Multivariable Calculus – Lecture #10 Notes

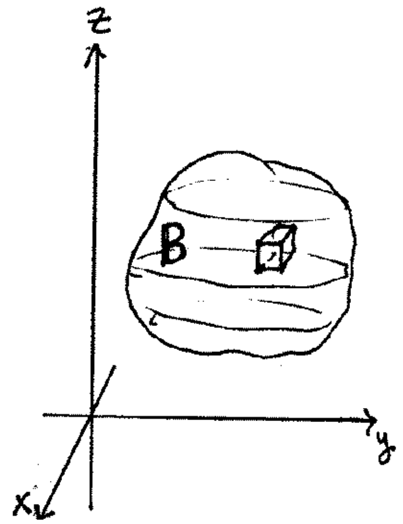
In this lecture we look at integration over regions in space, i.e. triple integrals, using Cartesian coordinates, cylindrical coordinates, and spherical coordinates. We'll show how to change coordinates generally in multiple integrals and also see some examples of integration on surfaces (a topic to be addressed more fully later).

Integration over regions in space (Triple Integrals)

All of the concepts and methods of calculation of integrals over regions in the plane carry over seamlessly to regions in space – with an additional layer of complexity. There are also additional applications (such as moment of inertia or gravitational attraction) that really only make sense in three dimensions.

A motivating example

When we were first introduced to the idea of the integral of a function of one variable over an integral, the motivating example was most likely finding the area under a graph, but we later saw other settings where integration is necessary to measure a quantity defined on a one-dimensional object such as an interval (mass of a rod, for example). When motivating double integrals the first illustration was finding the volume under a graph, though we later developed a range of other applications such as mass, charge, population, centroid, and center of mass. In motivating the idea of integration of a function of three variables, we really don't have the option of looking at its graph since that would require four dimensions to define it. So the best motivating examples are the measurement of total volume of a region or total mass (or charge or population) of a region in space.



Suppose B is a solid region in $\mathbf{R}^3$ in which a mass density function $\sigma(x,y,z)$ is defined. How shall we determine the total mass of such a region? [The "correct" answer, for practical purposes would be to weigh it on a scale, but this may not be a practical option, e.g. weighing a planet.] Proceeding using the Method of Riemann Sums, suppose we partitioned the solid in some systematic way into n small pieces, and define the ***mesh of the partition*** by $|\Delta| = \max\limits_{1\le k\le n}\left(\text{diameter}(B_k)\right)$. If B_k is any such piece, suppose we choose a sample point (x_k, y_k, z_k) in this piece and use $\sigma(x_k, y_k, z_k)$ to estimate the density in this small piece. If Δm_k represents the mass of this small piece and if ΔV_k represents its volume, then $\Delta m_k \cong \sigma(x_k, y_k, z_k)\Delta V_k$. The total mass can then be approximated by $\text{Mass}(B) = \sum_{k=1}^{n}\Delta m_k \cong \sum_{k=1}^{n}\sigma(x_k, y_k, z_k)\Delta V_k$. To make this more precise we would refine the partition and calculate the limit of this sum: $\lim\limits_{|\Delta|\to 0}\left[\sum_{k=1}^{n}\sigma(x_k, y_k, z_k)\Delta V_k\right] \equiv \iiint_B \sigma(x,y,z)\,dV$. If this limit exists independent of any choices, we call this function integrable over this region and call this limit the **triple integral of the function over this region**.

It's worth introducing some differential notation that will be useful in the applications. For the purposes of integration we can express the relationship between density, volume, and mass for an "infinitesimally small" piece as $\boxed{dm = \sigma dV}$.

As was the case for double integrals, there is a wide range of applications for such integrals. Here's a library of such applications.

1) If $\sigma(x,y,z)$ is a mass density function for an object that occupies a domain B in the $\mathbf{R}^3$, then the ***total mass*** of this object is given by: $\boxed{\text{Mass}(B) = \iiint_B dm = \iiint_B \sigma(x,y,z)\,dV}$.

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If the density measured electric charge, the integral $\iiint_B \sigma(x,y,z)dV$ would then give the ***total charge*** (summing both positive and negatively charged regions to produce the net charge). If $\sigma(x,y,z)$ measured population density in a region *B*, then $\iiint_B \sigma(x,y,z)dV$ would give the ***total population*** in the region. Generally, $\iiint_B \sigma(x,y,z)dV$ will give the ***total amount*** of any quantity with associated density function $\sigma(x,y,z)$.

2) The ***total volume of a region B*** can be calculated as: $\boxed{\text{Volume}(B) = \iiint_B dV}$.

3) In the case of a function of two variables $f(x,y,z)$ defined over a domain *B*, we can define the average value of this function over this region implicitly by the relation: $(\text{Volume of } B)\cdot \overline{f} = \iiint_B f(x,y,z)dV$. This is analogous to the similar definition for a function of two variables over a region in $\mathbf{R}^2$. Therefore the average value is given by: $\boxed{\overline{f} = \dfrac{\iiint_B f(x,y,z)dV}{\text{Volume}(B)}}$. Once again, the (unweighted) average is calculated by integrating the function of its domain and then *dividing by the geometric content of the domain*.

For example, if a region *B* in $\mathbf{R}^3$ has an associated mass density function $\sigma(x,y,z)$, we would calculate the ***average density*** as $\overline{\sigma} = \dfrac{\iiint_B \sigma(x,y,z)dV}{\text{Volume}(B)} = \dfrac{\text{Mass}(B)}{\text{Volume}(B)}$, and this coincides with the more familiar way of thinking about average density.

4) The ***centroid*** or geometric center of a region *B* in $\mathbf{R}^3$ is the point $(\overline{x},\overline{y},\overline{z})$ where

$$\boxed{\overline{x} = \frac{\iiint_B x\,dV}{\text{Volume}(B)}} \text{ and } \boxed{\overline{y} = \frac{\iiint_B y\,dV}{\text{Volume}(B)}} \text{ and } \boxed{\overline{z} = \frac{\iiint_B z\,dV}{\text{Volume}(B)}}.$$

5) In the case of a function of three variables $f(x,y,z)$ defined over a domain *B* in $\mathbf{R}^3$, if we have an associated density function $\sigma(x,y,z)$ for the region *B* that permits us to "weigh" some parts more than others, we can define the ***weighted average of the function*** implicitly by the condition that $(\text{Mass of } B)\cdot \overline{f}_{wtd} = \iiint_B f(x,y,z)dm$. This gives the definition

$$\boxed{\overline{f}_{wtd} = \frac{\iiint_B f(x,y,z)dm}{\text{Mass}(B)} = \frac{\iiint_B f(x,y,z)\sigma(x,y,z)dV}{\iiint_B \sigma(x,y,z)dV}}.$$

6) The **center of mass** of a region *B* in $\mathbf{R}^3$ with associated mass density function $\sigma(x,y,z)$ is the point $(\overline{x}_{cm},\overline{y}_{cm},\overline{z}_{cm})$ where $\boxed{\overline{x}_{cm} = \dfrac{\iiint_B x\,dm}{\text{Mass}(B)} = \dfrac{\iiint_B x\,\sigma(x,y,z)dV}{\iiint_B \sigma(x,y,z)dV}}$ and $\boxed{\overline{y}_{cm} = \dfrac{\iiint_B y\,dm}{\text{Mass}(B)} = \dfrac{\iiint_B y\,\sigma(x,y,z)dV}{\iiint_B \sigma(x,y,z)dV}}$ and

$$\boxed{\overline{z}_{cm} = \frac{\iiint_B z\,dm}{\text{Mass}(B)} = \frac{\iiint_B z\,\sigma(x,y,z)dV}{\iiint_B \sigma(x,y,z)dV}}.$$

It is worth noting that in the case of constant density the centroid and center of mass will coincide.

In physics, the moment of inertia of a "point mass" M located at a distance R from an axis of rotation is defined to be $I = MR^2$. Moment of inertia can be thought of as "rotational mass" in the sense that it measures the resistance to rotational acceleration about the given axis in the same manner that mass measures the resistance to linear acceleration. If mass happens to be distributed over a solid region B, i.e. not a "point mass", and if this mass is to be rotated about a given axis, we can use integration to define the moment of inertia of this distributed mass about an axis using Riemann Sums. If we partition the region into small pieces B_k and if we choose a sample point (x_k, y_k, z_k) in each piece and use $\sigma(x_k, y_k, z_k)$ to estimate the density in this small piece, then the mass of this piece will be $\Delta m_k \cong \sigma(x_k, y_k, z_k)\Delta V_k$. If this small piece is located at a distance R_k from the given axis, then the moment of inertia of this small piece will be approximately $\Delta I_k = R_k{}^2 \Delta m$. For the purposes of integration we can write this infinitesimal simply as $dI = R^2 dm$. The total moment of inertia about a given axis can then be measured as $\iiint_B R^2 dm$. The calculation will depend very much on the choice of axis of rotation. In particular, we can define the moments of inertia relative to the standard Cartesian coordinate axes as follows:

7) The **moments of inertia** (I_x, I_y, I_z) relative to the standard coordinate axes of a region B in $\mathbf{R}^3$ with associated mass density function $\sigma(x, y, z)$ are defined by :

$$I_x = \iiint_B R_x{}^2\, dm = \iiint_B (y^2 + z^2)\, dm = \iiint_B (y^2 + z^2)\sigma(x, y, z) dV$$

$$I_y = \iiint_B R_y{}^2\, dm = \iiint_B (x^2 + z^2)\, dm = \iiint_B (x^2 + z^2)\sigma(x, y, z) dV$$

$$I_z = \iiint_B R_z{}^2\, dm = \iiint_B (x^2 + y^2)\, dm = \iiint_B (x^2 + y^2)\sigma(x, y, z) dV.$$

In practice, we often will arrange to have the object situated symmetrically (if possible) about the z-axis so that we only have to calculate the last of these integrals, and we'll make coordinate choices to simply this further.

Calculation of triple integrals via iterated single integrals (Method of Successive Slicing)

The method of successive slicing that we used for calculating double integrals extends easily to the calculation of triple integrals. There's just an additional level of slicing necessary. The added difficulty of visualizing some solid regions will often make the determination of integral limits more challenging. The result will be three iterated single integrals where the integral limits of the middle and innermost integrals may involve variables. It is helpful to draw halfway decent sketches of the solid and perhaps use different colors for the slices in order to better determine the integral limits.

As was the case for double integrals, there will be different possible choices for how to do the slicing and different orders of integration as a result. The Fubini Theorem will still apply. In setting up a double integral we had two possible orders of integration. For triple integrals there will be six possible orders of integration, though usually only one or two of these will be practical.

In calculating an integral $\iiint_B f(x, y, z) dV$, suppose the solid region B is bounded in such a way that the x-variable has $a \le x \le b$ and that that for each x in this range, the cross-section is bounded between the variable limits $y = g(x)$ and $y = h(x)$. Further, by "slicing within the slice", suppose that with both x and y fixed in this slice, the z coordinate varies between a lower limit $z = k(x, y)$ and an upper limit $z = l(x, y)$. This may not be possible for a complicated region, but the region can generally be separated into a set of such simple regions and the results for each of these added to get the total value of the integral. In the order described, we would calculate this integral as:

$$\iiint_B f(x,y,z)\,dV = \int_{x=a}^{x=b}\left[\int_{y=g(x)}^{y=h(x)}\left[\int_{z=k(x,y)}^{z=l(x,y)} f(x,y,z)\,dz\right]dy\right]dx = \int_{x=a}^{x=b}\int_{y=g(x)}^{y=h(x)}\int_{z=k(x,y)}^{z=l(x,y)} f(x,y,z)\,dz\,dy\,dx .$$

We would say that this order of integration corresponds to the volume element $\boxed{dV = dz\,dy\,dx}$. This is just one of six permutations that are possible for the order of integration via iterated single integrals. The nature of the region and, to some degree, the ability to find antiderivatives in order to use the Fundamental Theorem of Calculus will influence the choice for the most simple and practical order of integration. As we'll soon see, the choice of coordinates will also be a very important decision in calculating a multiple integral.

Example 1: Use a triple integral to find the volume of the tetrahedron bounded by the three coordinate planes and the plane with equation $x+y+z=1$.

Solution: It is best practice to do two things at the start of a problem like this. First, draw a reasonable sketch of the region so that you'll be able to determine integral limits. Second, declare what integral you wish to calculate for the purpose at hand. In this example, we want to calculate $\text{Volume}(B) = \iiint_B dV$. Only after you have done these two initial steps should you begin to make any choices of either coordinates or the order of integration. Sometimes the order of integration will come down to a random choice.

In this example, if we use the order of integration indicated by the volume element $dV = dz\,dy\,dx$, we first note that the lowest extreme for *x* will be $x=0$ and the upper extreme will be $x=1$. For each *x* in between, the cross-section will be triangular with the *y*-variable running from $y=0$ to the variable upper limit determined by where the plane $x+y+z=1$ intersects the plane $z=0$. You should be able to see both geometrically and algebraically that $x+y=1$ at this intersection, so $y=1-x$ will be the upper integral limit. [It is often the case that determining the integral limits on the middle integral will be the most challenging.] Finally, if within this triangular slice we also fix the y-variable, then z will vary within this slice from the bottom plane $z=0$ to the plane where $x+y+z=1$, i.e. $z=1-x-y$. The resulting integral is

$$\iiint_B dV = \int_{x=0}^{x=1}\left[\int_{y=0}^{y=1-x}\left[\int_{z=0}^{z=1-x-y} dz\right]dy\right]dx .$$

The Inner Integral is: $\int_{z=0}^{z=1-x-y} dz = \left[z\right]_{z=0}^{z=1-x-y} = 1-x-y$.

The Middle Integral is: $\int_{y=0}^{y=1-x} (1-x-y)\,dy = \left[(1-x)y - \frac{1}{2}y^2\right]_{y=0}^{y=1-x} = (1-x)^2 - \frac{1}{2}(1-x)^2 = \frac{1}{2}(1-x)^2$.

The Outer Integral is: $\int_{x=0}^{x=1}\left[\frac{1}{2}(1-x)^2\right]dx = \left[-\frac{1}{6}(1-x)^3\right]_{x=0}^{x=1} = 0 + \frac{1}{6} = \boxed{\frac{1}{6}}$.

Example 2: For the same region in Example #1, find the average height above the *xy*-plane for points within this solid region.

Solution: The height of any point is just its *z*-coordinate, so we want to calculate $\overline{z} = \dfrac{\iiint_B z\,dV}{\text{Volume}(B)}$. If we use the same order of integration as we did in Example #1, then the integral limits will be exactly the same because the region is the same. The only thing that will be different will be the integrand and therefore the integrations. The denominator is what we calculated in Example 1, i.e. $\text{Volume}(B) = \boxed{\frac{1}{6}}$. The numerator will be $\iiint_B z\,dV = \int_{x=0}^{x=1}\left[\int_{y=0}^{y=1-x}\left[\int_{z=0}^{z=1-x-y} z\,dz\right]dy\right]dx$.

The Inner Integral is: $\int_{z=0}^{z=1-x-y} z\,dz = \left[\frac{1}{2}z^2\right]_{z=0}^{z=1-x-y} = \frac{1}{2}(1-x-y)^2$.

The Middle Integral is: $\int_{y=0}^{y=1-x} \left[\frac{1}{2}(1-x-y)^2\right]dy = \left[-\frac{1}{6}(1-x-y)^3\right]_{y=0}^{y=1-x} = 0 + \frac{1}{6}(1-x)^3 = \frac{1}{6}(1-x)^3$.

The Outer Integral is: $\int_{x=0}^{x=1} \left[\frac{1}{6}(1-x)^3\right]dx = \left[-\frac{1}{24}(1-x)^4\right]_{x=0}^{x=1} = 0 + \frac{1}{24} = \boxed{\frac{1}{24}}$.

Therefore $\bar{z} = \dfrac{\frac{1}{24}}{\frac{1}{6}} = \boxed{\frac{1}{4}}$. You should be able to convince yourself that this answer is plausible for this region.

Example 3: Find the center of mass of the half ball of radius *R* bounded above by the sphere $x^2+y^2+z^2=R^2$ with $z \geq 0$ if (a) the density is constant; and (b) the density is given by $\sigma(x,y,z)=z$.

Solution: In either case, the symmetry of the object as well as the symmetry of the density function means that the center of mass must lie on the z-axis, i.e. $\bar{x}_{cm}=0$ and $\bar{y}_{cm}=0$. It's important to make observations like this at the beginning of your solution so that you can avoid having to set up and evaluate any unnecessary integrals. So everything comes down to the calculation of $\bar{z}_{cm} = \dfrac{\iiint_B z\,dm}{\text{Mass}(B)} = \dfrac{\iiint_B z\,\sigma\,dV}{\iiint_B \sigma\,dV}$.

For part (a), the density is constant, so we can take it outside the integrals and cancel it, i.e. $\bar{z}_{cm} = \dfrac{\sigma\iiint_B z\,dV}{\sigma\iiint_B dV} = \dfrac{\iiint_B z\,dV}{\text{Vol}(B)}$. Furthermore, the denominator can be determined by a geometry formula without the need for integration, i.e. $\text{Vol}(B) = \boxed{\frac{2}{3}\pi R^3}$. So the only integral that's needed is the numerator $\iiint_B z\,dV$.

Since all we have to work with so far are Cartesian coordinates, this integral will be somewhat complicated. From the sketch we see that $-R \leq x \leq +R$ and, for each *x* in between, y will vary from one edge of the circle where $x^2+y^2=R^2$ to the other edge of this circle, i.e. from $y=-\sqrt{R^2-x^2}$ to $y=+\sqrt{R^2-x^2}$., Finally, if we fix both x and y, z will vary from $z=0$ up to the spherical cap, i.e. $z=+\sqrt{R^2-x^2-y^2}$. The iterated integral will therefore be $\iiint_B z\,dV = \int_{x=-R}^{x=+R}\left[\int_{y=-\sqrt{R^2-x^2}}^{y=+\sqrt{R^2-x^2}}\left[\int_{z=0}^{z=+\sqrt{R^2-x^2-y^2}} z\,dz\right]dy\right]dx$.

The Inner Integral is: $\int_{z=0}^{z=+\sqrt{R^2-x^2-y^2}} z\,dz = \left[\frac{1}{2}z^2\right]_{z=0}^{z=+\sqrt{R^2-x^2-y^2}} = \frac{1}{2}(R^2-x^2-y^2)$.

The Middle Integral is: $\int_{y=-\sqrt{R^2-x^2}}^{y=+\sqrt{R^2-x^2}} \left[\frac{1}{2}(R^2-x^2-y^2)\right]dy = \left[\frac{1}{2}(R^2-x^2)y - \frac{1}{6}y^3\right]_{y=-\sqrt{R^2-x^2}}^{y=+\sqrt{R^2-x^2}} = \frac{2}{3}(R^2-x^2)^{3/2}$.

The Outer Integral is: $\int_{x=-R}^{x=+R}\left[\frac{2}{3}(R^2-x^2)^{3/2}\right]dx$. This integral can be solved using a trigonometric substitution (or you can look it up). The resulting value is $\boxed{\frac{\pi}{4}R^4}$.

Note: If you feel that these calculations are unnecessarily complicated, you're quite right. As we'll soon see, a better choice of coordinates more suitable to this half-ball will make the integral set-up as well as the calculations much simpler.

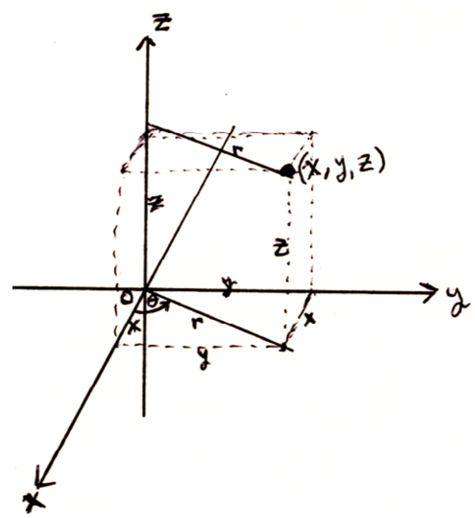
The centroid for part (a) is therefore $\bar{z}_{cm} = \dfrac{\iiint_B z\,dV}{\text{Vol}(B)} = \dfrac{\frac{\pi}{4}R^4}{\frac{2}{3}\pi R^3} = \boxed{\frac{3}{8}R}$. The center of mass is the point $(0,0,\frac{3}{8}R)$.

For part (b), we use the variable density $\sigma(x,y,z) = z$, so $dm = \sigma\,dV = z\,dV$ and the z-component of the center of mass will be $\bar{z}_{cm} = \dfrac{\iiint_B z\,dm}{\text{Mass}(B)} = \dfrac{\iiint_B z^2\,dV}{\iiint_B z\,dV}$. We have already calculated the integral in the denominator, so there's no need to do it again: $\iiint_B z\,dV = \boxed{\frac{\pi}{4}R^4}$. The only new calculation we must do is $\iiint_B z^2\,dV$. The object is the same as above, so the integral limits will also be the same as above. So $\iiint_B z^2\,dV = \int_{x=-R}^{x=+R}\left[\int_{y=-\sqrt{R^2-x^2}}^{y=+\sqrt{R^2-x^2}}\left[\int_{z=0}^{z=+\sqrt{R^2-x^2-y^2}} z^2\,dz\right]dy\right]dx$.

This iterated integral can be done with some perseverance and good integration skills, but it will be much simpler to introduce better coordinate choices that better exploit the symmetry of the object.

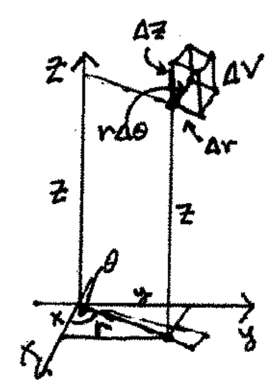
Integration using cylindrical coordinates

When using integration to measure something associated with an object that has some level of symmetry about an axis (generally the z-axis, but our methods can be adapted to other axes), the use of ***cylindrical coordinates*** will usually simplify the setup and calculation of the resulting integrals.



We were first introduced to cylindrical coordinates (r,θ,z) in Lecture #1 where we derived the coordinate transformation $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$. Any change in coordinates for the purpose of integration should be accompanied by such equations as they may be needed to substitute into the integrand in order to transform the integral. We will also need to understand what the level surfaces of the coordinate functions are and find an appropriate expression for the volume element dV for the purpose of setting up iterated single integrals.

In Cartesian (x,y,z) coordinates, "$x=\text{constant}$" surfaces are vertical planes, "$y=\text{constant}$" surfaces are also vertical planes perpendicular to the "$x=\text{constant}$" surfaces, and "$z=\text{constant}$" surfaces are horizontal planes. These surfaces can be used to carve up $\mathbf{R}^3$ into rectangular blocks.



In the case of cylindrical coordinates, an "$r=\text{constant}$" surface will be a cylinder wrapped around the z-axis within which θ and z can vary. A "$\theta=\text{constant}$" surface will be a half-plane coming out radially from the z-axis. Referencing the image of a book standing upright with its spine along the z-axis, it's helpful to view a "$\theta=\text{constant}$" surface as a "page" within which r and z can vary. A "$z=\text{constant}$" surface remains a horizontal plane within which r and θ can vary.

By looking at a small volume lying between the above referenced level surfaces, we will have a small "cell" with mutually perpendicular boundary surfaces (cylinders, pages, and horizontal planes). The respective side lengths (see diagram) will be Δr, $r\Delta\theta$, and Δz, and thanks to the perpendicularity of the boundary surfaces, the volume will be approximately $\Delta V \cong (\Delta r)(r\Delta\theta)(\Delta z) = r\Delta z\Delta r\Delta\theta$. When incorporated into a Riemann Sum and after we pass to the limit as the mesh of the partition shrinks to zero, this can be expressed within the resulting integral as the volume element $\boxed{dV = r\,dzdrd\theta}$. This, of course, presumes a particular order of integration, but this is only one of six possible permutations in the order of integration.

When working with cylindrical coordinates it's also useful to be reminded of the relation $x^2 + y^2 = r^2$. For example, a sphere of radius R centered at the origin will have equation $r^2 + z^2 = R^2$ rather than $x^2 + y^2 + z^2 = R^2$.

Example 4: Let's now calculate using cylindrical coordinates the integral $\iiint_B z^2\,dV$ from the previous example where *B* is the half ball of radius *R* bounded above by the sphere $x^2 + y^2 + z^2 = R^2$ with $z \ge 0$.

Solution: This region has obvious symmetry about the z-axis, so it's a good candidate for cylindrical coordinates. The integrand is already expressed in terms of cylindrical coordinates, and we can use the volume element $dV = r\,dzdrd\theta$ to set up the iterated single integrals. The integral limits will be determined by the observation that θ will go from 0 to 2π as we wrap around the *z*-axis. Each θ-slice will be a quarter-circle within which *r* will vary from 0 to *R*, and for every intermediate value of r within this slice, the z-coordinate will vary from $z = 0$ up to the circle with equation $r^2 + z^2 = R^2$, i.e. $z = \sqrt{R^2 - r^2}$. The resulting integral will therefore be $\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R}\int_{z=0}^{z=\sqrt{R^2-r^2}} z^2 r dzdrd\theta$. Using this order of integration, we have:

Inner Integral = $\int_{z=0}^{z=\sqrt{R^2-r^2}} z^2 r dz = \left[\frac{1}{3} r z^3\right]_{z=0}^{z=\sqrt{R^2-r^2}} = \frac{1}{3} r\left(R^2 - r^2\right)^{3/2}$

Middle Integral = $\int_{r=0}^{r=R} \frac{1}{3} r\left(R^2 - r^2\right)^{3/2} dr = \left[-\frac{1}{15}\left(R^2 - r^2\right)^{5/2}\right]_{r=0}^{r=R} = \frac{1}{15} R^5$

Outer Integral = $\int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{15} R^5\right) d\theta = \boxed{\frac{2\pi}{15} R^5}$

The integrations in this problem are actually a little bit simpler if we use the order of integration associated with the volume element $dV = r\,drdzd\theta$. Within the slice, *z* would vary between $z = 0$ and $z = R$ and for each intermediate choice of *z*, the variable *r* would vary from $r = 0$ out to $r = \sqrt{R^2 - z^2}$. The integral would therefore be $\int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=R}\int_{r=0}^{r=\sqrt{R^2-z^2}} z^2 r\,drdzd\theta$. This would give:

Inner Integral = $\int_{r=0}^{r=\sqrt{R^2-z^2}} z^2 r\,dr = \frac{1}{2} z^2 \left[r^2\right]_{r=0}^{r=\sqrt{R^2-z^2}} = \frac{1}{2} z^2 (R^2 - z^2) = \frac{1}{2}(z^2 R^2 - z^4)$

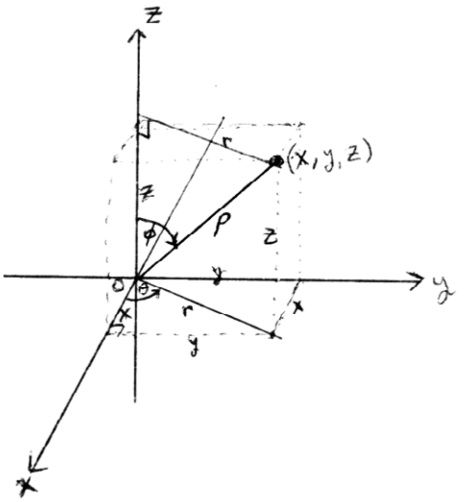
Middle Integral = $\int_{z=0}^{z=R} \frac{1}{2}(z^2 R^2 - z^4)\,dz = \left[\frac{1}{6} z^3 R^2 - \frac{1}{10} z^5\right]_{z=0}^{z=R} = \frac{1}{15} R^5$

Outer Integral = $\int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{15} R^5\right) d\theta = \boxed{\frac{2\pi}{15} R^5}$.

The Fubini Theorem ensured that these two different orders of integration would ultimately yield the same result.

Integration using spherical coordinates

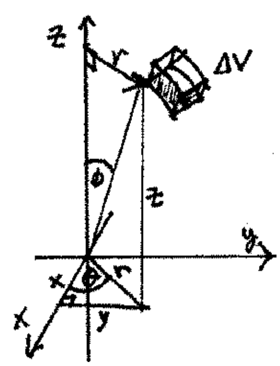
When dealing with an object in $\mathbf{R}^3$ that has some level of symmetry relative to a point (the origin), it is often best to use ***spherical coordinates***. We now consider the distance ρ between the origin and a given point (the spherical radius). This is not the same as the cylindrical radius r (which can be understood as the distance from the z-axis to a given point). We also measure the angle ϕ between the positive z-axis and a ray from the origin to a given point. This may be interpreted as the declination, but it's related to the familiar latitude except that the equatorial plane (our xy-plane) at 0° latitude corresponds to $\phi = 90°$ or $\pi/2$ radians. We also reuse the angle θ from cylindrical coordinates and call it the *azimuth* angle. The combination (ρ, ϕ, θ) constitutes the spherical coordinates.



Note that the constant surfaces for spherical coordinates are:
a "$\rho = \text{constant}$" surface is a sphere centered about the origin,
a "$\phi = \text{constant}$" surface is a half-cone, and
a "$\theta = \text{constant}$" surface is still a vertical half-plane (page) extending infinitely outward from the z-axis. When we do integration, these surfaces will enable us to carve up $\mathbf{R}^3$ into small cells lying between spheres, between cones, and between "pages".
The coordinate transformations between Cartesian coordinates and spherical coordinates are accomplished with some basic trigonometry.
From before we still have that $\begin{Bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{Bmatrix}$ and we can also see from the diagram that $\begin{Bmatrix} r = \rho\sin\phi \\ z = \rho\cos\phi \end{Bmatrix}$. Substitution then gives that $\begin{Bmatrix} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{Bmatrix}$.

Note, in particular, that $\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$, so $\rho = \sqrt{x^2 + y^2 + z^2}$.

For the volume element, note that for a small cell bounded between spheres, cones, and pages, the respective edge lengths will be $\Delta\rho$ between spheres, $\rho\Delta\phi$ between cones (and along a longitude), and $r\Delta\theta = \rho\sin\phi\Delta\theta$ between pages (and along a latitude). The level surfaces are all mutually perpendicular, so the volume of the cell will be approximately $\Delta V \cong (\Delta\rho)(\rho\Delta\phi)(\rho\sin\phi\Delta\theta) = \rho^2\sin\phi\Delta\rho\Delta\phi\Delta\theta$. When incorporated into a Riemann Sum and after we pass to the limit as the mesh of the partition shrinks to zero, this can be expressed within the resulting integral as the volume element $\boxed{dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$. Again, this presumes a particular order of integration, but this is only one of six possible permutations in the order of integration.



Example 5: Let's now calculate using <u>spherical</u> coordinates the integral $\iiint_B z^2\, dV$ from the previous example where B is the half ball of radius R bounded above by the sphere $x^2 + y^2 + z^2 = R^2$ with $z \ge 0$.

Solution: The integrand is not expressed in spherical coordinates, so we use $z = \rho\cos\phi$ to remedy this. We use the volume element $dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$ to set up the iterated single integrals. For the integral limits, θ will go from 0 to 2π as we wrap around the z-axis. Each θ-slice will be a quarter-circle within which ϕ will vary from 0 to $\pi/2$, and for every intermediate value of ϕ within this slice, ρ will vary from $\rho = 0$ out to the sphere with $\rho = R$. The resulting integral will therefore be

$$\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2}\int_{\rho=0}^{\rho=R}(\rho^2\cos^2\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2}\int_{\rho=0}^{\rho=R}\rho^4\cos^2\phi\sin\phi\, d\rho\, d\phi\, d\theta.$$

Note that all the limits of integration are now constant because the region is bounded entirely by level surfaces of the coordinate functions. Using this order of integration, we have:

Inner Integral = $\int_{\rho=0}^{\rho=R} \rho^4 \cos^2\phi \sin\phi \, d\rho = \left[\frac{1}{5}\rho^5\right]_0^R \cos^2\phi \sin\phi = \frac{1}{5}R^5 \cos^2\phi \sin\phi$

Middle Integral = $\int_{\phi=0}^{\phi=\pi/2} \frac{1}{5}R^5 \cos^2\phi \sin\phi \; d\phi = \left[-\frac{1}{15}R^5 \cos^3\phi\right]_{\phi=0}^{\phi=\pi/2} = \frac{1}{15}R^5$

Outer Integral = $\int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{15}R^5\right) d\theta = \boxed{\frac{2\pi}{15}R^5}$

Example 6: Use spherical coordinates to show that the volume of a spherical ball B of radius R is $\frac{4}{3}\pi R^3$.

Solution: $\text{Volume}(B) = \iiint_B dV = \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi}\int_{\rho=0}^{\rho=R} \rho^2 \sin\phi \, d\rho d\phi d\theta$.

Inner Integral = $\int_{\rho=0}^{\rho=R} \rho^2 \sin\phi \, d\rho = \left[\frac{1}{3}\rho^3\right]_0^R \sin\phi = \frac{1}{3}R^3 \sin\phi$

Middle Integral = $\int_{\phi=0}^{\phi=\pi} \frac{1}{3}R^3 \sin\phi \, d\phi = \frac{1}{3}R^3 \left[-\cos\phi\right]_{\phi=0}^{\phi=\pi} = \frac{2}{3}R^3$

Outer Integral = $\int_{\theta=0}^{\theta=2\pi} \frac{2}{3}R^3 d\theta = \frac{2}{3}R^3 \cdot 2\pi = \boxed{\frac{4}{3}\pi R^3}$

Example 7: Find the moment of inertia of a right circular cone B of constant density and total mass M with base radius R and height H around its central axis.

Solution: For convenience, situate the cone symmetrically around the positive z-axis with its vertex at the origin. This allows us to compute the moment of inertia as $I = I_z = \iiint_B r^2 dm$. If the constant density is σ, then $dm = \sigma dV$, so $I = \iiint_B r^2 \sigma dV$.

In cylindrical coordinates, this becomes: $\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R}\int_{z=\frac{H}{R}r}^{z=H} r^2 \sigma \, r dz dr d\theta = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R}\int_{z=\frac{H}{R}r}^{z=H} r^3 \sigma \, dz dr d\theta$

Inner Integral = $\int_{z=\frac{H}{R}r}^{z=H} r^3 \sigma \, dz = \sigma r^3 \left(H - \frac{H}{R}r\right) = \sigma H \left(r^3 - \frac{1}{R}r^4\right)$

Middle Integral = $\int_{r=0}^{r=R} \sigma H \left(r^3 - \frac{1}{R}r^4\right) dr = \sigma H \left[\frac{r^4}{4} - \frac{r^5}{5R}\right]_{r=0}^{r=R} = \frac{1}{20}\sigma H R^4$

Outer Integral = $\int_{\theta=0}^{\theta=2\pi} \frac{1}{20}\sigma H R^4 d\theta = \frac{\pi}{10}\sigma H R^4$

The total mass is $M = \sigma \cdot (\text{Volume}) = \sigma \cdot \frac{1}{3}\pi R^2 H = \frac{\pi}{3}\sigma R^2 H$, so we can write $\sigma = \dfrac{3M}{\pi R^2 H}$. If we substitute this into the above expression for the moment of inertia, we get $I = \frac{\pi}{10}\sigma H R^4 = \dfrac{\frac{\pi}{10} H R^4 \cdot 3M}{\pi R^2 H} = \boxed{\frac{3}{10}MR^2}$

Example 8: Newton's Law of Gravitation states that the attractive force between two bodies with masses M and m separated by a distance R is $F = \dfrac{GMm}{R^2}$ where $G = 6.67408 \times 10^{-11} \text{ meter}^3/\text{kg} \cdot \text{sec}^2$ is the universal gravitational constant. Show that for a uniform spherical ball with constant density and total mass m, the force exerted by this ball on a test mass M located on the surface of the ball is the same as if the entire mass m was concentrated at the center of the ball.

Solution: This problem can be solved in various ways, but let's do it as an exercise in the use of spherical coordinates. First, we have to recognize that this force is a ***vector***, so it will be helpful to set up this problem in such a way that it's relatively simple to determine the components of this vector force. Two ways are especially attractive: (a) locate the center of the ball at the origin and put the test mass on the positive z-axis, or (b) put the test mass at the origin and locate to ball above it symmetrically around the positive z-axis.

If we choose (b), this will work especially well in setting up the necessary integral in spherical coordinates. Since the ball B is presumed to have constant density σ and is symmetric about the z-axis, it should be clear that the gravitational force will be directed in the positive z-direction, i.e. the x and y components of the force will be zero. If, in the spirit of Riemann Sums, we imagine the ball to be chopped up into very small pieces of mass Δm at a distance ρ from the origin, then the force that this small piece will exert on the test mass M will be a vector with magnitude $\Delta F = \dfrac{GM\Delta m}{\rho^2}$, but we're only interested in the z-component of this vector and must therefore measure only its scalar projection onto the positive z-axis, i.e. $\Delta F_z = \dfrac{GM\Delta m}{\rho^2}\cos\phi$. If we sum up these contributions over the entire ball and pass to the limit as the mesh of the partition tends to zero, the resulting integral will be $F_z = \iiint_B \dfrac{GM\cos\phi}{\rho^2}\,dm$.

The sphere bounding the solid ball has Cartesian equation $x^2 + y^2 + (z-R)^2 = R^2$ which can be rewritten as $x^2 + y^2 + z^2 = 2Rz$ which can then be written in spherical coordinates as $\rho^2 = 2R\rho\cos\phi$ or $\rho = 2R\cos\phi$. Using $dm = \sigma dV = \sigma\rho^2 \sin\phi\, d\phi d\phi d\theta$, the integral in spherical coordinates becomes:

$$\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2}\int_{\rho=0}^{\rho=2R\cos\phi} \frac{GM\cos\phi}{\rho^2}\sigma\rho^2\sin\phi\, d\rho d\phi d\theta = GM\sigma\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2}\int_{\rho=0}^{\rho=2R\cos\phi}\cos\phi\sin\phi\, d\rho d\phi d\theta$$

Inner Integral = $\int_{\rho=0}^{\rho=2R\cos\phi}\cos\phi\sin\phi\, d\rho = 2R\cos^2\phi\sin\phi$

Middle Integral = $\int_{\phi=0}^{\phi=\pi/2} 2R\cos^2\phi\sin\phi d\phi = \frac{2}{3}R\left[-\cos^3\phi\right]_0^{\pi/2} = \frac{2}{3}R$

Outer Integral = $GM\sigma\int_{\theta=0}^{\theta=2\pi}\frac{2}{3}Rd\theta = \frac{4}{3}\pi RGM\sigma$

Now the density is just $\sigma = \dfrac{Mass(B)}{Vol(B)} = \dfrac{m}{\frac{4}{3}\pi R^3} = \dfrac{3m}{4\pi R^3}$, so the gravitational force becomes:

$F = F_z = \dfrac{4\pi RGM}{3}\dfrac{3m}{4\pi R^3} = \dfrac{GMm}{R^2}$, i.e. the same as we'd get for two point masses at a distance R.

In the next lecture, we'll discuss changing coordinates more generally in multiple integrals. We'll also discuss the idea of integration along a curve and the application of this idea in conjunction with vector fields to define the work done by a variable force along a specified curve (line integrals). This, in turn, will lead to a discussion of conservative vector fields vs. non-conservative vector fields, potential functions, and the Fundamental Theorem of Line Integrals.

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