Multivariable Calculus – Lecture #13 Notes

In this lecture, we'll look at parameterization of surfaces in $\mathbf{R}^3$ and integration on a parameterized surface. Applications will include surface area, mass of a surface or total electric charge on a surface given an associated density function; average value of a function on a surface; and the flux (flow) of a vector field through a surface. We will also state the Divergence Theorem (which relates flux of a vector field outward across a closed surface to the divergence of the vector field in the region bounded by this surface) and Stokes' Theorem (which relates the circulation of a vector field around a closed curve to the curl of that vector field on a surface bounded by this closed curve). We'll delay the proofs of these important theorems until the next lecture.

A motivating example

Suppose a surface S in $\mathbf{R}^3$ is composed of a material of varying (surface) density $\sigma(x,y,z)=\sigma(\mathbf{x})$ that is measured in units of mass per area. If we wanted to calculate the mass of such a surface *we would likely just weigh it*, but if weighing it is not an option, we can use Riemann Sums to develop an expression for its mass. To this end, suppose we partition this surface into small pieces S_k and choose a sample point $(x_k,y_k,z_k)=\mathbf{x}_k$ in each piece. Then the mass Δm_k of the k-th piece can be approximated as $\Delta m_k \cong \sigma(x_k,y_k,z_k)\Delta S_k = \sigma(\mathbf{x}_k)\Delta S_k$ where ΔS_k is the surface area of the k-th piece. Summing the masses of all of these pieces we can say that $\text{Mass}(S)=\sum_k \Delta m_k \cong \sum_k \sigma(x_k,y_k,z_k)\Delta S_k = \sum_k \sigma(\mathbf{x}_k)\Delta S_k$. If the mesh of this partition is $|\Delta| = \max_k \{diam(S_k)\}$, we can formally calculate the limit as the mesh approaches zero. If this limit exists independent of any choices made in the construction, we define $\lim_{|\Delta|\to 0}\left(\sum_k \sigma(x_k,y_k,z_k)\Delta S_k\right) = \iint_S \sigma(x,y,z)\,dS = \iint_S \sigma(\mathbf{x})\,dS$, the ***surface integral*** of this density function over the surface S.

It is often useful to use the differential shorthand notation $\boxed{dm=\sigma(x,y,z)dS=\sigma\, dS}$ when working with such in integral. This enables us to write:

1) $\boxed{\text{Mass}(S)=\iint_S dm = \iint_S \sigma\, dS}$. Of course, there's nothing special about mass here. We could just as easily have used a charge density to calculate the ***total charge*** distributed on the surface or a population density to find the ***total population*** on this surface. Indeed, we could calculate the total amount of anything distributed on the surface by integrating its associated density function.

There an even more basic application – the ***surface area*** of a given surface. The idea is the same as above, but without the density.

2) $\boxed{\text{Area}(S)=\iint_S dS}$. This is a simple expression, but the actual calculation will require us to understand how to express the element of surface area dS in a manner that lends itself to the previous methods we have discussed using iterated single integrals. More on that shortly.

3) We can also define the average value of a function defined on a surface in a manner analogous to how we've defined this in other contexts. Generally, we average a function over a geometric object by integrating the function over the object and diving by the geometric content of the object. In this context, the ***unweighted average*** of the function $f(x,y,z)$ over a surface S will be $\boxed{\overline{f}=\dfrac{\iint_S f\, dS}{\text{Area}(S)}=\dfrac{\iint_S f(x,y,z)dS}{\text{Area}(S)}}$.

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4) The ***centroid*** (geometric center) can be defined as we've done before by averaging the individual (Cartesian) coordinate functions, i.e. $(\bar{x},\bar{y},\bar{z})$. So $\bar{x}=\dfrac{\iint_S x\,dS}{\text{Area}(S)}$, $\bar{y}=\dfrac{\iint_S y\,dS}{\text{Area}(S)}$, and $\bar{z}=\dfrac{\iint_S z\,dS}{\text{Area}(S)}$.

We can also define weighted averages as we've done before if there is an associated density function that allows us to "weigh" some areas more than others.

5) If S is a surface on which a density function $\sigma(x,y,z)=\sigma(\mathbf{x})$ is defined, we define the ***weighted average*** of a function $f(x,y,z)$ over the surface S to be $\boxed{\bar{f}_{wtd}=\dfrac{\iint_S f\,dm}{\text{Mass}(S)}=\dfrac{\iint_S f\,\sigma\,dm}{\iint_S dm}=\dfrac{\iint_S f(x,y,z)\sigma(x,y,z)dS}{\iint_S \sigma(x,y,z)dS}}$.

6) If mass is distributed over a surface S with a surface density function $\sigma(x,y,z)$, we can determine its ***center of mass*** by calculating the weighted averages of the (Cartesian) coordinate functions. That is, we define $(\bar{x}_{cm},\bar{y}_{cm},\bar{z}_{cm})$ where $\bar{x}_{cm}=\dfrac{\iint_S x\,dm}{\text{Mass}(S)}$, $\bar{y}_{cm}=\dfrac{\iint_S y\,dm}{\text{Mass}(S)}$, and $\bar{z}_{cm}=\dfrac{\iint_S z\,dm}{\text{Mass}(S)}$.

Flux of a vector field across a surface

One of the most common applications of surface integrals is in the calculation of the ***flux of a vector field across (or through) a surface***. Suppose a piecewise smooth surface S is orientable in the sense that it's possible to consistently choose one distinguished side of the surface. (*Möbius bands* and *Klein bottles* need not apply.) Typically this might be the outside of a closed surface or, perhaps, the blue side if one side was painted blue and the other side red. For such an oriented surface we can determine a unit normal vector **n** at every point (except where distinct facets of the surface meet) that is perpendicular to this surface in the direction of the distinguished side. Let's further suppose that there is a vector field $\mathbf{F}=\mathbf{F}(x,y,z)$ defined everywhere in the vicinity of this surface. If this vector field represented some kind of flow such a water moving with varying direction and velocity, we might be interested in measuring how much water is flowing across some permeable (or virtual) surface, possibly per some unit of time. How would we do this?

If we partition this surface into small "patches" S_k, then the flux (flow) through any patch should be approximately proportional to the area of the patch and to the *normal component* $F_N=\mathbf{F}\cdot\mathbf{n}$ of the vector field at some sample point in this patch, i.e. $F_N\Delta S_k$. If we sum the contributions of all the patches in the partition, we get $\sum_k F_N\Delta S_k$, and if we pass to the limit as the mesh tends to zero we get:

7) $\boxed{\textbf{Flux of } \mathbf{F} \text{ through } S = \lim_{|\Delta|\to 0}\left(\sum_k F_N\Delta S_k\right)=\iint_S F_N\,dS=\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_S \mathbf{F}\cdot d\mathbf{S}}$

Here we use the notation $\boxed{d\mathbf{S}=\mathbf{n}\cdot dS}$ for the "vector element of surface area".

We now turn to the matter of how to calculate surface integrals. The primary method is to parameterize the surface and to then use this parameterization to "pull back" the integral to the parameter space to get an ordinary double integral which can be calculated using iterated single integrals.

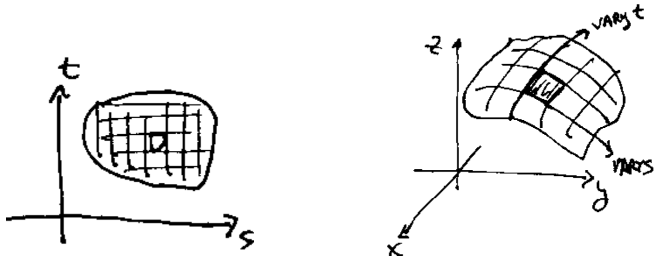
Integration on Surfaces - Toolkits

The main tools for calculating such integrals are (a) parameterization of the surface (or, equivalently, finding two coordinates defined on the surface that provide a "mesh" for the surface), and (b) an expression for the

"<u>element of surface area</u>" dS defined by the parameterization or coordinates. For flux integrals, it's also handy to have an expression for a <u>unit normal vector</u> ***n*** for the surface. Though there is an all-purpose method for calculating surface integrals for any parameterized surface, it is often easier and more geometrically clear to focus on the special cases of cylinders, spheres, and graphs. Each situation has its own **toolkit**.

General method for any parameterized surface

<u>Parameterization</u>: Suppose S is a surface parameterized by a vector-valued function $\boxed{\mathbf{r}(s,t)=\langle x(s,t),y(s,t),z(s,t)\rangle}$ where the parameters s and t vary over some domain in the parameter space D. The only requirement is that the curves in the surface produced by varying one parameter at a time provide a mesh on the surfaces, i.e. these curves should intersect "cleanly" (transversally) and produce a patchwork of small cells on the surface that can be used to build Riemann Sums on the surface.



<u>Surface area element</u>: If we vary s by an amount Δs, we move an approximate vector displacement along a cross-section of the graph of $\frac{\partial \mathbf{r}}{\partial s}\Delta s$, and if we also vary t by an amount Δt, we move an approximate vector displacement along a cross-section of the graph of $\frac{\partial \mathbf{r}}{\partial t}\Delta t$.

These two displacements will span a "patch" of the surface and the cross-product can be used to determine its approximate area. Specifically,

$$\left(\frac{\partial \mathbf{r}}{\partial s}\Delta s\right)\times\left(\frac{\partial \mathbf{r}}{\partial t}\Delta t\right)=\left(\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right)\Delta s\Delta t.$$

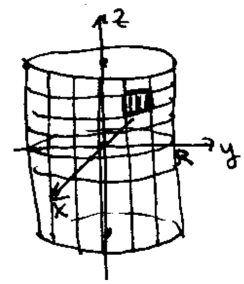
This is a vector with magnitude approximately equal to the area of a "patch" on the graph surface and direction normal to the surface (actually in the upward normal direction). The magnitude is $\Delta S\cong\left\|\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right\|\Delta s\Delta t$. Within a Riemann Sum expression as the limit of the mesh tends to zero, this yields the surface area element $\boxed{dS=\left\|\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right\|dsdt}$ for a general parameterized surface. We can also make use of the "vector element of surface area" $\boxed{d\mathbf{S}=\mathbf{n}dS=\left(\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right)dsdt}$ where $\mathbf{n}$ denotes the unit normal vector to the surface, oriented in a manner consistent with the cross-product.

<u>Unit normal vector</u>: From the above calculation, we also see that a unit normal vector to the surface with orientation consistent with the cross product is $\boxed{\mathbf{n}=\frac{\left(\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right)}{\left\|\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right\|}}$.

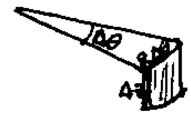
In particular, if $\mathbf{F}=\langle P,Q,R\rangle$ is a vector field defined on the surface, the flux of $\mathbf{F}$ through S can be calculated as $\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_D\left[\mathbf{F}\cdot\left(\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right)\right]dsdt$. [The integrand is the triple scalar product.]

Cylinder

Cartesian equation: $\boxed{x^2 + y^2 = R^2}$



Parameterization: $\boxed{\begin{array}{l} x = R\cos\theta \\ y = R\sin\theta \\ z = z \end{array}}$. The parameter θ allows movement around the cylinder with $0 \le \theta \le 2\pi$, and the parameter z (which does double-duty as both a coordinate and a parameter) allows movement up and down the cylinder.



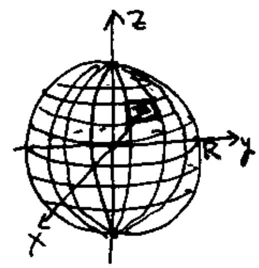
Surface area element: If we vary θ by an amount $\Delta\theta$, we move a distance $R\Delta\theta$ around the cylinder, and if we also move an amount Δz to span a "patch" of the surface, the area of this patch will be $\Delta S = (R\Delta z)(\Delta\theta)$. Within a Riemann Sum expression as the limit of the mesh tends to zero, this yields the surface area element $\boxed{dS = R\,dz\,d\theta}$ for a cylinder.

Unit normal vector: We can use gradient methods or observation to see that at any point (x, y, z) on the cylinder, the outward unit normal vector to the surface is $\boxed{\mathbf{n} = \frac{\langle x, y, 0\rangle}{R} = \langle \cos\theta, \sin\theta, 0\rangle}$.

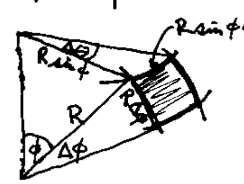
This method can be modified as necessary for cylinders around the x-axis or y-axis.

Sphere

Cartesian equation: $\boxed{x^2 + y^2 + z^2 = R^2}$



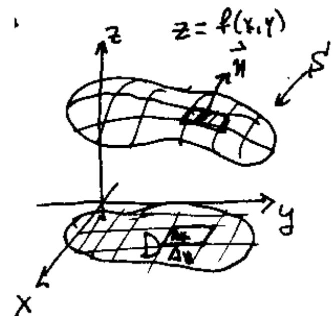
Parameterization: $\boxed{\begin{array}{l} x = R\cos\theta\sin\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\phi \end{array}}$. The parameter θ (longitude) allows movement around the sphere with $0 \le \theta \le 2\pi$, and the parameter ϕ (the inclination, related to latitude) allows movement up and down the sphere with $0 \le \phi \le \pi$.



Surface area element: If we vary ϕ by an amount $\Delta\phi$, we move a distance $R\Delta\phi$ along a longitude, and if we also move an amount $\Delta\theta$ along a latitude (at a radius from the z-axis of $r = R\sin\phi$), we will move a distance $R\sin\phi\,\Delta\theta$ along a latitude. Together, these will span a "patch" of the surface with area approximately $\Delta S \cong (R\sin\phi\,\Delta\theta)(R\Delta\phi) = R^2\sin\phi\,\Delta\theta\,\Delta\phi$. Within a Riemann Sum expression as the limit of the mesh tends to zero, this yields the surface area element $\boxed{dS = R^2\sin\phi\,d\phi\,d\theta}$ for a sphere.

Unit normal vector: We can use gradient methods or observation to see that at any point (x, y, z) on the sphere, the outward unit normal vector to the surface is $\boxed{\mathbf{n} = \frac{\langle x, y, z\rangle}{R} = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle}$.

Graph $z = f(x, y)$



Cartesian equation: $\boxed{z = f(x, y)}$

Parameterization: $\boxed{\begin{array}{l} x = x \\ y = y \\ z = f(x, y) \end{array}}$ or $\boxed{\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle}$. The variables x and y here do double-duty as both parameters and coordinates. They vary in the xy-plane over the domain of the function that describes this graph surface.

Surface area element: If we vary x by an amount Δx, we move an approximate vector displacement along a cross-section of the graph of $\frac{\partial \mathbf{r}}{\partial x}\Delta x = \langle 1, 0, f_x \rangle \Delta x$, and if we also vary y by an amount Δy, we move an approximate vector displacement along a cross-section of the graph of $\frac{\partial \mathbf{r}}{\partial y}\Delta y = \langle 0, 1, f_y \rangle \Delta y$. Together, these two displacements will span a "patch" of the surface and we can use the cross-product to determine its approximate area. Specifically,

$$\left(\frac{\partial \mathbf{r}}{\partial x}\Delta x\right) \times \left(\frac{\partial \mathbf{r}}{\partial y}\Delta y\right) = \left(\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\right)\Delta x \Delta y = \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle \Delta x \Delta y = \langle -f_x, -f_y, 1 \rangle \Delta x \Delta y .$$

This is a vector with magnitude approximately equal to the area of a "patch" on the graph surface and direction normal to the surface (actually in the upward normal direction). The magnitude is $\Delta S \cong \sqrt{1 + f_x^2 + f_y^2}\, \Delta x \Delta y$. Within a Riemann Sum expression as the limit of the mesh tends to zero, this yields the surface area element $\boxed{dS \cong \sqrt{1 + f_x^2 + f_y^2}\, dxdy}$ for a graph surface. We can also make use of the "vector element of surface area" $\boxed{d\mathbf{S} = \mathbf{n}dS = \langle -f_x, -f_y, 1 \rangle\, dxdy}$ where $\mathbf{n}$ denotes the (upward) unit normal vector to the graph surface.

Unit normal vector: From the above calculation, we see that a unit (upward) normal vector to a graph surface is $\boxed{\mathbf{n} = \frac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{1 + f_x^2 + f_y^2}}}$. An alternative geometric argument also shows that $\boxed{dS = \frac{dxdy}{\mathbf{n}\cdot\mathbf{k}} = \sqrt{1 + f_x^2 + f_y^2}\, dxdy}$.

Note: [This can be easily adapted to the case where $x = f(y,z)$ with $\boxed{dS = \frac{dydz}{\mathbf{n}\cdot\mathbf{i}} = \sqrt{1 + f_y^2 + f_z^2}\, dydz}$ or where $y = f(x,z)$ with $\boxed{dS = \frac{dxdz}{\mathbf{n}\cdot\mathbf{j}} = \sqrt{1 + f_x^2 + f_z^2}\, dxdz}$.]

Examples:

(1) Surface area of a sphere S of radius R:

$\text{Area}(S) = \iint_S dS = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\phi\, d\phi d\theta$. The inner integral gives $-R^2\cos\phi\Big|_{\phi=0}^{\phi=\pi} = -R^2(-1-1) = 2R^2$, and the outer integral gives $(2R^2)(2\pi) = 4\pi R^2$.

(2) The flux of the vector field $\mathbf{F} = \langle 3x, -y, z^2 \rangle$ outward through a sphere of radius 2 centered at the origin:

$\text{Flux} = \iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}dS$. We use the unit outward normal vector $\mathbf{n} = \frac{\langle x, y, z \rangle}{2}$, so

$\mathbf{F}\cdot\mathbf{n} = \langle 3x, -y, z^2 \rangle \cdot \frac{\langle x, y, z \rangle}{2} = \frac{3x^2 - y^2 + z^3}{2}$. Therefore, $\text{Flux} = \iint_S \left(\frac{3x^2 - y^2 + z^3}{2}\right) dS$. If we substitute the parameterization $\left\{\begin{array}{l} x = 2\cos\theta\sin\phi \\ y = 2\sin\theta\sin\phi \\ z = 2\cos\phi \end{array}\right\}$ and the area element $dS = 4\sin\phi\, d\phi d\theta$, we get

$\text{Flux} = \int_0^{2\pi}\int_0^{\pi} (24\cos^2\theta\sin^3\phi - 8\sin^2\theta\sin^3\phi + 16\cos^3\phi\sin\phi)\, d\phi d\theta$.

This may not be the simplest integral, but it's quite doable. The inner integral gives

$$\int_0^{\pi}(24\cos^2\theta(1-\cos^2\phi)\sin\phi - 8\sin^2\theta(1-\cos^2\phi)\sin\phi + 16\cos^3\phi\sin\phi)\,d\phi$$

$$=[24\cos^2\theta - 8\sin^2\theta]\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi} - 4\left[\cos^4\phi\right]_0^{\pi} = [24\cos^2\theta - 8\sin^2\theta]\left[\tfrac{4}{3}\right] - 4[0]$$

$$=\tfrac{4}{3}\left[24\left(\frac{1+\cos 2\theta}{2}\right) - 8\left(\frac{1-\cos 2\theta}{2}\right)\right] = \tfrac{4}{3}(8+16\cos 2\theta)$$

The outer integral is then $\frac{4}{3}\int_0^{2\pi}(8+16\cos 2\theta)\,d\theta = \frac{32}{3}\cdot 2\pi = \frac{64\pi}{3}$.

Note: This integral can also be (more simply) done using the **Divergence Theorem**. We calculate $\operatorname{div} F = 3-1+2z = 2+2z$, and

$$\text{Flux} = \oiint_{S=\text{Bnd}(B)} \mathbf{F}\cdot d\mathbf{S} = \iiint_B (\operatorname{div}\mathbf{F})\,dV = \iiint_B (2+2z)\,dV = \iiint_B 2\,dV = 2\cdot\text{Vol}(B) = 2\cdot(\tfrac{4}{3}\pi\cdot 2^3) = \tfrac{64\pi}{3}.$$

(3) Surface area of the paraboloid $z = x^2+y^2$ lying over the disk $x^2+y^2\le 4$ in the xy-plane.

There are several good approaches. If were to use (x,y) as parameters, we might describe the paraboloid parametrically by $\left\{\begin{array}{l} x = x \\ y = y \\ z = x^2+y^2 \end{array}\right\}$ or $\mathbf{r}(x,y) = \langle x, y, x^2+y^2\rangle$. The methods described above (with $z = f(x,y)$) give us that $dS = \sqrt{1+f_x^{\,2}+f_y^{\,2}}\,dxdy = \sqrt{1+(2x)^2+(2y)^2}\,dxdy = \sqrt{1+4(x^2+y^2)}\,dxdy$, so the surface area would be $\iint_D \sqrt{1+4(x^2+y^2)}\,dxdy$. The sensible thing is to change to polar coordinates to calculate this integral over the disk. This gives $\int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\,rdrd\theta = \cdots = \boxed{\tfrac{\pi}{6}(17\sqrt{17}-1)}$.

We could also have begun by using (r,θ) as parameters. This would give $\left\{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = r^2 \end{array}\right\}$ or

$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle$. We calculate $\frac{\partial\mathbf{r}}{\partial r} = \langle\cos\theta, \sin\theta, 2r\rangle$ and $\frac{\partial\mathbf{r}}{\partial\theta} = \langle -r\sin\theta, r\cos\theta, 0\rangle$, so

$d\mathbf{S} = \frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta} = \langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle = r\langle -2r\cos\theta, -2r\sin\theta, 1\rangle$ and

$dS = \left\|\frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta}\right\| drd\theta = r\left\|\langle -2r\cos\theta, -2r\sin\theta, 1\rangle\right\| drd\theta = r\sqrt{4r^2(\cos^2\theta+\sin^2\theta)+1}\,drd\theta = r\sqrt{1+4r^2}\,drd\theta$, so

we again get $\text{Area}(S) = \iint_S dS = \int_0^{2\pi}\int_0^2\sqrt{1+4r^2}\,rdrd\theta = \cdots = \boxed{\tfrac{\pi}{6}(17\sqrt{17}-1)}$.

Divergence and curl of vector fields in $\mathbf{R}^3$

Recall that if $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ defines a vector field in some region in $\mathbf{R}^3$ where the component functions $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ are differentiable, we define:

$$\text{div}(\mathbf{F}) = \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z} \qquad \text{(divergence of } \mathbf{F}\text{)}$$

$$\text{curl}(\mathbf{F}) = \left\langle \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right\rangle \qquad \text{(curl of } \mathbf{F}\text{)}$$

Divergence Theorem (also known as Gauss' Theorem)
Suppose S is a closed surface in $\mathbf{R}^3$ that bounds a solid region B and that this boundary S is oriented via an outward unit normal vector **n**. Further suppose that $\mathbf{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is a vector field defined and differentiable throughout B (and its boundary) with continuous first partial derivatives. Then:

$$\begin{pmatrix} \text{net flux of } \mathbf{F} \text{ outward} \\ \text{across } S = \partial B \end{pmatrix} = \oiint_{S=\partial B} F_N \, dS = \oiint_{S=\partial B} \mathbf{F} \cdot \mathbf{n} \, dS = \boxed{\oiint_{S=\partial B} \mathbf{F} \cdot \mathbf{dS} = \iiint_B \operatorname{div}(\mathbf{F}) \, dV}$$

This theorem provides some explanation for the interpretation of the divergence of a vector field as a ***source density***. Essentially, the total amount of "stuff" flowing outward across the boundary of a closed region should measure the total amount of the source of that "stuff" emanating from within the region.

Stokes' Theorem
Suppose S is an oriented surface in $\mathbf{R}^3$ (with unit normal vector **n** defined on the surface to choose a "side") with boundary curve C oriented in the counterclockwise sense, i.e. if the unit normal vector n represents "up", then you traverse the boundary in such a way that the surface is to your left [$\text{Bnd}(S) = \partial S = C$]. Further suppose that $\mathbf{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is a vector field defined and differentiable throughout S (and its boundary) with continuous first partial derivatives. Then:

$$\begin{pmatrix} \text{circulation of} \\ \mathbf{F} \text{ around } C = \partial S \end{pmatrix} = \boxed{\oint_{C=\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathbf{dS}} = \iint_S [\operatorname{curl}(\mathbf{F}) \cdot \mathbf{n}] \, dS$$

This theorem provides some explanation for the interpretation of the curl of a vector field as a ***circulation density***, i.e. a measure of local rotation of the vector field. Essentially, the circulation of the vector field around the perimeter is the same as the integral of the circulation density over the surface.

In the next lecture we'll prove these theorems and apply them in a variety of ways – both theoretical and practical. We'll also show that Green's Theorem is a corollary of Stokes' Theorem. We'll start by giving purely geometric, coordinate-free definitions of the divergence and curl of a vector field and show that the theorems are really just corollaries (integral versions) of these definitions. We'll then show that these geometric definitions yield the previously stated algebraic definitions of divergence and curl.

Notes by Robert Winters and Renée Chipman