Multivariable Calculus – Lecture #14 Notes

In this lecture, we'll state the Divergence Theorem (which relates flux of a vector field outward across a closed surface to the divergence of the vector field in the region bounded by this surface) and Stokes' Theorem (which relates the circulation of a vector field around a closed curve to the curl of that vector field on a surface bounded by this closed curve). We will prove both of these theorems by first giving coordinate-free geometric definitions of the divergence and curl of a vector field. We'll then show that these definitions yield the algebraic definitions in the case of Cartesian coordinates. We'll start by stating and providing context for five versions of the Fundamental Theorem of Calculus and make the case that they really constitute one theorem.

Five Versions of the Fundamental Theorem of Calculus

I – Fundamental Theorem of Calculus (FTC): Suppose a function $f(x)$ is a differentiable at all points within the interval $I=[a,b]$. Then $\boxed{f(b)-f(a)=\int_a^b f'(x)dx}$.

Note that in terms of differentials we can write $df = f'(x)dx$, so the right-hand side could be expressed as $\int_I df$. If we consider the interval to be oriented (from left to right, or in the direction of increasing values of x), the boundary of this interval consists of two points which we can also "orient" in the sense of the end point (which we'll consider to be "positively oriented") and the starting point (which we'll consider to be "negatively oriented"). If we liberalize our definition to include discrete sums and write $\text{Boundary}(I)=\partial I=\{b\}^+\cup\{a\}^-$ with $\int_{\partial I} f \equiv f(b)-f(a)$, the theorem may be then be expressed as $\boxed{\int_{\partial I} f=\int_I df}$.

II – Fundamental Theorem of Line Integrals (FTLI): Let C be a smooth curve given by the vector function $\mathbf{r}(t),\ a\le t\le b$ and let $\mathbf{F}=\overrightarrow{\nabla V}$ where $V(x,y)$ (or $V(x,y,z)$) is a differentiable function of two (or three) variables whose gradient $\mathbf{F}=\overrightarrow{\nabla V}$ is continuous on the curve C. Then:

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \overrightarrow{\nabla V}\cdot d\mathbf{r}=V(\mathbf{r}(b))-V(\mathbf{r}(a))=V(\text{end})-V(\text{start})$$

Note that in terms of differentials we can write $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=\overrightarrow{\nabla f}\cdot d\mathbf{r}$ for a function of two variables, or $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=\overrightarrow{\nabla f}\cdot d\mathbf{r}$ for a function of three variables (in Cartesian coordinates). [These may also be expressed in other coordinate systems.] Thus the right-hand side may be expressed as $\int_C dV$. The curve is oriented and we can also orient its boundary (consisting of two points) as $\text{Bnd}(C)=\partial C=\{r(b)\}^+\cup\{r(a)\}^-$ as above. If we write $\int_{\partial C} V\equiv V(\text{end})-V(\text{start})$, the theorem may be expressed as $\boxed{\int_{\partial C} V=\int_C dV}$ which is of the same form as the FTC.

III – Green's Theorem: Suppose $\mathbf{F}(x,y)=\langle P(x,y),Q(x,y)\rangle$ defines a vector field in some bounded region D in $\mathbf{R}^2$ where the component functions $P(x,y)$ and $Q(x,y)$ are differentiable. Let C be the boundary of this region oriented in the counterclockwise sense (this can be understood generally to mean that as you traverse the boundary the region D will always be to the left). We denote this by $\text{Bnd}(D)=\partial D=C$. Then:

$$\begin{pmatrix}\text{circulation of}\\ \mathbf{F}\text{ around } C=\partial D\end{pmatrix}=\oint_{C=\partial D}\mathbf{F}\cdot d\mathbf{r}=\boxed{\oint_{C=\partial D}Pdx+Qdy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA}$$

In this case, if we consider the "differential form" $\omega = Pdx + Qdy$. We can define the "wedge product" based on analogy with the cross product, i.e. we think of the "2-form" $dx \wedge dy$ as a vector-like quantity (they can be added and scaled) that represents an "oriented area" with magnitude $dxdy = dA$. In analogy with the cross product, the wedge product will also be antisymmetric in the sense that $dy \wedge dx = -dx \wedge dy$. Therefore $dx \wedge dx = 0$ and $dy \wedge dy = 0$. If we extend the 2-forms via scaling and adding with the corresponding distributive law, and if we calculate the differentials $dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$ and $dQ = \frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy$, we can then define:

$$d\omega = dP \wedge dx + dQ \wedge dy = \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right) \wedge dy$$

$$= \cancel{\frac{\partial P}{\partial x}dx \wedge dx} + \frac{\partial P}{\partial y}dy \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy + \cancel{\frac{\partial Q}{\partial y}dy \wedge dy} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx \wedge dy$$

With a little more formalism we can then also think of Green's Theorem as $\boxed{\int_{\partial D} \omega = \int_D d\omega}$.

IV – Divergence Theorem (also known as Gauss' Theorem)

Suppose S is a closed surface in $\mathbf{R}^3$ that bounds a solid region B and that this boundary S is oriented via an outward unit normal vector $\mathbf{n}$. Further suppose that $\mathbf{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ is a vector field defined and differentiable throughout B (and its boundary) with continuous first partial derivatives. Then:

$$\begin{pmatrix}\text{net flux of } \mathbf{F} \text{ outward} \\ \text{across } S = \partial B\end{pmatrix} = \oiint_{S=\partial B} F_N dS = \oiint_{S=\partial B} \mathbf{F}\cdot\mathbf{n}dS = \boxed{\oiint_{S=\partial B} \mathbf{F}\cdot d\mathbf{S} = \iiint_B \operatorname{div}(\mathbf{F})dV}$$

This theorem provides some explanation for the interpretation of the divergence of a vector field as a ***source density***. Essentially, the total amount of "stuff" flowing outward across the boundary of a closed region should measure the total amount of the source of that "stuff" emanating from within the region.

In terms of differential forms and extending some of the ideas previously expressed regarding the projections of the "vector element of surface area" in each of the coordinate directions as, respectively $\frac{dydz}{\mathbf{n}\cdot\mathbf{i}}$, $\frac{dxdz}{\mathbf{n}\cdot\mathbf{j}}$, and $\frac{dxdy}{\mathbf{n}\cdot\mathbf{k}}$, we can write $d\mathbf{S} = dydz\,\mathbf{i} + dzdx\,\mathbf{j} + dxdy\,\mathbf{k}$ in terms of the respective area elements in each of the coordinate planes. The integrand "$\mathbf{F}\cdot d\mathbf{S}$" may then be expressed, utilizing the appropriate orientations, as $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$ and:

$$d\omega = dP \wedge dy \wedge dz + dQ \wedge dz \wedge dx + dR \wedge dx \wedge dy$$

$$= \left(\frac{\partial P}{dx}dx + \cancel{\frac{\partial P}{dy}dy} + \cancel{\frac{\partial P}{dz}dz}\right) \wedge dy \wedge dz + \left(\cancel{\frac{\partial Q}{dx}dx} + \frac{\partial Q}{dy}dy + \cancel{\frac{\partial Q}{dz}dz}\right) \wedge dz \wedge dx + \left(\cancel{\frac{\partial R}{dx}dx} + \cancel{\frac{\partial R}{dy}dy} + \frac{\partial R}{dz}dz\right) \wedge dx \wedge dy$$

$$= \left(\frac{\partial P}{dx} + \frac{\partial Q}{dy} + \frac{\partial R}{dz}\right)dx \wedge dy \wedge dz$$

If we identify the "oriented volume element dV with the "3-form" $dx \wedge dy \wedge dz$, then the Divergence Theorem may be expressed as $\boxed{\int_{\partial B} \omega = \int_B d\omega}$.

V - Stokes' Theorem

Suppose S is an oriented surface in $\mathbf{R}^3$ (with unit normal vector **n** defined on the surface to choose a "side") with boundary curve C oriented in the counterclockwise sense, i.e. if the unit normal vector n represents "up", then you traverse the boundary in such a way that the surface is to your left [$\mathrm{Bnd}(S) = \partial S = C$]. Further suppose that $\mathbf{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ is a vector field defined and differentiable throughout S (and its boundary) with continuous first partial derivatives. Then:

$$\begin{pmatrix}\text{circulation of}\\ \mathbf{F}\text{ around } C=\partial S\end{pmatrix} = \boxed{\oint_{C=\partial S}\mathbf{F}\cdot d\mathbf{r} = \iint_S \mathrm{curl}(\mathbf{F})\cdot d\mathbf{S}} = \iint_S[\mathrm{curl}(\mathbf{F})\cdot\mathbf{n}]\,dS$$

This theorem provides some explanation for the interpretation of the curl of a vector field as a ***circulation density***, i.e. a measure of local rotation of the vector field. Essentially, the circulation of the vector field around the perimeter is the same as the integral of the circulation density over the surface.

As in the case of Green's Theorem, if we express the integrand as $\omega = Pdx + Qdy + Rdz$ we can calculate:

$$d\omega = dP\wedge dx + dQ\wedge dy + dR\wedge dz = \left(\cancel{\frac{\partial P}{\partial x}dx} + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right)\wedge dx + \left(\frac{\partial Q}{\partial x}dx + \cancel{\frac{\partial Q}{\partial y}dy} + \frac{\partial Q}{\partial z}dz\right)\wedge dy + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \cancel{\frac{\partial R}{\partial z}dz}\right)\wedge dz$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy\wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz\wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy$$

So, if we again translate Stokes' Theorem into the language of differential forms, it reads $\boxed{\int_{\partial S}\omega = \int_S d\omega}$. This result is known as the "**Generalized Stokes' Theorem**" and it's valid generally for all geometric objects known as "manifolds" in any dimension and using any coordinates with the corresponding assumptions that all functions involved are differentiable throughout the object.

We now proceed to the proofs of these last two theorems by first giving coordinate-free definitions of the divergence and curl of a vector field.

Geometric Definition of Divergence as a "Source Density"

Suppose **F** is vector field defined and sufficiently differentiable in the vicinity of a point $(x,y,z) = \mathbf{x}$. If we "build a bubble" B_k containing this point, we can measure the amount of flux that is emanating through the boundary $\partial B_k = S_k$ of this (small) bubble and create a density by dividing by the volume ΔV_k of the bubble. Note that this is a scalar quantity. To get an exact value for this density at the given point we shrink the bubble down to the point by lettings its diameter approach zero. Assuming that this limit is uniquely determined independent of all choices, we define this to be the divergence at the given point. That is:

$$\lim_{diam(B_k)\to 0}\left(\frac{\oiint_{S_k}\mathrm{F}\cdot d\mathbf{S}}{\Delta V_k}\right) \equiv (\mathrm{div}\,\mathbf{F})(\mathbf{x})$$

Proof of Divergence Theorem: Partition the solid region B into uniformly small cells $\{B_k\}$ and write the boundary of each cell as $S_k = \partial B_k$. Within each cell, choose a sample point $\mathbf{x}_k$. If each cell is small, we can say that $\dfrac{\oiint_{S_k}\mathrm{F}\cdot d\mathbf{S}}{\Delta V_k} \cong (\mathrm{div}\,\mathbf{F})(\mathbf{x}_k)$. Multiplication gives $\oiint_{S_k}\mathrm{F}\cdot d\mathbf{S} \cong (\mathrm{div}\,\mathbf{F})(\mathbf{x}_k)\Delta V_k$ for each cell, and if we sum these over all cells we get $\sum_k \oiint_{S_k}\mathrm{F}\cdot d\mathbf{S} \cong \sum_k (\mathrm{div}\,\mathbf{F})(\mathbf{x}_k)\Delta V_k$. In the left-hand sum, note that for any two adjacent cells the flux out of one will be the same as the flux into its neighbor, so all of the internal outward fluxes will cancel pairwise. The only flux that will remain is the flux passing out through the overall boundary $S=\partial B$. Thus $\oiint_{S=\partial B}\mathbf{F}\cdot\mathbf{dS}=\sum_k \oiint_{S_k}\mathbf{F}\cdot\mathbf{dS}\cong\sum_k (\operatorname{div}\mathbf{F})(\mathbf{x}_k)\Delta V_k$. This approximation will become more and more precise as the mesh $|\Delta|$ of the partition gets smaller and smaller, and in the limit we get that:

$$\oiint_{S=\partial B}\mathbf{F}\cdot\mathbf{dS}=\lim_{|\Delta|\to 0}\left(\sum_k (\operatorname{div}\mathbf{F})(\mathbf{x}_k)\Delta V_k\right)=\iiint_B (\operatorname{div}\mathbf{F})(\mathbf{x})\,dV$$

In essence, <u>the Divergence Theorem is just an integral version of the definition of divergence</u>.

Geometric definition of divergence yields the algebraic definition of divergence (Cartesian coordinates)

To show that $\operatorname{div}\mathbf{F}=\dfrac{\partial P}{\partial x}+\dfrac{\partial Q}{\partial y}+\dfrac{\partial R}{\partial z}$ for a differentiable vector field $\mathbf{F}=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$, choose any point (x,y,z) and build a (Cartesian) bubble around it, i.e. a rectangular bubble with edge lengths Δx, Δy, and Δz. For convenience, let (x,y,z) be at a corner of the bubble. This bubble will have six faces and we can calculate the net flux out of this (small) bubble by simply adding up the flux outward through each of these faces, i.e. $F_N\Delta S=(\mathbf{F}\cdot\mathbf{n})\Delta S$. We must take care when estimating F_N for each of the faces to choose a point that's actually on each of these faces. This information is best gathered into the following table:

Face	**n**	$F_N=\mathbf{F}\cdot\mathbf{n}$	ΔS
Front	$\mathbf{i}$	$P(x+\Delta x,y,z)$	$\Delta y\Delta z$
Back	$-\mathbf{i}$	$-P(x,y,z)$	$\Delta y\Delta z$
Right	$\mathbf{j}$	$Q(x,y+\Delta y,z)$	$\Delta x\Delta z$
Left	$-\mathbf{j}$	$-Q(x,y,z)$	$\Delta x\Delta z$
Top	$\mathbf{k}$	$R(x,y,z+\Delta z)$	$\Delta x\Delta y$
Bottom	$-\mathbf{k}$	$-R(x,y,z)$	$\Delta x\Delta y$

Summing over all six faces and dividing by the volume $\Delta V=\Delta x\Delta y\Delta z$, we get:

$$\frac{\sum F_N\Delta S}{\Delta V}\cong\frac{[P(x+\Delta x,y,z)-P(x,y,z)]\Delta y\Delta z+[Q(x,y+\Delta y,z)-Q(x,y,z)]\Delta x\Delta z+[R(x,y,z+\Delta z)-R(x,y,z)]\Delta x\Delta y}{\Delta x\Delta y\Delta z}$$
$$=\frac{[P(x+\Delta x,y,z)-P(x,y,z)]}{\Delta x}+\frac{[Q(x,y+\Delta y,z)-Q(x,y,z)]}{\Delta y}+\frac{[R(x,y,z+\Delta z)-R(x,y,z)]}{\Delta z}$$

Finally, we let the mesh approach zero by letting Δx, Δy, and Δz all approach zero. This gives:

$$(\operatorname{div}\mathbf{F})(x,y,z)=\lim_{\Delta x\to 0}\frac{[P(x+\Delta x,y,z)-P(x,y,z)]}{\Delta x}+\lim_{\Delta y\to 0}\frac{[Q(x,y+\Delta y,z)-Q(x,y,z)]}{\Delta y}+\lim_{\Delta z\to 0}\frac{[R(x,y,z+\Delta z)-R(x,y,z)]}{\Delta z}$$
$$=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

Example: Find the net flux of the vector field $\mathbf{F} = \langle x, y+z, x \rangle$ outward through the boundary of the upper half ball with $x^2 + y^2 + z^2 \le 4$ and $z \ge 0$.

Solution: If we calculate this directly, we first observe that the boundary consists of the upper hemisphere *S* with $x^2 + y^2 + z^2 = 4$ oriented with an upward (outward) unit normal vector $\mathbf{n} = \dfrac{\langle x, y, z \rangle}{2}$; and the disk *D* of radius 2 in the $z = 0$ plane oriented with downward (outward) unit normal vector $\mathbf{n} = -\mathbf{k} = \langle 0, 0, -1 \rangle$. On the spherical portion we have $\mathbf{F} \cdot \mathbf{n} = \langle x, y+z, x \rangle \cdot \dfrac{\langle x, y, z \rangle}{2} = \dfrac{x^2 + y^2 + yz + xz}{2}$. Thus the flux through the spherical portion will be $\iint_S \dfrac{x^2 + y^2 + yz + xz}{2} dS$. By symmetry, the last two terms will integrate to give zero, so we're left with the integral $\frac{1}{2}\iint_S (x^2 + y^2) dS$. In spherical coordinates with $dS = 4\sin\phi \, d\phi \, d\theta$ and $x^2 + y^2 = r^2 = 4\sin^2\phi$, we have:

$$\frac{1}{2}\int_0^{2\pi}\int_0^{\pi/2}(4\sin^2\phi)4\sin\phi \, d\phi \, d\theta = 8\int_0^{2\pi}\int_0^{\pi/2}(1-\cos^2\phi)\sin\phi \, d\phi \, d\theta = 8\cdot\left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi/2}\cdot 2\pi = \tfrac{32\pi}{3}.$$

For the bottom disk, $\mathbf{F} \cdot \mathbf{n} = \langle x, y+z, x \rangle \cdot \langle 0, 0, -1 \rangle = -x$, so the flux through the bottom disk will be $-\iint_S x \, dS$. By symmetry this will yield a flux of zero. So the overall outward flux is $\boxed{\tfrac{32\pi}{3}}$.

By the Divergence Theorem, we calculate $\operatorname{div}\mathbf{F} = 1 + 1 + 0 = 2$, so:

$$\iiint_B (\operatorname{div}\mathbf{F}) dV = \iiint_B 2 dV = 2\iiint_B dV = 2\cdot\operatorname{Vol}(B) = 2\cdot\tfrac{2}{3}\pi(2)^3 = \boxed{\tfrac{32\pi}{3}}.$$

Another interpretation of the Divergence Theorem (with applications)

Suppose S_1 and S_2 are two oriented surfaces that share a common boundary curve *C* but which do not otherwise intersect, and suppose *B* is a solid region having the union $S_1 \cup S_2$ as its boundary. If S_2 is oriented outward (think of it as the upper boundary), and S_1 is oriented inward, denote by $-S_1$ the same surface with downward (outward) orientation. Then $\partial B = \{S_2\} \cup \{-S_1\}$. If **F** is a differentiable vector field, we can then apply the Divergence Theorem to get $\oiint_{S=\partial B} \mathbf{F} \cdot \mathbf{dS} = \iint_{S_2} \mathbf{F} \cdot \mathbf{dS} - \iint_{S_1} \mathbf{F} \cdot \mathbf{dS} = \iiint_B \operatorname{div}(\mathbf{F}) dV$.

So $\iint_{S_2} \mathbf{F} \cdot \mathbf{dS} = \iint_{S_1} \mathbf{F} \cdot \mathbf{dS} + \iiint_B \operatorname{div}(\mathbf{F}) dV$. This can be interpreted as saying that the flow out through the top is equal to the flow in through the bottom plus the amount of new flow spawned by any sources in the interior region. This idea can often be used for calculating the flux through a non-closed surface by sewing in another surface (on which it's hopefully easy to calculate the flux).

Example: Calculate the flux of the vector field $\mathbf{F} = \langle y^2 z^2, 3y^2 z, z \rangle$ upward through the hemisphere $x^2 + y^2 + z^2 = 4$ with $z \ge 0$.

Solution: Carrying out the calculations directly for this surface *S* is cumbersome, but we can sew in the disk *D* of radius 2 in the *xy*-plane (with upward orientation) to close this surface and make it the boundary of the solid half-ball **B** within. Then $\partial B = \{S\} \cup \{-D\}$. We can then calculate $\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_D \mathbf{F} \cdot \mathbf{dS} + \iiint_B \operatorname{div}(\mathbf{F}) dV$. For the disk D, the unit (upward) normal is $\mathbf{n} = \mathbf{k}$ and $\iint_D \mathbf{F} \cdot \mathbf{dS} = \iint_D \mathbf{F} \cdot \mathbf{n} dS = \iint_D z dA = \iint_D 0 dA = 0$. We calculate $\operatorname{div}(\mathbf{F}) = 0 + 6yz + 1$, so $\iiint_B \operatorname{div}(\mathbf{F}) dV = \iiint_B (\cancel{6yz} + 1) dV = \iiint_B dV = \text{Volume}(B) = \tfrac{2}{3}\pi(2)^3 = \tfrac{16\pi}{3}$.

So $\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_D \mathbf{F} \cdot \mathbf{dS} + \iiint_B \operatorname{div}(\mathbf{F}) dV = 0 + \tfrac{16\pi}{3} = \boxed{\tfrac{16\pi}{3}}$.

Geometric Definition of Curl as a "Circulation Density" or measure of local rotation
Suppose **F** is vector field defined and sufficiently differentiable in the vicinity of a point $(x,y,z)=\mathbf{x}$. Defining the curl of this vector field is more involved that defining the divergence because curl is a vector quantity rather than a scalar quantity. Since a vector can be specified by providing its components, we'll focus of defining the component (scalar projection) of the curl in any specified direction. To do this, let's think of a "probe" as a small surface patch S_k with unit normal vector $\mathbf{n}$ ("the handle") inserted into the given vector field at the specified point. If the vector field has some local rotation, it will yield some circulation around the boundary of the probe $\partial S_k = C_k$, i.e. $\oint_{C_k} \mathbf{F}\cdot\mathbf{dr}$. We can create a density by dividing by the area ΔS_k of the probe. To get an exact value for this "circulation density" at the given point associated with the normal vector of the probe, we shrink the probe down to the point by lettings its diameter approach zero. Assuming that this limit is uniquely determined independent of all choices, we define this to be the curl at the given point. That is:

$$\lim_{diam(S_k)\to 0}\left(\frac{\oint_{C_k}\mathbf{F}\cdot\mathbf{dr}}{\Delta S_k}\right)\equiv[(\operatorname{curl}\mathbf{F})(\mathbf{x})]\cdot\mathbf{n}, \text{ i.e. the component of the curl in the direction of } \mathbf{n}$$

We'll shortly use this definition to find the *x*, *y*, and *z* components of the curl by choosing $\mathbf{n}$ to be, respectively, the unit vectors **i**, **j**, and **k**.

Proof of Stokes' Theorem: Partition the surface *S* into uniformly small "patches" $\{S_k\}$ with normal vector $\mathbf{n}$ provided by the orientation of the surface and write the boundary of each patch as $C_k=\partial S_k$. We can then think of each of these patches as the base of a "probe". Within each patch, choose a sample point $\mathbf{x}_k$. If each cell is small, we can say that $\dfrac{\oint_{C_k}\mathbf{F}\cdot\mathbf{dr}}{\Delta S_k}\cong[(\operatorname{curl}\mathbf{F})(\mathbf{x}_k)]\cdot\mathbf{n}$. Multiplication gives $\oint_{C_k}\mathbf{F}\cdot\mathbf{dr}\cong[(\operatorname{curl}\mathbf{F})(\mathbf{x}_k)]\cdot\mathbf{n}\Delta S_k$ for each patch, and if we sum these over all patches we get $\sum_k\oint_{C_k}\mathbf{F}\cdot\mathbf{dr}\cong\sum_k[(\operatorname{curl}\mathbf{F})(\mathbf{x}_k)]\cdot\mathbf{n}\Delta S_k$. In the left-hand sum, note that for any two adjacent patches the work along one edge will be the same as the work along the edge of its neighbor but oppositely directed, so the work associate with any such internal pairs will cancel. The only contributions to the sum that will remain will yield the circulation about the overall boundary of the surface $C=\partial S$. Thus $\oint_{C=\partial S}\mathbf{F}\cdot\mathbf{dr}=\sum_k\oint_{C_k}\mathbf{F}\cdot\mathbf{dr}\cong\sum_k[(\operatorname{curl}\mathbf{F})(\mathbf{x}_k)]\cdot\mathbf{n}\Delta S_k$. This approximation will become more and more precise as the mesh $|\Delta|$ of the partition gets smaller and smaller, and in the limit we get that:

$$\oint_{C=\partial S}\mathbf{F}\cdot\mathbf{dr}=\lim_{|\Delta|\to 0}\left[\sum_k[(\operatorname{curl}\mathbf{F})(\mathbf{x}_k)]\cdot\mathbf{n}\Delta S_k\right]=\iint_S[(\operatorname{curl}\mathbf{F})(\mathbf{x})]\cdot\mathbf{n}dS=\iint_S(\operatorname{curl}\mathbf{F})\cdot\mathbf{dS}$$

In essence, <u>Stokes' Theorem is just an integral version of the definition of curl</u>.

Geometric definition of curl yields the algebraic definition of curl (Cartesian coordinates)

We'll demonstrate that $\operatorname{curl}\mathbf{F}=\left\langle\dfrac{\partial R}{\partial y}-\dfrac{\partial Q}{\partial z},\dfrac{\partial P}{\partial z}-\dfrac{\partial R}{\partial x},\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}\right\rangle$ for a differentiable vector field $\mathbf{F}=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$ by calculating its z-component. The calculation of the other two components will be similar. We choose any point (x,y,z) and "insert the probe" with normal vector $\mathbf{k}$ and a small rectangular patch with side lengths Δx and Δy and with area $\Delta S=\Delta x\Delta y$. For convenience, let (x,y,z) be at a corner of the patch. This boundary of this patch will consist of four (oriented) edges and we can calculate the net circulation around the boundary of the patch by adding up the work along each of these edges, i.e. $F_T \Delta s = (\mathbf{F}\cdot\mathbf{T})\Delta s$ where **T** represents the unit tangent vector for each edge and Δs is the length of each edge. We must take care when estimating F_T for each of the edges to choose a point that's actually on each of these edges. This information is best gathered into the following table:

Edge	**T**	$F_T = \mathbf{F}\cdot\mathbf{T}$	Δs
Left	**i**	$P(x,y,z)$	Δx
Front	**j**	$Q(x+\Delta x,y,z)$	Δy
Right	**−i**	$-P(x,y+\Delta y,z)$	Δx
Back	**−j**	$-Q(x,y,z)$	Δy

Summing over all four edges faces and dividing by the area $\Delta S = \Delta x \Delta y$, we get:

$$\frac{\sum F_T \Delta s}{\Delta S} \cong \frac{[Q(x+\Delta x,y,z)-Q(x,y,z)]\Delta y - [P(x,y+\Delta y,z)-P(x,y,z)]\Delta x}{\Delta x \Delta y}$$
$$= \frac{Q(x+\Delta x,y,z)-Q(x,y,z)}{\Delta x} - \frac{P(x,y+\Delta y,z)-P(x,y,z)}{\Delta y}$$

Finally, we let the mesh approach zero by letting Δx and Δy both approach zero. This gives:

$$\lim_{\Delta x\to 0}\left[\frac{Q(x+\Delta x,y,z)-Q(x,y,z)}{\Delta x}\right] - \lim_{\Delta y\to 0}\left[\frac{P(x,y+\Delta y,z)-P(x,y,z)}{\Delta y}\right] = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

The other components are similar.

Example: Calculate the work done by the vector field $\mathbf{F} = \left\langle x^2, x, z\right\rangle$ around the curve where the plane $2x - y + z = 5$ intersects the cylinder $x^2 + y^2 = 4$ where the curve is oriented counterclockwise as viewed from the positive *z*-axis.

<u>By direct calculation</u>: The work integral is $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C x^2dx + xdy + zdz$. We can parameterize this curve *C* by circling around the cylinder while staying on the plane, i.e. $z = 5 - 2x + y$. A simple parameterization is $\left\{\begin{array}{l} x = 2\cos t \\ y = 2\sin t \\ z = 5 - 4\cos t + 2\sin t \end{array}\right\}$ from $t = 0$ to $t = 2\pi$. This gives $\left\{\begin{array}{l} dx = -2\sin t dt \\ dy = 2\cos t dt \\ dz = (4\sin t + 2\cos t)dt \end{array}\right\}$. Substitution then gives:

$$\oint_C x^2dx + xdy + zdz = \int_0^{2\pi}[(4\cos^2 t)(-2\sin t) + (2\cos t)(2\cos t) + (5 - 4\cos t + 2\sin t)(4\sin t + 2\cos t)]dt$$
$$= \int_0^{2\pi}[-8\cos^2 t\sin t + 4\cos^2 t + 20\sin t - 16\sin t\cos t + 8\sin^2 t + 10\cos t - 8\cos^2 t + 4\sin t\cos t]dt$$
$$= \int_0^{2\pi}[-\cancel{8\cos^2 t\sin t} - 4\cos^2 t + 8\sin^2 t - \cancel{12\sin t\cos t} + \cancel{20\sin t} + \cancel{10\cos t}]dt$$

(crossed-out terms integrate to 0 over a full cycle)

$$= \int_0^{2\pi}\left[-4\left(\frac{1+\cos 2t}{2}\right) + 8\left(\frac{1-\cos 2t}{2}\right)\right]dt = \int_0^{2\pi}\left[-2(1+\cancel{\cos 2t}) + 4(1-\cancel{\cos 2t})\right]dt = \int_0^{2\pi} 2dt = \boxed{4\pi}$$

By Stokes' Theorem: We calculate $\operatorname{curl}\mathbf{F} = \langle 0-0, 0-0, 1-0\rangle = \langle 0,0,1\rangle$. If we consider C as the boundary of the oval-shaped region in the plane $2x - y + z = 5$ then we can identify the unit normal as $\mathbf{n} = \frac{\langle 2,-1,1\rangle}{\sqrt{6}}$ and the surface area element as $dS = \frac{dxdy}{\mathbf{n}\cdot\mathbf{k}} = \sqrt{6}\,dxdy$, and $(\operatorname{curl}\mathbf{F})\cdot\mathbf{n} = \langle 0,0,1\rangle \cdot \frac{\langle 2,-1,1\rangle}{\sqrt{6}} = \frac{1}{\sqrt{6}}$. So, denoting the region in the *xy*-plane below the surface S as the circular disk D, we have:

$$\oint_{C=\partial S} \mathbf{F}\cdot d\mathbf{r} = \iint_S [\operatorname{curl}(\mathbf{F})\cdot\mathbf{n}]\,dS = \iint_D \left(\frac{1}{\sqrt{6}}\right)\sqrt{6}\,dxdy = \iint_D dxdy = \text{Area}(D) = \boxed{4\pi}$$

Green's Theorem is a corollary of Stokes' Theorem

If we consider the region D in the *xy*-plane as a surface embedded in $\mathbf{R}^3$ with (upward) unit normal vector $\mathbf{n} = \mathbf{k}$ and if we extend the vector field to $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ in the *xy*-plane to a vector field $\mathbf{F}(x,y,z) = \langle P(x,y), Q(x,y), 0\rangle$ in $\mathbf{R}^3$, then $\operatorname{curl}\mathbf{F} = \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle$, and $(\operatorname{curl}\mathbf{F})\cdot\mathbf{n} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, and $dS = dA = dxdy$, so the right-hand side of Stokes' Theorem becomes simply $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$. Furthermore, the left-hand side is just $\oint_{C=\partial D} \mathbf{F}\cdot d\mathbf{r} = \oint_{C=\partial D} P(x,y)dx + Q(x,y)dy$ which is the left-hand side of Green's Theorem.

Notes by Robert Winters and Renée Chipman