Multivariable Calculus – Lecture #11 Notes

In this lecture, we'll discuss changing coordinates more generally in multiple integrals. We'll also discuss the idea of integration along a curve and the application of this idea in conjunction with vector fields to define the work done by a variable force along a specified curve (line integrals). This, in turn, will lead to a discussion of conservative vector fields vs. non-conservative vector fields, potential functions, and the Fundamental Theorem of Line Integrals.

Change of Variables in Multiple Integrals

In the previous two lectures we used direct geometric observations to derive methods for setting up double integrals in polar coordinates and for setting up triple integrals in cylindrical and spherical coordinates. There are some circumstances in which we may want to invent our own coordinates in order to set up and calculate integrals. The approach is analogous to the *u*-substitution method in single-variable calculus. There are three aspects to such a change: (a) changing the integrand, (b) changing the "measure", and (c) changing the integral limits.

Transforming a double integral

Suppose you need to calculate an integral $\iint_D f(x,y)dA = \iint_D f(x,y)dxdy$ over a region D in $\mathbf{R}^2$ for which standard (x,y) Cartesian coordinates may not be the ideal choice. Further suppose that you have decided on using new coordinates (u,v) and that there is a coordinate transformation expressed by $\left\{\begin{matrix} x = x(u,v) \\ y = y(u,v) \end{matrix}\right\}$. We may wish to express this as $F(u,v) = (x(u,v), y(u,v))$ where (u,v) represents points in a region D' in the *uv*-plane that is transformed in a one-to-one manner to the region *D* in the *xy*-plane. Transforming the integrand into the new (u,v) coordinates is straightforward substitution, but we need to better understand how the "area element" will transform.

The key to understanding this is to focus on the parameterized curves obtained by varying one parameter at a time. If we start at a point $(u,v) \in D'$ and vary *u* only (and think of this variable *u* like "time"), then a curve in *D* will be traced out and the vector $\frac{\partial F}{\partial u}$ will be like a "velocity vector" that is tangent to this curve. A small change Δu will then yield an approximate "displacement" $\left[\frac{\partial F}{\partial u}\Delta u\right]$. Similarly, if we vary *v* only (and again think of this variable *v* like "time"), then a curve in *D* will be traced out and the vector $\frac{\partial F}{\partial v}$ will be tangent to this curve. A small change Δv will then yield an approximate "displacement" $\left[\frac{\partial F}{\partial v}\Delta v\right]$. These two displacement vectors will determine a small approximate parallelogram in *D* that is in one-to-one correspondence with the rectangle in D' with side lengths Δu and Δv. If we relate the area $\Delta u \Delta v$ of the rectangle in D' and the area of the corresponding approximate parallelogram in *D*, we can use a linear algebra fact that this is given by the (absolute value of) the determinant of the matrix whose columns are the vectors that determine the parallelogram. That is $\Delta A \cong \left|\det\begin{bmatrix} \frac{\partial F}{\partial u}\Delta u & \frac{\partial F}{\partial v}\Delta v \end{bmatrix}\right| = \left|\det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right| \Delta u \Delta v$. We introduce the

Revised November 26, 2017

notation $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$ for the ***Jacobian determinant***. The matrix itself $J_F = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$ is the ***Jacobian matrix*** associated with this coordinate transformation.

The above calculation enables us to relate a small area $\Delta A' = \Delta u \Delta v$ in D' to the corresponding area in D by $\Delta A \cong \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v$. For the purposes of integration, this enables us, in the limit, to relate the area elements by $dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$. Basically, the absolute value of the Jacobian determinant $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ acts as a scaling factor for small areas that enables us to relate the measures in the respective regions.

Using these facts, we conclude that $\boxed{\iint_D f(x,y)dxdy = \iint_{D'} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv}$.

For the purpose of calculation it is sometimes easier to deal with the inverse coordinate transformation. Understanding the Jacobian determinant to be a local area scaling factor, it follows that $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$, i.e. the reciprocal.

Example 1: The change from Cartesian coordinates to polar coordinates is accomplished via the transformation given by $\begin{Bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{Bmatrix}$. Its Jacobian matrix is $\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$ and the Jacobian determinant is $\frac{\partial(x,y)}{\partial(r,\theta)} = r(\cos^2\theta + \sin^2\theta) = r$, so $dxdy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta = r\,drd\theta$ as expected.

Example 2: Find the area of the region *D* bounded by the ellipse with semi-axes *a* and *b* and given by equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: $Area(D) = \iint_D dA = \iint_D dxdy$. If we express the equation of the ellipse as $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, this suggests a coordinate change by rescaling the axes, i.e. $\begin{Bmatrix} \frac{x}{a} = u \\ \frac{y}{b} = v \end{Bmatrix}$ or $\begin{Bmatrix} x = au \\ y = bv \end{Bmatrix}$. The bounding curve of the transformed region D' is then the unit circle with equation $u^2 + v^2 = 1$.
The Jacobian determinant is $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = ab$, so the area is

$Area(D) = \iint_D dxdy = \iint_D \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \iint_{D'} ab\, dudv = ab\iint_{D'} dudv = ab \cdot Area(D') = \boxed{\pi ab}$ since the area of the circular unit disk D' is just π (no integration necessary).

Example 3: Calculate the integral $\iint_D x^2 y\, dA = \iint_D x^2 y\, dxdy$ where D is the parallelogram region bounded by the lines $y = x$, $y = x+2$, $y = -2x+2$, and $y = -2x+6$.

Solution: We can realize these boundary lines as level sets of coordinate functions by rewriting them as $-x+y=0$, $-x+y=2$, $2x+y=2$, and $2x+y=6$ and defining the coordinate change by $\left\{\begin{matrix} u = 2x+y \\ v = -x+y \end{matrix}\right\}$.

The Jacobian determinant is then $\dfrac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = 2+1 = 3$. Therefore $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{3}$. To transform the integrand, we solve for $\left\{\begin{matrix} x = \dfrac{u-v}{3} \\ y = \dfrac{u+2v}{3} \end{matrix}\right\}$. The boundary curves (going counterclockwise around the boundary) are: (1) $2x+y=2 \Rightarrow \boxed{u=2}$; (2) $-x+y=0 \Rightarrow \boxed{v=0}$; (3) $2x+y=6 \Rightarrow \boxed{u=6}$; and $-x+y=2 \Rightarrow \boxed{v=2}$.
The resulting region D' is therefore just a rectangle and the integral becomes

$$\iint_D x^2 y\, dxdy = \int_{v=0}^{v=2}\int_{u=2}^{u=6}\left(\frac{u-v}{3}\right)^2\left(\frac{u+2v}{3}\right)\cdot\tfrac{1}{3}\,dudv = \tfrac{1}{81}\int_{v=0}^{v=2}\int_{u=2}^{u=6}(u^3 - 3uv^2 + 2v^3)\,dudv.$$

Inner Integral =

$$\int_{u=2}^{u=6}(u^3 - 3uv^2 + 2v^3)\,du = \left[\tfrac{1}{4}u^4 - \tfrac{3}{2}u^2v^2 + 2uv^3\right]_{u=2}^{u=6} = \tfrac{1}{4}(1296-16) - \tfrac{3}{2}(36-4)v^2 + 2(6-2)v^3 = 320 - 48v^2 + 8v^3$$

Outer Integral = $\tfrac{1}{81}\int_{v=0}^{v=2}\left(320 - 48v^2 + 8v^3\right)dv = \tfrac{1}{81}\left[320v - 16v^3 + 2v^4\right]_{v=0}^{v=2} = \tfrac{1}{81}(640 - 128 + 32) = \boxed{\dfrac{544}{81}}$

Transforming a triple integral

Suppose you need to calculate an integral $\iiint_B f(x,y,z)dV = \iiint_B f(x,y,z)dxdydz$ over a region B in $\mathbf{R}^3$ for which standard (x,y,z) Cartesian coordinates may not be the ideal choice. Further suppose that you have decided on using new coordinates (u,v,w) and that there is a coordinate transformation expressed by $\left\{\begin{matrix} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{matrix}\right\}$. We may wish to express this as $F(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ where (u,v,w) represents points in a region B' in *uvw*-space that is transformed in a one-to-one manner to the region B in *xyz*-space. Transforming the integrand into the new (u,v,w) coordinates is straightforward substitution, but we need to better understand how the "volume element" will transform.

Again, the key to understanding this is to focus on the parameterized curves obtained by varying one parameter at a time. If we start at a point $(u,v,w) \in B'$ and vary u only (and think of this variable u like "time"), then a curve in B will be traced out and the vector $\dfrac{\partial F}{\partial u}$ will be like a "velocity vector" that is tangent to this curve. A small change Δu will then yield an approximate "displacement" $\left[\dfrac{\partial F}{\partial u}\Delta u\right]$. If we vary v only (and again think of this variable v like "time"), then a curve in B will be traced out and the vector $\dfrac{\partial F}{\partial v}$ will be tangent to this curve. A small change Δv will then yield an approximate "displacement" $\left[\frac{\partial F}{\partial v}\Delta v\right]$. If we vary *w* only (and again think of this variable *w* like "time"), then a curve in *B* will be traced out and the vector $\frac{\partial F}{\partial w}$ will be tangent to this curve. A small change Δw will then yield an approximate "displacement" $\left[\frac{\partial F}{\partial w}\Delta w\right]$. These three displacement vectors will determine a small approximate parallelepiped in *B* that is in one-to-one correspondence with the rectangular solid region in B' with side lengths Δu, Δv, and Δw. If we relate the volume $\Delta u\Delta v\Delta w$ of the rectangle in B' and the volume of the corresponding approximate parallelepiped in *B*, we can use a linear algebra fact that this is given by the (absolute value of) the determinant of the matrix whose columns are the vectors that determine the parallelepiped or, equivalently, the triple scalar product. That is

$$\Delta V \cong \left|\det\left[\frac{\partial F}{\partial u}\Delta u \quad \frac{\partial F}{\partial v}\Delta v \quad \frac{\partial F}{\partial w}\Delta w\right]\right| = \left|\det\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{bmatrix}\right|\Delta u\Delta v\Delta w$$

. We introduce the notation

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{bmatrix}$$

for the ***Jacobian determinant***. The matrix itself $J_F = \begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{bmatrix}$ is the ***Jacobian matrix*** associated with this coordinate transformation.

The above calculation enables us to relate a small area $\Delta V' = \Delta u\Delta v\Delta w$ in B' to the corresponding area in *B* by $\Delta V \cong \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\Delta u\Delta v\Delta w$. For the purposes of integration, this enables us, in the limit, to relate the volume elements by $dxdydz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dudvdw$. The absolute value of the Jacobian determinant $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$ acts as a scaling factor for small volumes that enables us to relate the measures in the respective regions.

Using these facts, we conclude that

$$\boxed{\iiint_B f(x,y,z)dxdydz = \iiint_{B'} f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dudvdw}.$$

For the purpose of calculation it is sometimes easier to deal with the inverse coordinate transformation. Understanding the Jacobian determinant to be a local volume scaling factor, it follows that $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|^{-1}$, i.e. the reciprocal.

Example 4: Consider the spherical coordinate change given by the equations $\left\{\begin{array}{c} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{array}\right\}$. The Jacobian determinant is:

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta\sin\phi & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\sin\theta\cos\phi & \rho\cos\theta\sin\phi \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}$$

$$= \cos\theta\sin\phi(\rho^2\cos\theta\sin^2\phi) + \rho\cos\theta\cos\phi(\rho\cos\theta\sin\phi\cos\phi) + \rho\sin\theta\sin\phi[\rho\sin\theta(\sin^2\phi+\cos^2\phi)]$$

$$= \rho^2\sin\phi[\cos^2\theta(\sin^2\phi+\cos^2\phi)+\sin^2\theta] = \rho^2\sin\phi[\cos^2\theta+\sin^2\theta] = \boxed{\rho^2\sin\phi}$$

So $dV = dxdydz = \left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| d\rho d\phi d\theta = \boxed{\rho^2\sin\phi\, d\rho d\phi d\theta}$ as expected.

Example 5: Find the volume of the region *B* bounded by the ellipsoid with semi-axes *a*, *b*, and *c* and given by equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution: $Volume(B) = \iiint_B dV = \iiint_B dxdydz$. If we express the equation of the ellipse as

$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$, this suggests a coordinate change by rescaling the axes, i.e. $\begin{Bmatrix} \frac{x}{a}=u \\ \frac{y}{b}=v \\ \frac{z}{c}=w \end{Bmatrix}$ or $\begin{Bmatrix} x=au \\ y=bv \\ z=cw \end{Bmatrix}$.

The bounding surface of the transformed region B' is then the unit sphere with equation $u^2+v^2+w^2=1$.

The Jacobian determinant is $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = abc$, so the volume is

$$Volume(B) = \iiint_B dxdydz = \iiint_{B'} \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw = \iiint_{B'} abc\, dudvdw = abc\iiint_{B'} dudvdw$$

$$= abc\cdot Volume(B') = \boxed{\tfrac{4}{3}\pi abc}$$

since the volume of the unit sphere B' is just $\frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$ (no integration necessary)

Integration along a curve

We now turn our attention to situations where we may want to measure some quantity that's defined for points along a curve. As with all of our previous integral constructions there is a range of situations where might need to do this.

A motivating example – mass of a wire

Suppose a wire is located in either $\mathbf{R}^2$ or $\mathbf{R}^3$ with endpoints *A* and *B* and following a curve *C* between these endpoints. Further suppose that there is a mass density function $\sigma(x,y)$ or $\sigma(x,y,z)$ defined for all points along the wire and measured in appropriate units such as grams/centimeter (a linear density). If we partition the curve C into small pieces C_k where the length of the *k*-th piece is Δs_k, then we can estimate the mass of this piece as $\Delta m_k \cong \sigma(\mathbf{x}_k)\Delta s_k$ where $\mathbf{x}_k$ is a sample point chosen within this *k*-th piece. Summing these, we get that the total mass is given (approximately) by $Mass(C) = \sum_k \Delta m_k \cong \sum_k \sigma(\mathbf{x}_k)\Delta s_k$. As we refine the partition and pass to the limit as the mesh $|\Delta| = \max_k[diam(C_k)]$ tends to zero and if this limit exists independent of any choices, then we define $\lim_{|\Delta|\to 0}\left[\sum_k \sigma(\mathbf{x}_k)\Delta s_k\right]=\int_C \sigma(\mathbf{x})ds$. We can, for the sake of simplicity when working with these integrals, write $\boxed{dm=\sigma ds}$ which allows us to write $Mass(C)=\int_C dm=\int_C \sigma\, ds$.

There is, of course, nothing special about using mass as the quantity to be measured. If, for example, $\sigma(x,y,z)$ measured the electric charge density at points along a charged wire, then we would write $Charge(C)=\int_C dQ=\int_C \sigma(x,y,z)\,ds$. If $\sigma(x,y,z)$ measured the population density at points along a road, then we would write $Population(C)=\int_C dP=\int_C \sigma(x,y,z)\,ds$. In this manner, we can measure the ***total amount*** of any quantity defined along such a curve from its associated density function.

Many of the same applications that we described in terms of double integrals or triple integrals can also formulated for curves. Here is a list of some of these applications with minimal derivation:

1) If $\sigma(x,y)$ or $\sigma(x,y,z)$ is a mass density function for an object that occupies a curve *C*, then the ***total mass*** of this object is given by: $\boxed{Mass(C)=\int_C dm=\int_C \sigma\, ds}$.

   There is, of course, nothing special about using mass as the quantity to be measured. If, for example, $\sigma(x,y,z)$ measured the electric charge density at points along a charged wire, then we would write $\boxed{Charge(C)=\int_C dQ=\int_C \sigma(x,y,z)\,ds}$. If $\sigma(x,y,z)$ measured the population density at points along a road, then we would write $\boxed{Population(C)=\int_C dP=\int_C \sigma(x,y,z)\,ds}$. In this manner, we can measure the ***total amount*** of any quantity defined along such a curve from its associated density function.

2) The ***arclength of a curve C*** can be calculated as: $\boxed{Length(C)=\int_C ds}$.

   We'll elaborate on this shortly when we discuss methods of calculation for these integrals.

3) In the case of a function f defined along a curve *C*, we can define the (unweighted) average value of this function over this curve implicitly by the relation: $(\text{Length of } C)\cdot \overline{f}=\int_C f\, ds$. Therefore the average value is given by: $\boxed{\overline{f}=\dfrac{\int_C f\, ds}{\text{Length}\,(C)}}$. Once again, the (unweighted) average is calculated by integrating the function over its domain and then *dividing by the geometric content of the domain*.

4) The ***centroid*** or geometric center of a curve *C* in $\mathbf{R}^3$ is the point $(\overline{x},\overline{y},\overline{z})$ where

   $\boxed{\overline{x}=\dfrac{\int_C x\, ds}{\text{Length}\,(C)}}$ and $\boxed{\overline{y}=\dfrac{\int_C y\, ds}{\text{Length}\,(C)}}$ and $\boxed{\overline{z}=\dfrac{\int_C z\, ds}{\text{Length}\,(C)}}$.

   We can also do this for curves in the *xy*-plane using just the first two expressions. The centroid need not actually lie on the curve *C*.

5) In the case of a function of three variables f defined along a curve *C*, if we have an associated density function σ defined on this curve that permits us to "weigh" some parts more than others, we can define the ***weighted average of the function*** implicitly by the condition that $(\text{Mass of } C)\cdot \overline{f}_{wtd}=\int_C f\, dm$.

This gives the definition $\boxed{\overline{f}_{wtd} = \dfrac{\int_C f\,dm}{\text{Mass}(C)} = \dfrac{\int_C f\,\sigma\,ds}{\int_C \sigma\,ds}}$.

6) The **center of mass** of a region C in $\mathbf{R}^3$ with associated mass density function $\sigma(x,y,z)$ is the point $(\overline{x}_{cm},\overline{y}_{cm},\overline{z}_{cm})$ where $\boxed{\overline{x}_{cm} = \dfrac{\int_C x\,dm}{\text{Mass}(C)} = \dfrac{\int_C x\,\sigma\,ds}{\int_C \sigma\,ds}}$, $\boxed{\overline{y}_{cm} = \dfrac{\int_C y\,dm}{\text{Mass}(C)} = \dfrac{\int_C y\,\sigma\,ds}{\int_C \sigma\,ds}}$, $\boxed{\overline{z}_{cm} = \dfrac{\int_C z\,dm}{\text{Mass}(C)} = \dfrac{\int_C z\,\sigma\,ds}{\int_C \sigma\,ds}}$.

It is worth noting that in the case of constant density the centroid and center of mass will coincide.

Example 6: Find the center of mass of the half-circular region of radius 1 with $x^2+y^2=1$ with $y\ge 0$ if the density is (a) constant; and (b) $\sigma(x,y)=2-y$.

Solution: In either case, the symmetry of the curve as well as the density function enables us to conclude that $\overline{x}_{cm}=0$. In case (a), the density is constant, so $\overline{y}_{cm} = \dfrac{\int_C y\,\sigma\,ds}{\int_C \sigma\,ds} = \dfrac{\sigma\int_C y\,ds}{\sigma\int_C ds} = \dfrac{\int_C y\,ds}{Length(C)} = \overline{y}$. The denominator is just π, and we can relate arclength to the central angle θ by $ds = Rd\theta = d\theta$. So the numerator becomes $\int_C y\,ds = \int_0^{\pi}(\sin\theta)d\theta = \left[-\cos\theta\right]_0^{\pi} = 2$. So $\overline{y}_{cm} = \dfrac{\int_C y\,ds}{Length(C)} = \frac{2}{\pi} \cong .6366$.

In case (b), $\overline{y}_{cm} = \dfrac{\int_C y\,\sigma\,ds}{\int_C \sigma\,ds} = \dfrac{\int_C y(2-y)\,ds}{\int_C (2-y)\,ds}$. Again using $y=\sin\theta$ and $ds=d\theta$, we have:

Denominator = $\int_C (2-y)\,ds = \int_0^{\pi}(2-\sin\theta)d\theta = \left[2\theta+\cos\theta\right]_0^{\pi} = 2\pi-2$

Numerator = $\int_C (2y-y^2)\,ds = \int_0^{\pi}(2\sin\theta-\sin^2\theta)d\theta = \cdots = 4-\frac{\pi}{2} = \frac{8-\pi}{2}$

Therefore $\overline{y}_{cm} = \dfrac{\int_C y(2-y)\,ds}{\int_C (2-y)\,ds} = \dfrac{\frac{8-\pi}{2}}{2\pi-2} = \dfrac{8-\pi}{4\pi-4} \cong .5672$. Comparing this with part (a), since the density is greatest toward lower y-values, we expect the center of mass to be lower in (b).

General Method: If the given curve is be parameterized by $\mathbf{r}(t)$ where $a\le t\le b$ and if $\mathbf{v}(t)=\mathbf{r}'(t)$ is the velocity vector, we can use the fact that the speed is $\|\mathbf{v}(t)\| = \dfrac{ds}{dt}$ to write $ds = \|\mathbf{v}(t)\|dt = \|\mathbf{v}\|dt$. This will enable us to "pull back" a given integral to produce an ordinary definite integral in the parameter *t*. That is, $\boxed{\int_C f(\mathbf{x})\,ds = \int_{t=a}^{t=b} f(\mathbf{r}(t))\|\mathbf{v}(t)\|dt}$.

Example 7: Find the length and mass of a wire configured along the portion of the parabola $y=x^2$ from the point $(-1,1)$ to the point $(2,4)$ if the density is given by $\sigma(x,y)=y^2$.

Solution: We can parameterize the curve by $\begin{Bmatrix} x=t \\ y=t^2 \end{Bmatrix}$ with $-1\le t\le 2$. The parameterization function is $\mathbf{r}(t)=\langle t,t^2\rangle$. The velocity is $\mathbf{v}(t)=\langle 1,2t\rangle$, and the speed is $\|\mathbf{v}(t)\|=\sqrt{1+4t^2}$.

Using the above method, the length will be $Length(C) = \int_C ds = \int_{-1}^{2}\|v(t)\|dt = \int_{-1}^{2}\sqrt{1+4t^2}\,dt \cong 6.1257$.

The mass is given by $Mass(C) = \int_C \sigma\, ds = \int_C y^2 ds = \int_{-1}^{2} t^4 \sqrt{1+4t^2}\, dt \cong 22.695$

Vector Fields in $\mathbf{R}^2$ and $\mathbf{R}^3$

One of the most important concepts in such disparate fields as physics, ecological modeling, and economics is the concept of vector field. The term itself is relatively self-explanatory. To every point in whatever space we are situated, we assign a vector. This might be done via formulas or perhaps defined by the situation, e.g. the velocity vector at a point in space associated with the wind at any given moment.

If we take a Cartesian point of view, we might define a vector field $\mathbf{F}$ in $\mathbf{R}^2$ by assigning to each point (x, y) a corresponding vector $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ where we'll refer to the functions $P(x, y)$ and $Q(x, y)$ as the ***component functions*** of this vector field. We will often require such conditions as continuity or differentiability of the component functions, but this depends on the application. For a vector field $\mathbf{F}$ in $\mathbf{R}^3$ we would assign to each point (x, y, z) a corresponding vector $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$.

Example 8: We define the ***radial vector field*** $\mathbf{r}(x, y) = \langle x, y\rangle$ in $\mathbf{R}^2$. This simply assigns to every point its own position vector but relocated to the given point. This vector field forms a good starting point for constructing other interesting vector fields. In $\mathbf{R}^3$ we would have $\mathbf{r}(x, y, z) = \langle x, y, z\rangle$.

Example 9: We define the ***unit radial vector field*** $\hat{\mathbf{r}} = \dfrac{\mathbf{r}}{\|\mathbf{r}\|} = \dfrac{\langle x, y\rangle}{\sqrt{x^2+y^2}} = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$ in $\mathbf{R}^2$. This assigns to every point (except the origin) a unit vector pointing radially outward. In $\mathbf{R}^3$ we would have $\hat{\mathbf{r}} = \dfrac{\mathbf{r}}{\|\mathbf{r}\|} = \dfrac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}} = \left\langle \dfrac{x}{\sqrt{x^2+y^2+z^2}}, \dfrac{y}{\sqrt{x^2+y^2+z^2}}, \dfrac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$. As you may surmise, this is a vector field that's easy to understand conceptually but doesn't mesh well with Cartesian coordinates.

Example 10: We can define a vector field $\hat{\theta}$ in $\mathbf{R}^2$ that assigns to each point (except the origin) a unit vector in the direction of increasing polar angle θ. First, note that if $\mathbf{v} = \langle a, b\rangle$, then we can rotate this vector counterclockwise (retaining its magnitude) to get the vector $\langle -b, a\rangle$. Using this idea, we can define $\hat{\theta} = \dfrac{\langle -y, x\rangle}{\sqrt{x^2+y^2}} = \left\langle \dfrac{-y}{\sqrt{x^2+y^2}}, \dfrac{x}{\sqrt{x^2+y^2}} \right\rangle$.

Example 11: We know from physics that the gravitational attraction associated with a mass *M* located at the origin on another mass *m* will be directed radially inward toward the origin with magnitude given by the inverse square law $\dfrac{GMm}{R^2}$ where *R* is the distance between the masses. Using the above constructions with appropriate scaling and reversal of sign, we can express the gravitational force by:

$$\mathbf{F} = -\left(\frac{GMm}{R^2}\right)\hat{\mathbf{r}} = -\left(\frac{GMm}{\|\mathbf{r}\|^2}\right)\frac{\mathbf{r}}{\|\mathbf{r}\|} = -\left(\frac{GMm}{\|\mathbf{r}\|^3}\right)\mathbf{r} = -\frac{GMm\langle x, y, z\rangle}{(x^2+y^2+y^2)^{3/2}}$$

Example 12: Given any differentiable function $V(x, y)$, the gradient of this function actually defines a vector field that assigns to every point $(x, y) \in \mathbf{R}^2$ the gradient vector at that point, i.e. $\overrightarrow{\nabla V}(x, y) = \left\langle \dfrac{\partial V}{\partial x}, \dfrac{\partial V}{\partial y} \right\rangle$. You should recall that the gradient vector at any given point will be perpendicular to the level set passing through that point and will be directed toward increasing values. For a differentiable function $V(x,y,z)$, the gradient of this function actually defines a vector field that assigns to every point $(x,y,z)\in\mathbf{R}^3$ the gradient vector at that point, i.e. $\overrightarrow{\nabla V}(x,y,z)=\left\langle \frac{\partial V}{\partial x},\frac{\partial V}{\partial y},\frac{\partial V}{\partial z}\right\rangle$. At any point this vector will be perpendicular to the level surface passing through that point.

Work done by a variable force along a curve (line integrals)

In physics, work done by a constant force F over a distance Δs is just the product $F\Delta s$. If the force is a vector, then only the tangential part of this vector F_T will contribute to the work, i.e. $F_T\Delta s$. Now suppose that **F** is a variable force field acting along a path C that goes from a starting point A to an ending point B. If we partition the path into small pieces C_k where the length of the k-th piece is Δs_k, then we can estimate the work done along this small segment as $\Delta W_k \cong F_T\Delta s_k$ where F_T is evaluated at a sample point $\mathbf{x}_k$ within this k-th piece. Summing these, we get that the total work is given (approximately) by $W=\sum_k \Delta W_k \cong \sum_k F_T\Delta s_k$. As we refine the partition and pass to the limit as the mesh $|\Delta|=\max_k[diam(C_k)]$ tends to zero we get $\lim_{|\Delta|\to 0}\left[\sum_k F_T\Delta s_k\right]=\int_C F_T\,ds$. This is the fundamental definition of work, but it is often expressed in other forms.

Suppose we parameterize the curve C in such a way that the velocity vector never vanishes, i.e. "keep moving". If the parameterization function is $\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle$ where $a\le t\le b$, the velocity vector will then be $\mathbf{v}(t)=\frac{d\mathbf{r}}{dt}=\left\langle \frac{dx}{dt},\frac{dy}{dt}\right\rangle$ in $\mathbf{R}^2$ or $\mathbf{v}(t)=\frac{d\mathbf{r}}{dt}=\left\langle \frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right\rangle$ in $\mathbf{R}^3$, the "speed" will be $\|\mathbf{v}(t)\|$ and we can rewrite the fact that $\|\mathbf{v}(t)\|=\frac{ds}{dt}$ as $ds=\|\mathbf{v}(t)\|\,dt=\|\mathbf{v}\|\,dt$. The unit tangent vector can be calculated as $\mathbf{T}=\frac{\mathbf{v}}{\|\mathbf{v}\|}$, and $F_T=\mathbf{F}\cdot\mathbf{T}$. We can also formally express the fact that $\frac{d\mathbf{r}}{dt}=\mathbf{v}(t)$ as $d\mathbf{r}=\mathbf{v}\,dt=\langle dx,dy\rangle$ in $\mathbf{R}^2$ or $d\mathbf{r}=\mathbf{v}\,dt=\langle dx,dy,dz\rangle$ in $\mathbf{R}^3$. Using these relations, we can write:

$$Work=\int_C F_T\,ds=\int_C \mathbf{F}\cdot\mathbf{T}\,ds=\int_a^b \mathbf{F}\cdot\frac{\mathbf{v}}{\|\mathbf{v}\|}\|\mathbf{v}\|\,dt=\int_a^b \mathbf{F}\cdot\mathbf{v}\,dt=\int_C \mathbf{F}\cdot d\mathbf{r}$$

If $\mathbf{F}(x,y)=\langle P(x,y),Q(x,y)\rangle$, then this can be expressed as $Work=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C P(x,y)dx+Q(x,y)dy$, though we will often express this simply as $\int_C P\,dx+Q\,dy$.

If $\mathbf{F}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$, then this can be expressed as $Work=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz$, though we will often express this simply as $\int_C P\,dx+Q\,dy+R\,dz$.

Example 13: Calculate the work done by the force $\mathbf{F}=\langle xy,-x^2\rangle$ along the straight line path from $(1,2)$ to $(5,-3)$.

Solution: Using the above formalism, we can express the work in this case as $\int_C xy\,dx - x^2 dy$. The path can be parameterized as $\begin{Bmatrix} x = 1+4t \\ y = 2-5t \end{Bmatrix}$ where $0 \le t \le 1$. We calculate $\begin{Bmatrix} dx = 4dt \\ dy = -5dt \end{Bmatrix}$, and substitution into the integral gives $\int_0^1 [(1+4t)(2-5t)(4)-(1+4t)^2(-5)]dt = \int_0^1 (13+52t)dt = \left[13t+26t^2\right]_0^1 = \boxed{39}$.

Example 14: Calculate the work done by the same force $\mathbf{F} = \langle xy, -x^2 \rangle$ along the path consisting of the horizontal segment C_1 from $(1,2)$ to $(5,2)$ followed by the vertical segment C_2 from $(5,2)$ to $(5,-3)$.

Solution: $\int_{C=C_1\cup C_2} xy\,dx - x^2 dy = \int_{C_1} (xy\,dx - x^2 dy) + \int_{C_2} (xy\,dx - x^2 dy)$.

We can parameterize the first segment using x as both parameter and coordinate, $\begin{Bmatrix} x = x \\ y = 2 \end{Bmatrix}$, $\begin{Bmatrix} dx = dx \\ dy = 0 \end{Bmatrix}$, and we get $\int_{C_1} (xy\,dx - x^2 dy) = \int_{x=1}^{x=5} 2x dx = \left[x^2\right]_1^5 = 25 - 1 = 24$

For the second segment, we can use y as both parameter and coordinate, $\begin{Bmatrix} x = 5 \\ y = y \end{Bmatrix}$, $\begin{Bmatrix} dx = 0 \\ dy = dy \end{Bmatrix}$, and we get

$\int_{C_2} (xy\,dx - x^2 dy) = \int_{y=2}^{y=-3} -25 dy = \left[-25y\right]_2^{-3} = 125$

Therefore $\int_{C=C_1\cup C_2} xy\,dx - x^2 dy = 24 + 125 = 149$

Note that for this vector field, the work integral gives different results for different paths connecting the same two endpoints.

Conservative Vector Fields and the Fundamental Theorem of Line Integrals

It's essential to keep in mind that when calculating a path integral of the form $\int_C \mathbf{F}\cdot d\mathbf{r}$ (known as a "line integral" even though it has nothing to do with lines), the value of this integral from one point to another will generally depend on which path is taken. The idea is derived from physics and basically means that the work (energy) done by a vector field (representing a force) as you travel from one point to another may well depend on the path followed.

Definition: A vector field **F** is called ***conservative*** if the work integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path, i.e. it depends only on the endpoints of the path.

We calculated a work integral above following two different paths and found two different values for the work. This raises the question of when might the work integral be **independent of path**? We'll see that, in fact, <u>conservative vector fields and gradient vector fields are the same thing</u>. It's easy to see why a gradient vector field is conservative based on the following:

Fundamental Theorem of Line Integrals: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$ and let $\mathbf{F} = \overrightarrow{\nabla V}$ where $V(x,y)$ (or $V(x,y,z)$) is a differentiable function of two (or three) variables whose gradient $\mathbf{F} = \overrightarrow{\nabla V}$ is continuous on the curve C. Then:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \overrightarrow{\nabla V}\cdot d\mathbf{r} = V(\mathbf{r}(b)) - V(\mathbf{r}(a)) = V(\text{end}) - V(\text{start})$$

The function V is generally called a **potential function**, and the Fundamental Theorem of Line Integrals essentially says that the work done by a **conservative vector field** in following a given path is the **potential difference**. The fact that a gradient vector field is conservative should be clear from the statement of this theorem. In the case of a gradient vector field, the work depends only on the values of the potential function at the endpoints – not on any particular path followed from the starting point to the endpoint.

Proof of the Fundamental Theorem of Line Integrals: This is just a blending of the **Fundamental Theorem of Calculus** and the **Chain Rule**. In the case of $\mathbf{F} = \overrightarrow{\nabla V}$ where $V = V(x, y, z)$ in $\mathbf{R}^3$, we have:

$$\int_C \overrightarrow{\nabla V} \cdot d\mathbf{r} = \int_C \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle \cdot \langle dx, dy, dz \rangle = \int_C \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) = \int_a^b \left(\frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt}(V(\mathbf{r}(t))\, dt = V(\mathbf{r}(b)) - V(\mathbf{r}(a))$$

There are two other important aspects to this topic, namely:

(a) How do you know when a given vector field is a gradient (conservative) vector field?

(b) If you know that a vector field is conservative (gradient), how do you find a potential function?

We can partially answer the first question by invoking **Clairaut's Theorem** (equality of mixed partials). Specifically, if $\mathbf{F} = \overrightarrow{\nabla V} = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y} \right\rangle = \langle P(x, y), Q(x, y) \rangle$ where $V(x, y)$ is sufficiently differentiable, then:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x} \right) = \frac{\partial^2 V}{\partial y \partial x} = \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y} \right) = \frac{\partial Q}{\partial x}$$

So it would have to be the case that $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$. If not, then the vector field could not be a gradient vector field.

In the case where $\mathbf{F} = \overrightarrow{\nabla V} = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle = \langle P, Q, R \rangle$, there are three such relations that would have to hold:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x} \right) = \frac{\partial^2 V}{\partial y \partial x} = \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y} \right) = \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial x} \right) = \frac{\partial^2 V}{\partial z \partial x} = \frac{\partial^2 V}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z} \right) = \frac{\partial R}{\partial x} \quad \Rightarrow \quad \boxed{\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}}$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial y} \right) = \frac{\partial^2 V}{\partial z \partial y} = \frac{\partial^2 V}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z} \right) = \frac{\partial R}{\partial y} \quad \Rightarrow \quad \boxed{\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}}$$

These are known as the exactness conditions or the "test for exactness". If any of these three conditions fails to be the case, then the vector field could not be a gradient vector field.

These calculations provide necessary conditions for a vector field to be conservative, but they do not provide sufficient conditions. For that we'll need either **Green's Theorem** (in $\mathbf{R}^2$) or **Stokes' Theorem** (in $\mathbf{R}^3$). However, if we can find an everywhere differentiable **potential function**, then this will be sufficient.

This brings us to the second question: How do we find a potential function after we have established that the conditions above have been met? This is really just a matter of finding antiderivatives and doing a little detective work, though often it comes down simply to "guess and check."

For example, suppose we are given the vector field $\mathbf{F} = \langle 2xy^3, 3x^2y^2 + 8y \rangle = 2xy^3\,\mathbf{i} + (3x^2y^2 + 8y)\,\mathbf{j}$.
Before doing anything else, we check to see if the required condition is met:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy^3) = 6xy^2 \text{ and } \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(3x^2y^2 + 8y) = 6xy^2\text{, so } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ and we're good to go!}$$

We're looking for a function $V(x, y)$ such that $\frac{\partial V}{\partial x} = P(x, y) = 2xy^3$ and $\frac{\partial V}{\partial y} = Q(x, y) = 3x^2y^2 + 8y$.

The first condition implies that $V(x, y) = x^2y^3 + g(y)$ where $g(y)$ is an arbitrary function of y alone.

Differentiation then gives that $\frac{\partial V}{\partial y} = 3x^2y^2 + g'(y) = Q(x, y) = 3x^2y^2 + 8y$, so we must have $g'(y) = 8y$.

Therefore $g(y)$ must be of the form $g(y) = 4y^2 + C$ where C is an arbitrary constant. The potential function must then necessarily be of the form $V(x, y) = x^2y^3 + 4y^2 + C$. We actually only need one potential function, so we just take the arbitrary constant to be $C = 0$ and we use $V(x, y) = x^2y^3 + 4y^2$.

It's important to note that you could have looked at both of the components of **F** and **guessed** a potential function, but if you do this you must take the partial derivatives to **check** that it gives the correct gradient.

Example 15: Find the work done by the vector field $\mathbf{F} = 2xy^3\,\mathbf{i} + (3x^2y^2 + 8y)\,\mathbf{j}$ along some wild and crazy path from the starting point $(1,1)$ to the endpoint $(2,3)$. [Had we given a specific path, the method would still be the same.]

Solution: First, you have to check whether or not the vector field is conservative (gradient). If it isn't, then you have no choice but to parameterize the given path. However, in this case, we've already shown that this vector field is the gradient of the potential function $V(x, y) = x^2y^3 + 4y^2$. Therefore, by the Fundamental Theorem of Line Integrals, we have:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \overrightarrow{\nabla V}\cdot d\mathbf{r} = V(\text{end}) - V(\text{start}) = V(2,3) - V(1,1) = [108 + 36] - [1 + 4] = 144 - 5 = 139$$

Examples involving vector fields and paths in $\mathbf{R}^3$ work pretty much the same way except that you have to check three conditions in order to verify whether a given vector field could possibly be a gradient vector field, and then you have to use a bit more deduction or careful guessing and checking to find the potential function.

Example 16: Calculate the work done by the vector field $\mathbf{F} = yz^2\,\mathbf{i} + (xz^2 + 6y)\,\mathbf{j} + (2xyz + 5)\,\mathbf{k}$ along any path from the point $(0,0,0)$ to the point $(3,1,-1)$.

Solution: We first check that the exactness conditions hold:

$$\frac{\partial P}{\partial y} = z^2 = \frac{\partial Q}{\partial x}; \quad \frac{\partial P}{\partial z} = 2yz = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial z} = 2xz = \frac{\partial R}{\partial y}$$

So $\mathbf{F} = \overrightarrow{\nabla V} = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle = \langle yz^2, xz^2 + 6y, 2xyz + 5 \rangle$. $\quad \frac{\partial V}{\partial x} = yz^2 \Rightarrow V(x, y, z) = xyz^2 + f(y, z)$

So $\frac{\partial V}{\partial y} = xz^2 + \frac{\partial f}{\partial y} = xz^2 + 6y \Rightarrow \frac{\partial f}{\partial y} = 6y \Rightarrow f(y, z) = 3y^2 + g(z) \Rightarrow V = xyz^2 + 3y^2 + g(z)$

So $\frac{\partial V}{\partial z} = 2xyz + g'(z) = 2xyz + 5 \Rightarrow g'(z) = 5 \Rightarrow g(z) = 5z \Rightarrow \boxed{V(x, y, z) = xyz^2 + 3y^2 + 5z}$ will do.

By the Fundamental Theorem of Line Integrals, $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \overrightarrow{\nabla V}\cdot d\mathbf{r} = V(3,1,-1) - V(0,0,0) = 1 - 0 = 1$.

In the next lecture we'll develop a list of statements equivalent to a vector field being conservative. We'll also state and prove Green's Theorem to help connect these facts.

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Revised November 26, 2017