Multivariable Calculus – Lecture #12 Notes

In this lecture, we will develop a list of statements equivalent to a vector field being conservative and state and prove Green's Theorem to help connect these facts. We'll also state and prove a Normal Form of Green's Theorem that will be a two-dimensional preview of the Divergence Theorem. We'll start by defining (both algebraically and geometrically) the divergence and curl of a vector field in $\mathbf{R}^3$ as well as 2-dimensional versions of these that are relevant for Green's Theorem (both versions).

Divergence and curl of vector fields in $\mathbf{R}^2$ and $\mathbf{R}^3$

Suppose $\mathbf{F}(x,y)=\langle P(x,y),Q(x,y)\rangle$ defines a vector field in some region in $\mathbf{R}^2$ where the component functions $P(x,y)$ and $Q(x,y)$ are differentiable. We define:

$$\text{div}(\mathbf{F})=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y} \text{ (two-dimensional divergence of } \mathbf{F})$$

$$\text{2D-curl}(\mathbf{F})=\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y} \text{ (two-dimensional curl of } \mathbf{F})$$

These are just formal algebraic definitions for these quantities. We will later redefine these in a coordinate-free, geometric manner and show that the above algebraic definitions are equivalent.

If $\mathbf{F}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$ defines a vector field in some region in $\mathbf{R}^3$ where the component functions $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ are differentiable, we define:

$$\text{div}(\mathbf{F})=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z} \text{ (divergence of } \mathbf{F})$$

$$\text{curl}(\mathbf{F})=\left\langle \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right\rangle \text{ (curl of } \mathbf{F})$$

The latter definition may be *formally* expressed in terms of a determinant as $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$ where we apply the derivatives appropriately to the component functions rather than calculate any actual products. Again, these are just formal algebraic definitions. We will later redefine these in a coordinate-free, geometric manner and show that the above algebraic definitions are equivalent.

Note that if we display all possible partial derivatives of these component functions in the 3×3 matrix $\begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} & \frac{\partial Q}{\partial z} \\ \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial z} \end{bmatrix}$, then the divergence is just the trace of this matrix (sum of the main diagonal entries) and the curl is constructed (in perhaps a mysterious way) from the remaining six entries.

It must be emphasized that the divergence of a vector field is a scalar-valued function, and the curl of a vector field is also a vector field.

Equivalent statements to a vector field being conservative

Suppose that $\mathbf{F}=\langle P,Q\rangle$ defines a vector field in some *simply connected* region D in $\mathbf{R}^2$ where the component functions P and Q are differentiable; or that $\mathbf{F}=\langle P,Q,R\rangle$ defines a vector field in some region D in $\mathbf{R}^3$ where the component functions P, Q and R are differentiable. [A region is called *simply connected* if any closed path (loop) with the region can be continuously contracted down to a single point while remaining in the region.] Then the following statements are equivalent:

(1) **F** is conservative.

(2) The work integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path *C* between two fixed points *A* and *B* in the region *D*.

(3) The circulation $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ around any closed path (loop) *C* in the region *D*.

(4) $\mathbf{F} = \overrightarrow{\nabla V}$ for some differentiable function V. [$V(x,y)$ in the $\mathbf{R}^2$ case, and $V(x,y,z)$ in the $\mathbf{R}^3$ case.]

(5) 2D-curl$(\mathbf{F}) = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$ or, equivalently, $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ in the $\mathbf{R}^2$ case,

curl$(\mathbf{F}) = \left\langle \dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}, \dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}, \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right\rangle = \mathbf{0}$ or, equivalently, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$, and $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$ in the $\mathbf{R}^3$ case. We previously referred to this (these) condition(s) as the "test for exactness".

(1) and (2) are equivalent by the definition of a conservative vector field. It's easy to see that (2) implies (3) by inserting any two points along the closed curve C and noting that the work will be the same following the two possible routes between these points together with the fact that following one in reverse will simply reverse the sign for the work. This argument is reversible, so we also have that (3) implies (2).

<u>Gradient implies conservative</u>: We have already shown that (4) implies (1) by the Fundamental Theorem of Line Integrals, and that (4) implies (5) by Clairaut's Theorem.

<u>Conservative implies gradient</u>: To show that (1) implies (4), suppose **F** is conservative. We'll do this in the $\mathbf{R}^3$ case. Pick any fixed point ("the ground") (x_0, y_0, z_0) in the given region and for any other point (x,y,z) in the region define $V(x,y,z) = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$ where C is any curve from (x_0, y_0, z_0) to (x,y,z). This is well-defined because the vector field **F** is presumed to be conservative. We'll show that $\dfrac{\partial V}{\partial x} = P$. The calculation for the other components is similar. Suppose we vary *x* only by an amount Δx starting at (x,y,z) and ending at $(x+\Delta x, y, z)$. Over this segment we'll have $dy = 0$ and $dz = 0$, so the change ΔV will be given by $\Delta V = V(x+\Delta x, y, z) - V(x,y,z) = \int_{C'} \mathbf{F}\cdot d\mathbf{r} = \int_{C'} P\,dx$ where C' is the aforementioned short segment. The integral is given approximately by $P(\tilde{x}, y, z)\Delta x$ where $\tilde{x}$ is between x and $x+\Delta x$. So we have $\Delta V = V(x+\Delta x, y, z) - V(x,y,z) \cong P(\tilde{x}, y, z)\Delta x$. Division by Δx then gives $\dfrac{\Delta V}{\Delta x} = \dfrac{V(x+\Delta x, y, z) - V(x,y,z)}{\Delta x} \cong P(\tilde{x}, y, z)$, and if we then pass to the limit as $\Delta x \to 0$, $\tilde{x}$ will be squeezed toward x and we'll have $\dfrac{\partial V}{\partial x} = \lim\limits_{\Delta x \to 0}\left[\dfrac{V(x+\Delta x, y, z) - V(x,y,z)}{\Delta x}\right] = P(x,y,z)$. Similarly, $\dfrac{\partial V}{\partial y} = Q$ and $\dfrac{\partial V}{\partial z} = R$.

All that's left to prove is that (5) implies any of the other statements. We'll show that (5) implies (3), but to do so will require another theorem or two. In the $\mathbf{R}^2$ case we'll need **Green's Theorem**, and in the $\mathbf{R}^3$ case we'll need **Stokes' Theorem** – both of which are, in fact, just different versions of the Fundamental Theorem of Calculus.

Green's Theorem: Suppose $\mathbf{F}(x,y)=\langle P(x,y),Q(x,y)\rangle$ defines a vector field in some bounded region *D* in $\mathbf{R}^2$ where the component functions $P(x,y)$ and $Q(x,y)$ are differentiable. Let *C* be the boundary of this region oriented in the counterclockwise sense (this can be understood generally to mean that as you transfer the boundary the region *D* will always be to the left). We denote this by $\text{Bnd}(D)=\partial D=C$. Then:

$$\begin{pmatrix}\text{circulation of}\\ \mathbf{F}\text{ around } C=\partial D\end{pmatrix}=\oint_{C=\partial D}\mathbf{F}\cdot d\mathbf{r}=\oint_{C=\partial D}Pdx+Qdy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$

In other words, the circulation around the boundary is the same as the integral of the 2D-curl over the interior. All versions of the Fundamental Theorem of Calculus share this same theme of trading in a boundary for some kind of derivative and integrating over the interior.

Corollary: Suppose $\mathbf{F}(x,y)=\langle P(x,y),Q(x,y)\rangle$ defines a vector field in some bounded, simply connected region *D* in $\mathbf{R}^2$ where the component functions $P(x,y)$ and $Q(x,y)$ are differentiable and let $\text{Bnd}(D)=C$. If $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ through the region *D*, then $\oint_C\mathbf{F}\cdot d\mathbf{r}=0$. That is, (5) implies (3).

Proof: If $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$, then $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$ throughout *D*. Therefore $\oint_{C=\partial D}\mathbf{F}\cdot d\mathbf{r}=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=\iint_D 0\,dA=0$.

We'll delay the proof that (5) implies (3) in the $\mathbf{R}^3$ case until we state and prove Stokes' Theorem.

Geometric, coordinate-free definition of 2D-curl: The curl of a vector field can best be understood as a ***circulation density***. If we choose any point $(x,y)\in\mathbf{R}^2$, let D_k be any (small) region that contains this point and let $C_k=\partial D_k$ be its boundary (oriented in the counterclockwise sense). We define the circulation density of F at the point (x,y) by calculating the circulation of F about $C_k=\partial D_k$ as a fraction of the $\text{Area}(D_k)=\Delta A_k$ of the small region, and then take the limit as this small region shrinks down to the single point (x,y) - assuming, for the moment, that this limit exists independent of any choices during the construction. That is:

$$[\text{2D-curl}(\mathbf{F})](x,y)=\lim_{\text{diam}(D_k)\to 0}\left[\frac{\oint_{C_k=\partial D_k}\mathbf{F}\cdot d\mathbf{r}}{\Delta A_k}\right]$$

Proof of Green's Theorem: Partition the region *D* into small cells D_k and let $C_k=\partial D_k$ be the boundary of the *k*-th cell. Choose a sample point $(x_k,y_k)\in D_k$ for each cell. Then from the geometric limit definition above we can say that for all *k*, $\frac{\oint_{C_k=\partial D_k}\mathbf{F}\cdot d\mathbf{r}}{\Delta A_k}\cong[\text{2D-curl}(\mathbf{F})](x_k,y_k)$, so $\oint_{C_k=\partial D_k}\mathbf{F}\cdot d\mathbf{r}\cong\{[\text{2D-curl}(\mathbf{F})](x_k,y_k)\}\,\Delta A_k$.

Summing over *k* we have that $\sum_k\left(\oint_{C_k=\partial D_k}\mathbf{F}\cdot d\mathbf{r}\right)\cong\sum_k\{[\text{2D-curl}(\mathbf{F})](x_k,y_k)\}\,\Delta A_k$.

Observe that in the left-hand sum <u>the contribution from any adjacent cells will cancel pairwise</u> since those portions of the boundaries will be in opposite directions. Therefore the only contributions will be from the cells with boundaries on the overall boundary of the region *D*. That is, $\oint_{C=\partial D}\mathbf{F}\cdot d\mathbf{r}\cong\sum_k\{[\text{2D-curl}(\mathbf{F})](x_k,y_k)\}\,\Delta A_k$.

Finally, by refining the partition and passing to the limit as the mesh of the partition tends to zero, the approximation will approach an equality, so we'll have:

$$\oint_{C=\partial D} \mathbf{F}\cdot d\mathbf{r} = \lim_{|\Delta|\to 0}\left(\sum_k \{[\text{2D-curl}(\mathbf{F})](x_k, y_k)\}\,\Delta A_k\right) = \iint_D \{[\text{2D-curl}(\mathbf{F})](x,y)\}\,dA$$

Basically, the proof of Green's Theorem is really just a corollary of the geometric definition of the curl.

There is one missing piece that we still need to resolve – namely that the geometric definition of the 2D-curl implies the algebraic definition, i.e. $\text{2D-curl}(\mathbf{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. To do this, recall that the work integral around any small cell D_k is $\oint_{C_k=\partial D_k} \mathbf{F}\cdot d\mathbf{r} = \oint_{C_k=\partial D_k} F_T\, ds$ where $F_T = \mathbf{F}\cdot\mathbf{T}$ and **T** is the unit tangent vector for the boundary curve. To derive the Cartesian expression for the 2D-curl, we choose a small rectangular cell with side lengths Δx and Δy and, for convenience, locate the point (x, y) at the (lower left) corner of this cell. The integral can then be approximated by adding up the contributions from the four sides of this cell. We can assemble the necessary information in a convenient table:

Side	**T**	$F_T = \mathbf{F}\cdot\mathbf{T}$	Δs
Bottom	**i**	$P(x,y)$	Δx
Right	**j**	$Q(x+\Delta x, y)$	Δy
Top	–**i**	$-P(x, y+\Delta y)$	Δx
Left	−**j**	$-Q(x,y)$	Δy

For each side we chose the most convenient point on that side for our approximate values. If we sum these four contributions and divide by the area of the cell, we get:

$$\frac{\oint_{C_k=\partial D_k} F_T\, ds}{\Delta A_k} \cong \frac{\sum F_T\,\Delta s}{\Delta A_k} = \frac{[Q(x+\Delta x, y) - Q(x,y)]\Delta y - [P(x, y+\Delta y) - P(x,y)]\Delta x}{\Delta x \Delta y}$$

$$= \frac{[Q(x+\Delta x, y) - Q(x,y)]}{\Delta x} - \frac{[P(x, y+\Delta y) - P(x,y)]}{\Delta y}$$

Finally, if we pass to the limit as both Δx and Δy approach zero, we'll have:

$$[\text{2D-curl}](\mathbf{F}) = \lim_{|\Delta|\to 0}\left[\frac{\oint_{C_k=\partial D_k} F_T\, ds}{\Delta A_k}\right] = \lim_{\Delta x\to 0}\left[\frac{[Q(x+\Delta x, y) - Q(x,y)]}{\Delta x}\right] - \lim_{\Delta y\to 0}\left[\frac{[P(x, y+\Delta y) - P(x,y)]}{\Delta y}\right] = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Curious Corollary of Green's Theorem: Suppose *D* is any region in the xy-plane and that $C = \partial D$ is its boundary. Then $Area(D) = \oint_{C=\partial D} x\,dy$.

As a practical matter, this means that we can either parameterize the boundary curve and calculate the given integral OR choose closely-spaced waypoints (x_k, y_k) all along the boundary and approximate the integral by calculating the sum $\sum_k x_k \Delta y_k$. This is relatively easy to carry out using a GPS device.

Proof of Corollary: $\oint_{C=\partial D} x\,dy = \oint_{C=\partial D} 0\,dx + x\,dy = \iint_D (1-0)\,dA = \iint_D dA = Area(D)$

Normal Form of Green's Theorem

The standard form of Green's Theorem is derived by considering the work integral around the counterclockwise boundary of a region, i.e. $\oint_{C=\partial D} F_T\, ds = \oint_{C=\partial D} \mathbf{F}\cdot\mathbf{T}\, ds$. We can alternatively rotate the unit tangent vector **T** clockwise $90°$ to produce an outward unit normal vector **N** at every point of this curve (or any curve). For a small segment of the boundary with length Δs_k, we can measure the flux (or flow) of the vector field across the curve as $F_N \Delta s_k$ where $F_N = \mathbf{F}\cdot\mathbf{N}$ is the outward normal component of the vector field at any given point. If we sum these up over the entire curve and pass to the limit as these pieces become arbitrarily small, we can define the ***flux of F across the curve C*** as $\int_C F_N\, ds = \int_C \mathbf{F}\cdot\mathbf{N}\, ds$.

In Cartesian coordinates, if $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ and if we parameterize the curve C as $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, then $\mathbf{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle$ and formally $\mathbf{T}\, ds = \frac{\mathbf{v}}{\|\mathbf{v}\|}\|\mathbf{v}\|\, dt = \mathbf{v}dt = \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt = \langle dx, dy\rangle = d\mathbf{r}$

If we rotate this clockwise $90°$, we get $\mathbf{N}\, ds = \left\langle \frac{dy}{dt}, -\frac{dx}{dt}\right\rangle dt = \langle dy, -dx\rangle$, so:

$$\int_C F_N\, ds = \int_C \mathbf{F}\cdot\mathbf{N}\, ds = \int_C \langle P, Q\rangle\cdot\langle dy, -dx\rangle = \int_C -Q(x,y)dx + P(x,y)dy$$

Normal Form of Green's Theorem: If $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ defines a vector field in some bounded region D in $\mathbf{R}^2$ where the component functions $P(x,y)$ and $Q(x,y)$ are differentiable and where $C = \partial D$ is the boundary of this region oriented in the counterclockwise sense, then:

$$\begin{pmatrix}\text{net flux of } \mathbf{F} \text{ outward} \\ \text{across } C = \partial D\end{pmatrix} = \oint_{C=\partial D} F_N ds = \oint_{C=\partial D} -Qdx + Pdy = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \iint_D \operatorname{div}(\mathbf{F})\, dA$$

where $\operatorname{div}(\mathbf{F})$ is the two-dimension divergence of this vector field.

This provides some explanation for the interpretation of the divergence of a vector field as a ***source density***. Essentially, the total amount of "stuff" flowing outward across the boundary of a closed region should measure the total amount of the source of that "stuff" emanating from within the region. The three-dimensional version of this will be the Divergence Theorem.

In the next lecture we'll look at integration on surfaces.

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