π units of time means that the vertical velocity must be $\frac{10}{2\pi} = \frac{5}{\pi}$. The motion will thus be described by the parametric equations $\left\{ \begin{array}{l} x = 2\cos t \\ y = 2\sin t \\ z = \frac{5}{\pi}t \end{array} \right\}_{t\in[0,2\pi]}$. The position vector is therefore $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, \frac{5}{\pi}t \rangle$.

Velocity vectors and tangent vectors

Perhaps the greatest advantage of using parameterized curves is how we can borrow basic physics concepts to assist in the geometry. The idea of a **velocity vector** is at the top of our list. If we imagine a parameterized curve where the component functions vary as *differentiable* functions of the parameter *t*, then by comparing the position vector $\mathbf{r}(t)$ at "time" t and the position vector $\mathbf{r}(t+\Delta t)$ a moment later, the difference of these vectors $\Delta\mathbf{r} = \mathbf{r}(t+\Delta t) - \mathbf{r}(t)$ will line up essentially tangent to the curve traced out by the position vector. Using some physics ideas, we can then divide this by the brief change in time to get $\dfrac{\Delta\mathbf{r}}{\Delta t} = \dfrac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}$ which might be identified for an actual physical motion as the "average velocity vector". Taking the limit as $\Delta t \to 0$ gives the "instantaneous velocity vector" $\lim\limits_{\Delta t\to 0}\left(\dfrac{\Delta\mathbf{r}}{\Delta t}\right) = \lim\limits_{\Delta t\to 0}\left(\dfrac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\right) \equiv \dfrac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \mathbf{v}(t)$. This **velocity vector** yields a vector that will be tangent to the path traced out by the position vector at every point along the path. This is very useful for finding tangent lines, for example. In terms of the component functions, if $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, then $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$.



Definition: The **speed** is the magnitude of the velocity vector, i.e. $\|\mathbf{v}(t)\|$.

Example (motion around a circle): For our circle parameterized as $\mathbf{r}(t) = \langle R\cos\omega t, R\sin\omega t\rangle = R\langle \cos\omega t, \sin\omega t\rangle$, we calculate that $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -R\omega\sin\omega t, R\omega\cos\omega t\rangle = R\omega\langle -\sin\omega t, \cos\omega t\rangle$. The speed is $\|\mathbf{v}(t)\| = R\omega$ which in physics might be described by saying that the "tangential velocity" is the product of the radius of the circle and the angular velocity. Note that for all t, $\mathbf{v}(t) \perp \mathbf{r}(t)$ as is expected for a circle about the origin.

Example (helix): In our example with $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, \frac{5}{\pi}t\rangle$, we calculate that $\mathbf{v}(t) = \langle -2\sin t, 2\cos t, \frac{5}{\pi}\rangle$.

The speed is $\|\mathbf{v}(t)\| = \sqrt{4(\sin^2 t + \cos^2 t) + \frac{25}{\pi^2}} = \sqrt{4 + \frac{25}{\pi^2}} = \dfrac{\sqrt{4\pi^2 + 25}}{\pi}$, a constant.

Both the position vector $\mathbf{r}(t)$ and the velocity vector $\mathbf{v}(t)$ are vector-valued functions. It is straightforward to prove the following:

Basic Rules of Differentiation for Vector-Valued Functions:

If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are vector-valued functions, $f(t)$ is any scalar-valued function, and c is any constant, then:

1) $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$ [The derivative of a sum is the sum of the derivatives.]
2) $\frac{d}{dt}[c\,\mathbf{u}(t)] = c\,\mathbf{u}'(t)$ [Scalars pass through derivatives just as with ordinary differentiation.]
3) $\frac{d}{dt}[f(t)\,\mathbf{u}(t)] = f(t)\,\mathbf{u}'(t) + f'(t)\cdot\mathbf{u}(t)$ [Product Rule for scalar multiplication.]
4) $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}(t)\cdot\mathbf{v}'(t) + \mathbf{u}'(t)\cdot\mathbf{v}(t)$ [Product Rule for dot products.]
5) $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}(t)\times\mathbf{v}'(t) + \mathbf{u}'(t)\times\mathbf{v}(t)$ [Product Rule for cross products.]
6) $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\,\mathbf{u}'(f(t))$ [Chain Rule for compositions.]

Each of these follows by applying the familiar rules of differentiation in the components. In particular, note that since the cross product is not commutative (it's anti-commutative), the order of the factors matters.

Notes by Robert Winters