Multivariable Calculus – Lecture #1 Notes

This introductory lecture will focus on basic ideas about coordinates and coordinate transformations, vectors and vector algebra, and some simple equations for lines, circles, spheres, and more.

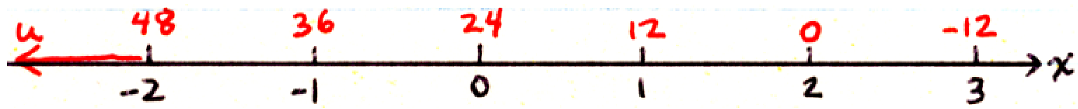
Coordinates – $\mathbf{R}^1$, $\mathbf{R}^2$, $\mathbf{R}^3$,…, $\mathbf{R}^n$.

Though everyone may be familiar with the use of coordinates, it's worth noting the primary purpose of coordinates and some of the choices made in the use of coordinates. The most basic purpose of coordinates is simply to distinguish different points, i.e. two points are distinct if they have different coordinates.

$\mathbf{R}^1$ – "the Real Line"

There are many perspectives on this very basic example, but a uniform coordinate system requires basically the choice of a distinguished point (the origin), a scale, and an orientation. These are the essential elements when we "draw the x-axis":

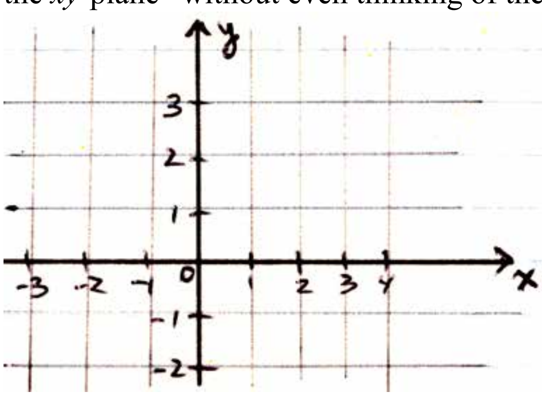
Something not often emphasized when first learning the use of coordinates is the arbitrariness of it all. People will often just "draw the x-axis" as if it was a done deal. It's not. Imagine choosing a different origin (perhaps corresponding to where $x = 0$ is labeled above) and, imagining that the original scale was measured in units of feet, let's now use inches. Let's also reverse the orientation so that we're now counting from right to left. This gives us a competing coordinate (labeled u in the following diagram):



Either coordinate choice will do fine for the purpose of distinguishing points. The only important thing then is to be able to relate these coordinate choices. This can be done in several ways. For example, shifting the origin to the right is accomplished by the substitution $x \mapsto (x-2)$. Changing from feet to inches requires scaling by a factor 12 and reversing the orientation is a simple sign change. Thus we have $u = -12(x-2)$ to change from x to u, and we can solve for $x = -\frac{u}{12} + 2 = -\left(\frac{u-24}{12}\right)$ in order to go back and forth between our two coordinate choices. This is the notion of a **coordinate transformation** which (ideally) should give a one-to-one correspondence between respective coordinates, i.e. it should be invertible.

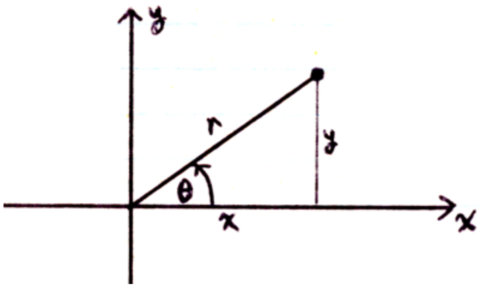
$\mathbf{R}^2$ – "the xy-plane"

When there are two degrees of freedom, a single coordinate is inadequate for distinguishing points. In creating "the xy-plane" we again need to make some arbitrary choices, i.e. a distinguished point (the origin), a choice of axes (perpendicular axes have some advantages, but this is not necessary), scales (there are advantages to using the same scale for both axes), and an orientation for the axes. As in the above illustration, we often just "draw the xy-plane" without even thinking of the arbitrary choices involved.



We could easily have chosen a different origin and different axes with a different scale giving (u,v) coordinates instead of the usual (x,y) coordinates shown. As long as the chosen coordinates can distinguish distinct points, it will be a good choice. As in the previous illustration we would be obliged to determine the **coordinate transformation** relating these choices, i.e. would have to be able to solve for $\begin{Bmatrix} u = u(x,y) \\ v = v(x,y) \end{Bmatrix}$ and its inverse $\begin{Bmatrix} x = x(u,v) \\ y = y(u,v) \end{Bmatrix}$.

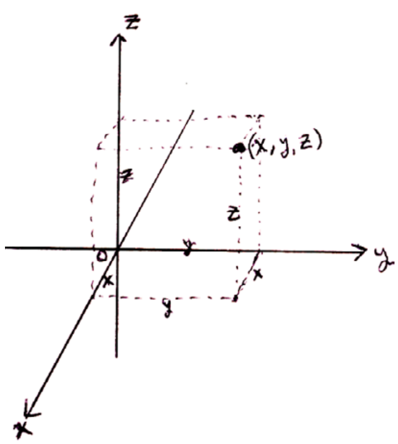
There are other options for coordinates besides those associated with axes (perpendicular or otherwise. One such choice with which you may be familiar is the choice of *polar coordinates*. If we agree to use the same origin and measure distance from that origin as the coordinate r and measure the angle counting counter-clockwise from the positive x-axis as the coordinate θ (you will, of course, have to decide whether to use degrees, radians or some other unit), we can graphically relate the (x, y) and (r, θ) coordinates as shown in the following diagram:



Thanks to the choice of perpendicular axes, the Pythagorean Theorem, and basic trigonometry, we can easily express the coordinate transformation as $\begin{Bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{Bmatrix}$ and $\begin{Bmatrix} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{Bmatrix}$. There is some ambiguity in this, but at least for points away from the origin and if we decide on an appropriate range for the coordinate θ, it does the job of relating the coordinates.

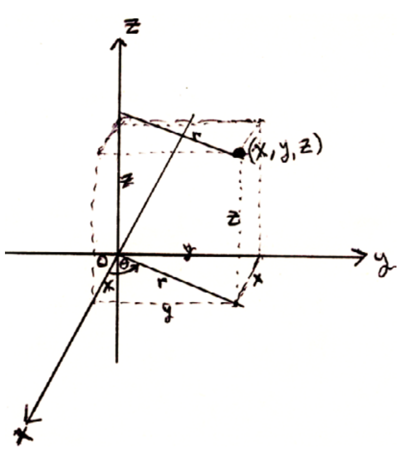
<u>$\mathbf{R}^3$ – "*xyz*-space"</u>

If we have to work in a context where there are three degrees of freedom, we need three distinct coordinates in order to adequately distinguish points using a coordinate system. Representing such choices on two-dimensional paper (or a chalkboard) presents some challenges, but learning to do so will be very helpful in developing a deeper understanding of multivariable calculus and related topics. As in the previous illustrations, there are arbitrary choices to be made. If we choose a distinguished point (the origin) and conveniently choose perpendicular axes and a common scale, we still have to decide whether we want to use x and y for the horizontal axes and z for the vertical axis or use some other perfectly acceptable scheme. We also have to make some decisions about orientation. The most common convention for these ***Cartesian coordinates*** is illustrated as follows:



The illustration suggests that the x and y coordinate axes are horizontal (*how far out* and *how far over*) and the z-axis is vertical (*how far up*). The use of dotted lines parallel to the coordinate axes is useful for establishing some perspective on the location of any given point. The ordering of the coordinates (x, y, z) is also oriented in a right-hand manner in the sense that you can form the positive x, y, and z directions with your right hand using, respectively, your thumb, index finger, and middle finger. Another convention popular with physicists is to say that if you point all of the fingers of your right hand in the direction of the positive x-axis, then curl your fingers toward the positive y-axis, then your thumb will be pointing in the direction of the positive z-axis.

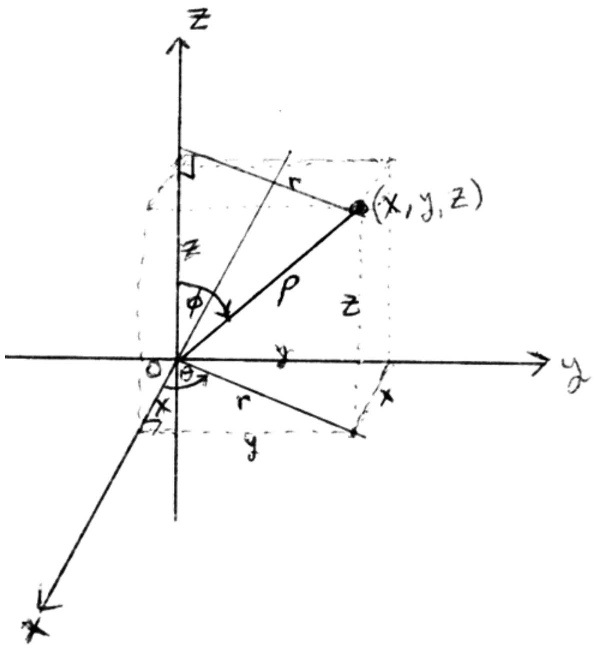
An alternative coordinate system for $\mathbf{R}^3$ can be created by using polar coordinates in the horizontal xy-plane and retaining the z-coordinate for vertical measurement. This choice is known as cylindrical polar coordinates or, more simply, ***cylindrical coordinates***.



Borrowing from the previous illustration, this yields the coordinate transformation $\begin{Bmatrix} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{Bmatrix}$. The third relation may seem superfluous, but it captures the essential fact that we're using the same z-coordinate for both systems and enables us to properly express $\begin{Bmatrix} x = x(r,\theta,z) \\ y = y(r,\theta,z) \\ z = z(r,\theta,z) \end{Bmatrix}$ even though there may not be an explicit dependence on each of the respective variables.

It's helpful now (and will be even more helpful later when we talk about integration) to think about the nature of the constant surfaces for these various coordinates. For example, in the Cartesian (x, y, z) coordinates, an "$x = \text{constant}$" surface is a vertical plane. A "$y = \text{constant}$" surface is also a vertical plane perpendicular to any "$x = \text{constant}$" surface. A "$z = \text{constant}$" surface is a horizontal plane. If you think of these constant surfaces like cutting blades, they can be used to carve up $\mathbf{R}^3$ into rectangular blocks. This will be very useful later in the course when we talk about partitions and integration.

If we instead consider the constant surfaces for cylindrical coordinates we see that an "$r = \text{constant}$" surface is actually a vertical cylinder, a "$\theta = \text{constant}$" surface is a vertical half-plane extending infinitely outward from the *z*-axis (which can be thought of as a "page"), and a "$z = \text{constant}$" surface is still a horizontal plane. When we do integration, these surfaces will enable us to carve up $\mathbf{R}^3$ into small cells lying between upright cylinders, between "pages", and between horizontal planes.



Another important coordinate system for $\mathbf{R}^3$ is ***spherical coordinates***. We now consider the distance between the origin and a given point and denote this by the Greek letter ρ (the spherical radius). This is not the same as the cylindrical radius *r* (which can be understood as the distance from the *z*-axis to a given point). We also measure the angle between the positive z-axis and a ray from the origin to a given point and denote this angle by the Greek letter ϕ. This may be interpreted as the declination, but it's related to the familiar latitude except that the equatorial plane (our *xy*-plane) at 0° latitude corresponds to $\phi = 90°$ or $\pi/2$ radians. We also reuse the angle θ from cylindrical coordinates and call it the *azimuth* angle. The combination (ρ, ϕ, θ) constitutes the spherical coordinates.

Note that the constant surfaces for spherical coordinates are that a "$\rho = \text{constant}$" surface now a sphere centered about the origin, a "$\phi = \text{constant}$" surface is a half-cone, and a "$\theta = \text{constant}$" surface is still a vertical half-plane (page) extending infinitely outward from the *z*-axis. When we do integration, these surfaces will enable us to carve up $\mathbf{R}^3$ into small cells lying between spheres, between cones, and between "pages".

The coordinate transformations between Cartesian coordinates and spherical coordinates can be accomplished with some basic trigonometry. From before we still have that $\begin{Bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{Bmatrix}$ and we can also see from the diagram that $\begin{Bmatrix} r = \rho\sin\phi \\ z = \rho\cos\phi \end{Bmatrix}$. Substitution then gives that $\begin{Bmatrix} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{Bmatrix}$. You can also solve for the inverse coordinate transformation. Note, in particular, that $\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$, so $\rho = \sqrt{x^2 + y^2 + z^2}$.

<u>$\mathbf{R}^n$ – "Euclidean *n*-space"</u>

We can analogously construct Cartesian coordinates when there are n degrees of freedom – though visualization will clearly be a challenge. By analogy, we could introduce coordinates $(x_1, x_2, \cdots, x_n)$ and even apply the Pythagorean Theorem repeatedly to define the distance *D* from the origin $(0, 0, \cdots, 0)$ to $(x_1, x_2, \cdots, x_n)$ by $D^2 = {x_1}^2 + {x_2}^2 + \cdots + {x_n}^2$ or $D = \sqrt{{x_1}^2 + {x_2}^2 + \cdots + {x_n}^2}$. We could also introduce a whole family of angles between the respective positive axes and a ray from the origin to a given point to devise something analogous to spherical coordinates, but we'll save those details until such time as might actually need them.

Points vs. Vectors

Coordinates and coordinate systems are excellent tools for labelling points, but they don't naturally allow us to do much in the way of algebra. Labels like "Boston" and "Chicago" enable us to distinguish distinct cities, but we would never think of defining anything like "Boston + Chicago" or "3 times Boston". For these kinds of algebraic needs we need to go beyond coordinate labels. For this we'll introduce vectors and vector algebra. There are many ways to formulate this, but we'll choose an approach that's especially useful for multivariable calculus.

Algebraic definition of vectors in $\mathbf{R}^n$: A vector in $\mathbf{R}^n$ is an "ordered n-tuple" expressed as $\mathbf{x} = \langle x_1, x_2, \cdots, x_n \rangle$. We also sometimes use the notation $\vec{x}$. Vectors do not exist in isolation. They yearn to be scaled and added. To this end, we define the **vector sum** by $\mathbf{x}+\mathbf{y} = \langle x_1, x_2, \cdots, x_n \rangle + \langle y_1, y_2, \cdots, y_n \rangle = \langle x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n \rangle$, i.e. you add like vectors by adding their respective **components**. We can also **scale a vector** by any scalar t by defining this to be $t\mathbf{x} = t\langle x_1, x_2, \cdots, x_n \rangle = \langle tx_1, tx_2, \cdots, tx_n \rangle$. That is, we scale a vector by uniformly scaling each of its components by that same scalar.

Examples: In $\mathbf{R}^2$, we have that $2\langle 3,2\rangle - 3\langle 1,5\rangle = \langle 6,4\rangle - \langle 3,15\rangle = \langle 3,-11\rangle$.

In $\mathbf{R}^3$, we have that $2\langle 3,2,1\rangle - 3\langle 1,5,-2\rangle = \langle 6,4,2\rangle - \langle 3,15,-6\rangle = \langle 3,-11,8\rangle$.

We define the **zero vector** in $\mathbf{R}^n$ to be the vector in $\mathbf{R}^n$ with all components equal to 0, i.e. $\mathbf{0} = \langle 0,0,\cdots,0\rangle$. We also denote $(-1)\mathbf{u} = -\mathbf{u}$.

It's easy to show from the above definitions the following **Rules for Vector Algebra**:

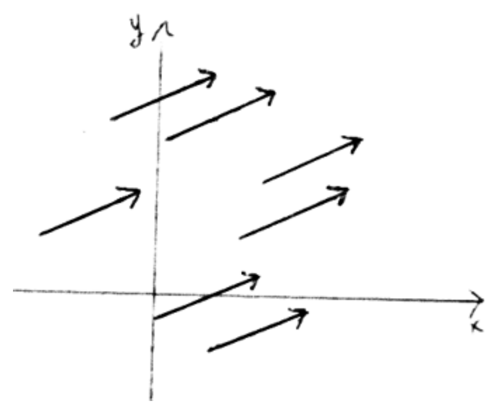
If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors in $\mathbf{R}^n$ and s and t are scalars, then:

$$\mathbf{v}+\mathbf{u} = \mathbf{u}+\mathbf{v} \qquad \mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w} \qquad t(\mathbf{u}+\mathbf{v}) = t\mathbf{u}+t\mathbf{v} \qquad (s+t)\mathbf{u} = s\mathbf{u}+t\mathbf{u}$$

$$\mathbf{u}+\mathbf{0} = \mathbf{u} \qquad \mathbf{u}+(-\mathbf{u}) = \mathbf{0} \qquad s(t\mathbf{u}) = (st)\mathbf{u} \qquad 1\mathbf{u} = \mathbf{u}$$

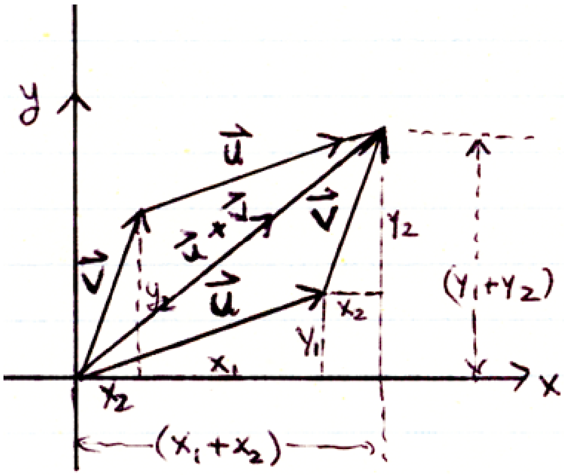
All of these basic rules of vector algebra derive from and are analogous to familiar rules for working with real numbers (commutativity, associativity, distributive laws, etc.).

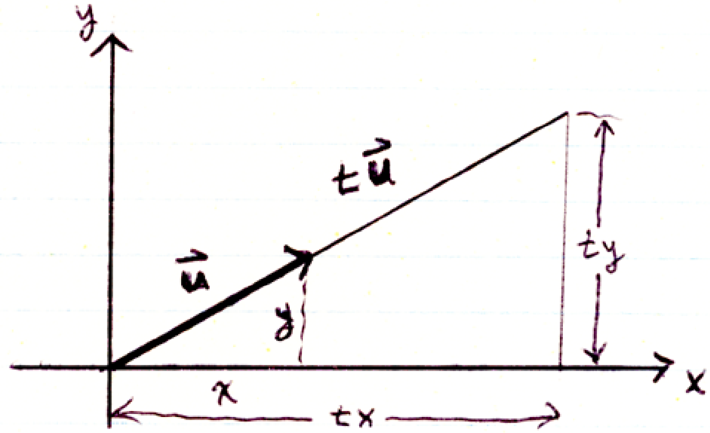
Geometric definition of vectors in $\mathbf{R}^n$:



The key idea in bridging the algebra of vectors and the coordinates of points is the consideration of **directed line segments**. For example, in $\mathbf{R}^2$, if we draw a directed line segment from the point $(1,3)$ to the point $(3,4)$, the incremental changes in the respective coordinates are 2 and 1 which we might represent as $\langle 2,1\rangle$. These same increments would be realized a directed line segments from $(4,6)$ to $(6,7)$ or from $(0,2)$ to $(2,3)$. In each case we would move horizontally 2 units and upward 1 unit. As a "movement", each of these are fundamentally the same even though the starting points differ. If we simply declare two directed line segments to be ***equivalent*** if they have the same increments, then we can think of an entire class of such directed line segments to be the same vector. Using this approach, we then think of a vector (at least for now) as an ***equivalence class of directed line segments***, and we define vector addition and scalar multiplication as before.

One of the great advantages of this approach is that allows us to translate vectors freely to any convenient starting point and have an equivalent representative of this vector. This allows us to realize simple ways to geometrically understand both vector addition and scalar multiplication. Consider the following illustrations in $\mathbf{R}^2$ which work the same way in any dimension:





The left diagram illustrations why the algebraic definition for vector addition corresponds to **"tip-to-tail" addition** of the respective vectors when thought of as (equivalence classes of) directed line segments. This is also known as the **Parallelogram Rule** because this **resultant vector** is realized by following consecutive edges of the parallelogram whose parallel sides are determined by the respective vectors.

The right diagram illustrates how scalar multiplication can easily be understood in terms of similar triangles. Indeed, the diagram also suggests the following definition.

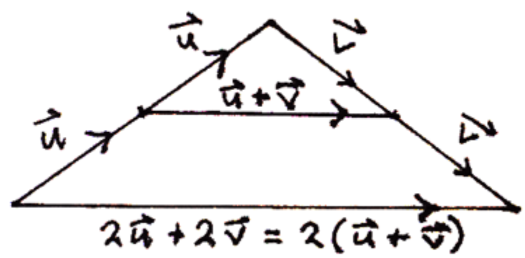
Definition: Two vectors **u** and **v** are called parallel if $\mathbf{v} = t\mathbf{u}$ for some (nonzero) scalar *t*.

Note that two vectors will still be called parallel even if they are *oppositely* directed, i.e. one is obtained from the other by scaling by a negative number.

Coordinate-free vector proofs

One especially interesting application of our formulation of vectors is in proving many facts about geometry that can otherwise be cumbersome when done just using coordinates, lengths, etc. The freedom to move vectors while remaining "the same" vector can be very powerful.

Example: Show that for any triangle the line connecting the midpoints of two adjacent sides of the triangle is parallel to the third side and has half the length of the third side.



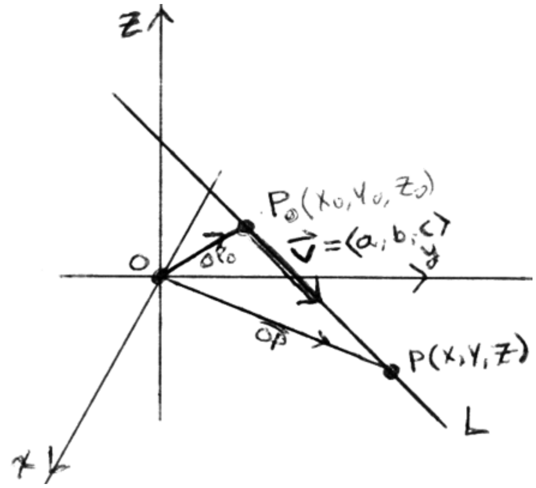
Solution: Consider the diagram shown (right). If we denote the respective vectors from vertex to midpoint (and midpoint to the next vertex) as vectors **u** and **v** as shown, we are making liberal use of the ability to translate vectors to different locations while treating them as the same vector. The Parallelogram Rule for vector addition then enables us to identify the directed line segment connecting the midpoints as the vector $\mathbf{u}+\mathbf{v}$. Adjacent sides of the full triangle are determined by the vectors $2\mathbf{u}$ and $2\mathbf{v}$ by virtue of the fact that **u** and **v** constitute only half the sides. The third side of the full triangle is then (again using the Parallelogram Rule) identified with the vector $2\mathbf{u}+2\mathbf{v} = 2(\mathbf{u}+\mathbf{v})$. This gives us both of the desired results since this vector is clearly a multiple of $\mathbf{u}+\mathbf{v}$ (hence is parallel) and is twice the length.

Note: In practice, when doing these kinds of vector proofs it is usually best to introduce as few independent vectors as possible and to then express all other vectors in terms of these chosen vectors. In the above example, we really needed only the two vectors to label every desired part of the triangle.

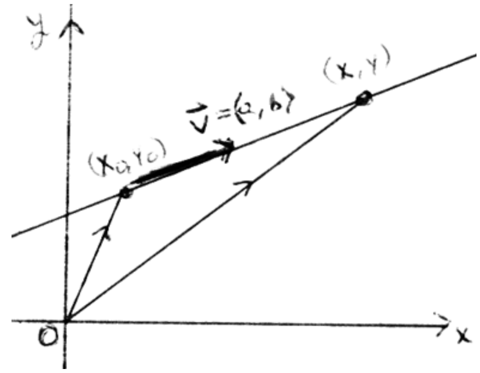
Difference vector

Perhaps the most common way we derive vectors from coordinates is as a **difference vector**, i.e. we form a vector by finding the specific directed line segment from one point to another. In $\mathbf{R}^2$, this translates into the fact that the difference vector from $P(x_1, y_1)$ to $Q(x_2, y_2)$ is the vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$. In particular, we determine the position vector of a given point by construction a vector from the origin to that point. For example, in $\mathbf{R}^3$, the position vector of the point $R(x, y, z)$ is the vector $\overrightarrow{OR} = \langle x, y, z \rangle$. Note that the ***components of the position vector*** correspond exactly to the ***coordinates of the given point***. This can often cause people to confuse points and vectors, but this ambiguity can also be used to some advantage.

Vector and Parametric Equations of a Line in $\mathbf{R}^n$



Though everyone may have some familiarity with equations for lines in the *xy*-plane, the use of vector algebra makes it easy to describe lines generally. Referring to the illustration in $\mathbf{R}^3$, note that if any single point $P_0(x_0, y_0, z_0)$ on a line L is specified and if a direction vector $\mathbf{v} = \langle a, b, c \rangle$ is also specified for this line (a vector parallel to the line), then we see that for any other point $P(x, y, z)$ on the line, we have $\boxed{\overrightarrow{OP} = \overrightarrow{OP_0} + t\mathbf{v}}$ for some scalar (parameter) *t*. This is known as the **vector form of a line**. If we translate this into components, this reads $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$ which can also be expressed as $\langle x, y, z \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$. This can also be expressed as separate (parametric) equations, i.e. $\left\{\begin{array}{l} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{array}\right\}_{t \in \mathbf{R}}$.



This construction can be done in any dimension, including $\mathbf{R}^2$. In this case, the parametric equations become simply $\left\{\begin{array}{l} x = x_0 + at \\ y = y_0 + bt \end{array}\right\}_{t \in \mathbf{R}}$. Note that with direction vector $\mathbf{v} = \langle a, b \rangle$, the slope of the line would be $m = \frac{b}{a}$ (if $a \neq 0$) and we can solve for the common parameter to get $t = \boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b}}$. This is known as the symmetric equation for the line (assuming neither *a* nor *b* is equal to zero). We can take this one step further and solve for $y - y_0 = \frac{b}{a}(x - x_0)$ or, equivalently, $\boxed{y - y_0 = m(x - x_0)}$ better known as the point-slope form of a line.

Though we see that we can recover familiar expressions for the equation of a line in $\mathbf{R}^2$, it's important to note that those expressions do not easily generalize to high dimensions, but we can always describe a line parametrically.

Example: Find parametric equations for the line passing through the points $P(1, -2, 3)$ and $Q(4, 1, -3)$.

Solution: We start by noting that a direction vector for this line is given by the difference vector $\overrightarrow{PQ} = \langle 4 - 1, 1 - (-2), -3 - 3 \rangle = \langle 3, 3, -6 \rangle = 3\langle 1, 1, -2 \rangle$. Any parallel vector will suffice, so for simplicity choose the direction vector $\mathbf{v} = \langle 1, 1, -2 \rangle$. We can use either of the given points as our reference point on the line, so let's use $P(1, -2, 3)$. The vector equation for this line will then be $\langle x, y, z \rangle = \langle 1, -2, 3 \rangle + t\langle 1, 1, -2 \rangle$, and this in turn gives the parametric equations $\left\{\begin{array}{l} x = 1 + t \\ y = -2 + t \\ z = 3 - 2t \end{array}\right\}_{t \in \mathbf{R}}$.

Note that $t = 0$ corresponds to the point *P*, and $t = 1$ corresponds to the point *Q*.

Norm (or magnitude) of a vector
Using the Pythagorean Theorem, we can define the norm (or magnitude) of a vector by measuring it as a directed line segment. For a vector in $\mathbf{R}^2$, this translates into $\|\mathbf{v}\| = \|\langle v_1, v_2\rangle\| = \sqrt{v_1^{\,2} + v_2^{\,2}}$. In $\mathbf{R}^2$, this translates into $\|\mathbf{v}\| = \|\langle v_1, v_2, v_3\rangle\| = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}}$, and in $\mathbf{R}^n$, we would have $\|\mathbf{v}\| = \|\langle v_1, v_2, \cdots, v_n\rangle\| = \sqrt{v_1^{\,2} + v_2^{\,2} + \cdots + v_n^{\,2}}$.
One reason for using the term "norm" or "magnitude" of a vector is because vectors are also used in physics and elsewhere for velocity, force, angular momentum, and other vector quantities. It would be most awkward to refer to "the length of the force", so saying "the magnitude of the force" is preferred. Interpreting the magnitude of a vector as a length is really only appropriate in the context of vectors as (equivalence classes of) directed line segments or, more specifically, for a difference vector between points.

Distance between points *P* and *Q*
Note that the distance between two points coincides with the norm of the difference vector between those points. In $\mathbf{R}^2$, the distance from $P(x_1, y_1)$ to $Q(x_2, y_2)$ is $\|\overrightarrow{PQ}\| = \|\langle x_2 - x_1, y_2 - y_1\rangle\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. In $\mathbf{R}^3$, the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is $\|\overrightarrow{PQ}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Definition: A **unit vector** is a vector with magnitude equal to 1.

Unit vectors are often used to specify directions with a standard magnitude of 1. For example, in $\mathbf{R}^2$, we have the unit vectors in the directions of the positive *x* and *y* axes given by $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$. This gives an alternate way of expressing a vector as $\mathbf{v} = \langle x, y\rangle = \langle x, 0\rangle + \langle 0, y\rangle = x\langle 1, 0\rangle + y\langle 0, 1\rangle = x\mathbf{i} + y\mathbf{j}$. We can do the analogous thing in $\mathbf{R}^3$ using the vectors $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$ (note that these are <u>not</u> the same **i** and **j** vectors as in the $\mathbf{R}^2$ case). We will then have $\mathbf{v} = \langle x, y, z\rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ as an alternate notation.

Given any vector **v**, we can find a parallel unit vector by **normalizing the vector**, i.e. scaling the given vector by the reciprocal of its magnitude: $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$.

Example: Find the distance between the points $P(1, -2, 3)$ and $Q(4, 1, -3)$ and a unit vector **u** in the direction connecting these points.

Solution: We start by finding the difference vector $\overrightarrow{PQ} = \langle 4-1, 1-(-2), -3-3\rangle = \langle 3, 3, -6\rangle = 3\langle 1, 1, -2\rangle$. The distance is then $\|\overrightarrow{PQ}\| = 3\sqrt{1^2 + 1^2 + (-2)^2} = 3\sqrt{6}$.

The desired unit vector is then $\mathbf{u} = \dfrac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \dfrac{3\langle 1, 1, -2\rangle}{3\sqrt{6}} = \dfrac{\langle 1, 1, -2\rangle}{\sqrt{6}} = \frac{1}{\sqrt{6}}\langle 1, 1, -2\rangle$.

<u>**Equations for Circles and Spheres**</u>
A **circle** (in $\mathbf{R}^2$) consists of the set of points equidistant from a given point. This is easily described using vectors. For a circle centered at the origin (designated as *O*), the set of all such points $P(x, y)$ can be characterized by the fact that $\|\overrightarrow{OP}\| = R$ where *R* is the (constant) radius of the circle. This gives $\sqrt{x^2 + y^2} = R$ or, more simply, $\boxed{x^2 + y^2 = R^2}$. In the case where the circle is centered on a different point $C(x_0, y_0)$, i.e. a translated circle, this becomes $\|\overrightarrow{CP}\| = R$ or $\sqrt{(x - x_0)^2 + (y - y_0)^2} = R$ or $\boxed{(x - x_0)^2 + (y - y_0)^2 = R^2}$.

In $\mathbf{R}^3$, a **sphere** consists of the set of all points equidistant from a given point. If the sphere is centered at the origin (designated as *O*), the set of all such points $P(x, y, z)$ can again be characterized by the fact that

$\|\overrightarrow{OP}\| = R$ where R is the (constant) radius of the sphere. This gives $\sqrt{x^2 + y^2 + z^2} = R$ or, more simply, $\boxed{x^2 + y^2 + z^2 = R^2}$. In the case where the circle is centered on a different point $C(x_0, y_0, z_0)$, i.e. a translated sphere, this becomes $\|\overrightarrow{CP}\| = R$ or $\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = R$ or $\boxed{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = R^2}$.

Example: Find an equation of the sphere of radius 2 centered around the point $(1,2,-3)$.

Solution: Using the above formulation we quickly get the equation $(x-1)^2 + (y-2)^2 + (z+3)^2 = 4$.
If we were inclined to multiply this out, we'd get $x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 + 6z + 9 = 4$ which can be simplified to $x^2 + y^2 + z^2 - 2x - 4y + 6z + 10 = 0$.

Now suppose that we had instead been given this equation and were asked to identify it. In that case, we can proceed by "**completing the square**." That is, we start by expressing the equation as $(x^2 - 2x + \_) + (y^2 - 4y + \_) + (z^2 + 6z + \_) = -10$ and then add (to both side of the equation, of course) whatever is required to produce complete squares.

In this case we get $(x^2 - 2x + 1) + (y^2 - 4y + 4) + (z^2 + 6z + 9) = -10 + 1 + 4 + 9 = 4$ or, more simply, $(x-1)^2 + (y-2)^2 + (z+3)^2 = 4$. At this point we can simply recall the standard form of a translated sphere and note that its center is at $(1,2,-3)$ and its radius is 2.

<u>Missing Variables</u>
There is something inherently ambiguous about expressing algebraically-defined objects just by an equation or equations. For example, the equation $x = 2$ represents a single point in $\mathbf{R}^1$ (when considered as the x-axis), a vertical line in $\mathbf{R}^2$ (when considered as the xy-plane), and a vertical plane in $\mathbf{R}^3$ (when considered using the standard (x, y, z) Cartesian coordinates). It is always necessary to clarify the context in which an equation is stated. For example, in the latter case we might better define the plane as $\{(x, y, z) : x = 2\}$ to avoid ambiguity.

Similarly, the equation $x^2 + y^2 = 4$ represents a circle in the xy-plane but it also represents a vertical cylinder in $\mathbf{R}^3$. One useful way to think about this is to note that if any coordinate does not appear in the equation that defines an object, then "you are free to roam" in that variable, i.e. it is <u>unconstrained</u>. Thus the fact that there is no mention of the z-variable in the equation $x^2 + y^2 = 4$ means that as a subset of $\mathbf{R}^3$ the cylinder extends vertically infinitely in both directions. If we wish to define a circle in $\mathbf{R}^3$ this requires *two* constraints. For example, the horizontal circle of radius 2 centered around the z-axis at the level where $z = 3$ may be expressed as $\{(x, y, z):\ x^2 + y^2 = 4,\ z = 3\}$.

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