Differential Equations Supplement – Lecture #5a

Definition: A differential equation of the form $\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t) = q(t)$, where $p_{n-1}(t),\ldots, p_1(t), p_0(t), q(t)$ are functions of the independent variable *t*, is called an ***n*th order linear ordinary differential equation**. In the case where $q(t)=0$ for all *t*, we call the equation **homogeneous**. Otherwise we call it **inhomogeneous**.

Linearity

In the context of functions of one variable, linearity is an often abused word. In fact, a function of the form $f(x) = mx + b$ is NOT a linear function. It is more appropriately called a 1st order *affine* function. Linearity is a property most simply characterized by the fact that linear functions preserve scaling and adding. The linear functions of one variable consist only of those of the form $f(x) = mx$. Note that $f(ax) = m(ax) = a(mx) = af(x)$, i.e. it preserves scaling, and $f(x+y) = m(x+y) = mx + my = f(x) + f(y)$, i.e. it preserves addition.

Definition: Formally we say that a function is **linear** if for all inputs x_1, x_2 and constants c_1, c_2 we must have $f(c_1x_1 + c_2x_2) = c_1 f(x_1) + c_2 f(x_2)$.

In the case of functions $T : \mathbf{R}^n \to \mathbf{R}^m$, linearity means that the scaling of vectors and the addition of vectors is preserved via a linear transformation. All such transformations are of the form $T(\mathbf{x}) = \mathbf{Ax}$ where **A** is an $m \times n$ matrix with constant entries. Linearity then translates into the matrix algebra facts that $\mathbf{A}(k\mathbf{x}) = k(\mathbf{Ax})$ and $\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{Ax} + \mathbf{Ay}$, or (combined) $\mathbf{A}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{Ax} + \beta\mathbf{Ay}$ for all scalars α, β and all vectors $\mathbf{x}, \mathbf{y}$.

Our current situation involves working with functions in the same way that we looked at vectors in $\mathbf{R}^n$. Just as we can scale and add vectors, we can also scale and add functions. A transformation that acts on functions in a manner analogous to the way matrices act on vectors is known as a **linear (differential) operator**. The basic examples are differentiation and multiplication by a fixed function. We can then compose these basic operators and add them to form more complicated operators.

There are many spaces of functions in which we can seek solutions to differential equations. Perhaps the most common such space is the space of functions that are differentiable to all orders, i.e. $C^\infty(\mathbf{R})$.

Multiplication by a fixed function is a linear operator.

Suppose we have a fixed function $p(t)$ and we define a transformation of functions by $\left[T(f)\right](t) = p(t)f(t)$. We can easily see that for any constant *c*, $\left[T(cf)\right](t) = p(t)cf(t) = cp(t)f(t) = c\left[T(f)\right](t)$, so $T(cf) = cT(f)$, i.e. *T* preserves scaling. Similarly, if f_1 and f_2 are two functions, then $\left[T(f_1+f_2)\right](t) = p(t)(f_1+f_2)(t) = p(t)(f_1(t)+f_2(t)) = p(t)f_1(t) + p(t)f_2(t) = \left[T(f_1)\right](t) + \left[T(f_2)\right](t)$.
This is really just the distributive law, but the result is that formally $T(f_1+f_2) = T(f_1) + T(f_2)$, i.e. *T* preserves addition of functions. Together, this shows that *T* is a linear operator.

Differentiation of functions is a linear operator.

Let *D* be the transformation defined by $D(f) = f'$, i.e. $D = \frac{d}{dt}$. That is, $\left[D(f)\right](t) = \frac{df}{dt} = f'(t)$. The old refrains you learned in first semester calculus are precisely what makes this a linear operator: (a) The derivative of a constant times a function is the constant times the derivative of the function; and (b) The derivative of a sum is the sum of the derivatives. In symbolic terms, $D(cf) = cf'$ and $D(f+g) = f' + g'$. We can put these together as a single linearity rule: $D(c_1f_1 + c_2f_2) = c_1D(f_1) + c_2D(f_2)$.

The composition of linear operators (or any linear function), where defined, is also linear.
If S and T are both linear operators and if the composition $S \circ T$ is defined, then using the linearity properties of both we have that for all scalars c_1, c_2 and functions f_1, f_2,

$$\left(S \circ T\right)(c_1 f_1 + c_2 f_2) = S(T(c_1 f_1 + c_2 f_2)) = S(c_1 T(f_1) + c_2 T(f_2))$$
$$= c_1 S(T(f_1) + c_2 S(T(f_2)) = c_1 \left(S \circ T\right)(f_1) + c_2 \left(S \circ T\right)(f_2)$$

For example, since differentiation acts linearly, we can compose this with itself to get the 2nd derivative and this also acts linearly. The same holds for higher order derivatives.

The sum of two linear operators is also a linear operator.
The sum of two operators is defined in the same way we add any functions, i.e. $(S+T)(f) = S(f) + T(f)$.
If S and T are both linear operators, then we'll have that for all scalars c_1, c_2 and functions f_1, f_2,

$$\left[S+T\right](c_1 f_1 + c_2 f_2) = S(c_1 f_1 + c_2 f_2) + T(c_1 f_1 + c_2 f_2) = c_1 S(f_1) + c_2 S(f_2) + c_1 T(f_1) + c_2 T(f_2)$$
$$= c_1 S(f_1) + c_1 T(f_1) + c_2 S(f_2) + c_2 T(f_2) = c_1 \left[S(f_1) + T(f_1)\right] + c_2 \left[S(f_2) + T(f_2)\right] = c_1 \left[S+T\right](f_1) + c_2 \left[S+T\right](f_2)$$

If we put together the facts that composition of linear operators and the addition of linear operators yields another linear operator, we see that the expression $\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t)$ for functions $p_{n-1}(t), \ldots, p_1(t), p_0(t)$ represents a linear operator acting on an undetermined function $x(t)$. If we write this operator as $T(x(t)) = \frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t)$, we then know by linearity that $T(x_1(t) + x_2(t)) = T(x_1(t)) + T(x_2(t))$ and $T(c\,x(t)) = c\,T(x(t))$ and, more generally, $T(c_1\,x_1(t) + c_2\,x_2(t)) = c_1\,T(x_1(t)) + c_2\,T(x_2(t))$.

Now that we have paved the road to Linearity, we can apply this idea to solving linear differential equations.

Linearity method using homogeneous solutions and particular solutions
Suppose we have an inhomogeneous linear ODE of the form $T(f) = g$ where T is an *n*th order linear differential operator. We can produce ALL solutions to $T(f) = g$ as follows:

(1) First solve the homogeneous equation $T(f) = 0$ to find a general expression for all such solutions. We call these the <u>homogeneous solutions</u> f_h. It will generally involve *n* arbitrary constants.

(2) Find a single <u>particular solution</u> to the inhomogeneous equation $T(f) = g$. Call this particular solution f_p.

(3) The <u>general solution</u> to $T(f) = g$ is then $\boxed{f = f_h + f_p}$.

(4) Use the initial condition(s) to determine the unique solution to the given initial value problem (IVP).

Proof of the method: We know that $T(f_p) = g$, so suppose f is any other solution to $T(f) = g$. Then, by linearity, $T(f - f_p) = T(f) - T(f_p) = g - g = 0$. So $f - f_p$ solves the homogeneous equation and must be included among all homogeneous solution, i.e. $f - f_p = f_h$. Therefore $f = f_h + f_p$.

This fact is really the same thing that we see when solving a consistent, inhomogeneous system of linear algebraic equations. In matrix form, if the system is represented as $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is an $m \times n$ matrix, and if $\mathbf{x}_h$ represents all solutions to the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{x}_p$ is a single solution to $\mathbf{Ax} = \mathbf{b}$, then all solutions to $\mathbf{Ax} = \mathbf{b}$ will be of the form $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$. Typically, these homogeneous solutions are lines, planes or higher-dimensional analogues (subspaces) passing through the origin. This just says that the inhomogeneous solutions are parallel translates of these subspaces.

So, let's solve the problem $\frac{dx}{dt} = 5x + 3, \quad x(0) = 4$ using linearity methods:

We start by wring the ODE in the form $\frac{dx}{dt} - 5x = 3$, a first order, linear, inhomogeneous ODE.

(1) The homogeneous equation is just $\frac{dx}{dt} - 5x = 0$ or $\frac{dx}{dt} = 5x$. This is easily solved to get all solutions in the form $x_h(t) = Ae^{5t}$.

(2) We can find an inhomogeneous solution by educated guessing (formally called the **method of undetermined coefficients**). Try a solution of the form $x = at + b$. Calculate $\frac{dx}{dt} = a$ and substitute into the ODE to get $\frac{dx}{dt} - 5x = a - 5(at+b) = (a-5b) - 5at = 3$ (for <u>all</u> *t*). We can solve this by choosing $a - 5b = 0$ and $-5a = 0$. So $\boxed{a=0}$ and $-5b = 3$, so $\boxed{b = -\frac{3}{5}}$ and a particular solution is therefore $\boxed{x_p(t) = -\frac{3}{5}}$ which we could have guessed directly.

(3) By linearity, all solutions are therefore of the form $x(t) = x_h(t) + x_p(t) = Ae^{5t} - \frac{3}{5}$.

(4) Substitution of the initial condition $x(0) = 4$ then gives $A = \frac{23}{5}$ and $\boxed{x(t) = \frac{1}{5}(23e^{5t} - 3)}$.

Analogy with solving inhomogeneous systems of linear equations

Suppose we want to solve a consistent, inhomogeneous system of linear algebraic equations. In matrix form, if the system is represented as $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is an $m \times n$ matrix, and if $\mathbf{x}_h$ represents all solutions to the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{x}_p$ is a single solution to $\mathbf{Ax} = \mathbf{b}$, then all solutions to $\mathbf{Ax} = \mathbf{b}$ will be of the form $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$. Typically, these homogeneous solutions are lines, planes or higher-dimensional analogues (subspaces) passing through the origin. This just says that the inhomogeneous solutions are parallel translates of these subspaces.

Example: Find all solutions of the linear system $\begin{Bmatrix} x \quad\;\; - 2z = 3 \\ 2x - y - 5z = 2 \\ 3x - y - 7z = 5 \end{Bmatrix}$. We can solve this most easily by row reduction to get an equivalent system from which we can readily express all solutions. Specifically, we have:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 2 & -1 & -5 & 2 \\ 3 & -1 & -7 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{Bmatrix} x - 2z = 3 \\ y + z = 4 \\ z \text{ arbitrary} \end{Bmatrix} \Rightarrow \begin{Bmatrix} x = 3 + 2t \\ y = 4 - t \\ z = t \end{Bmatrix} \Rightarrow \boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}$$

If we were to solve the corresponding homogeneous linear system $\begin{Bmatrix} x \quad\;\; - 2z = 0 \\ 2x - y - 5z = 0 \\ 3x - y - 7z = 0 \end{Bmatrix}$, the process is similar:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & -1 & -5 & 0 \\ 3 & -1 & -7 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{Bmatrix} x - 2z = 0 \\ y + z = 0 \\ z \text{ arbitrary} \end{Bmatrix} \Rightarrow \begin{Bmatrix} x = 2t \\ y = t \\ z = t \end{Bmatrix} \Rightarrow \boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}$$

The only difference is that the inhomogeneous solutions differ from the homogeneous solutions by a particular solution (which corresponds to $t = 0$).

Back to solving differential equations

Example #1: Solve the initial value problem $\frac{dy}{dx} + xy = 2x$; $\quad y(0) = 5$.

(1) First, we solve the homogeneous equation $\frac{dy}{dx} + xy = 0$. This will always be separable. We get $\frac{dy}{dx} = -xy$ and $\frac{dy}{y} = -x\,dx$, so $\int \frac{dy}{y} = -\int x\,dx \quad \Rightarrow \quad \ln|y| = -\frac{1}{2}x^2 + C \quad \Rightarrow \quad y_h = Ae^{-\frac{1}{2}x^2}$.

(2) Next, we seek a particular solution to $\frac{dy}{dx}+xy=2x$. The Method of Undetermined Coefficients is a good choice here based on the relatively simple functions involved. If we try a solution of the form $y_p=ax^2+bx+c$ (which is actually more general that we really need), we have $\frac{dy}{dx}=2ax+b$, so substitution into the ODE gives:

$$(2ax+b)+x(ax^2+bx+c)=ax^3+bx^2+(2a+c)x+b=2x$$

So we must have $a=0, b=0, 2a+c=2, b=0 \;\Rightarrow\; a=0, b=0, c=2 \;\Rightarrow\; y_p=2$.

(3) So, the general solution must be $y=y_h+y_p=Ae^{-\frac{1}{2}x^2}+2$. The initial value gives $y(0)=A+2=5$, so $A=3$ and the unique solution to the initial value problem is $\boxed{y=3e^{-\frac{1}{2}x^2}+2}$.

Example #2: Solve the initial value problem $\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x=\sin t,\; x(0)=1,\; x'(0)=2=\dot{x}(0)$.

(1) First we seek homogeneous solutions, i.e. solutions of $\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x=0$. We're getting a little ahead of ourselves here, but for a linear ODE with constant coefficients we begin by seeking exponential solutions of the form $x=e^{rt}$. The full logic behind this choice is developed elsewhere (Math E-21c), but differentiation gives $\frac{dx}{dt}=re^{rt}$ and $\frac{d^2x}{dt^2}=r^2e^{rt}$. Substitution into the ODE gives $r^2e^{rt}-3re^{rt}+2e^{rt}=(r^2-3r+2)e^{rt}=0$. This can only vanish when $r^2-3r+2=(r-1)(r-2)=0$, so either $r=1$ or $r=2$. Therefore $x_1(t)=e^t$ and $x_2(t)=e^{2t}$ are solutions.

Now here's where linearity becomes especially useful. If $T[x(t)]=0$ is the form of the homogeneous equation (so $T[x_1(t)]=0$ and $T[x_2(t)]=0$ for all *t*), then any function of the form $c_1x_1(t)+c_2x_2(t)$ will also satisfy the homogeneous equation, i.e. $T[c_1x_1+c_2x_2]=c_1T(x_1)+c_2T(x_2)=c_1\cdot 0+c_2\cdot 0=0$. So $\boxed{x_h(t)=c_1e^t+c_2e^{2t}}$ will give homogeneous solutions for any scalars c_1, c_2. Though we have not fully proven this fact here, the fact is that these do give all of the homogeneous solutions.

(2) Now let's concentrate on getting a particular solution to the original inhomogeneous equation. If you think about what kinds of functions might be such that when combined with its 1st and 2nd derivatives in the manner prescribed by the ODE to yield the function $\sin x$, it should be pretty clear that something of the form $x_p=A\sin t+B\cos t$ is a likely candidate. We have $\begin{Bmatrix} x_p=A\sin t+B\cos t \\ x_p'=-B\sin t+A\cos t \\ x_p''=-A\sin t-B\cos t \end{Bmatrix}$, so:

$$x_p''-3x_p'+2x_p=(-A+3B+2A)\sin t+(-B-3A+2B)\cos t=(A+3B)\sin t+(-3A+B)\cos t=\sin t$$

This implies that $\begin{Bmatrix} A+3B=1 \\ -3A+B=0 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} A \\ B \end{bmatrix}=\frac{1}{10}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} \frac{1}{10} \\ \frac{3}{10} \end{bmatrix} \Rightarrow \begin{Bmatrix} A=\frac{1}{10} \\ B=\frac{3}{10} \end{Bmatrix}$.

So $x_p(t)=\frac{1}{10}\sin t+\frac{3}{10}\cos t$ is a particular solution.

(3) Therefore all solutions are of the form $\boxed{x(t)=c_1e^t+c_2e^{2t}+\frac{1}{10}\sin t+\frac{3}{10}\cos t}$.

(4) Finally, to solve the given initial value problem, note that $x'(t)=c_1e^t+2c_2e^{2t}+\frac{1}{10}\cos t-\frac{3}{10}\sin t$, so:

$$\begin{Bmatrix} x(0)=c_1+c_2+\frac{3}{10}=1 \\ x'(0)=c_1+2c_2+\frac{1}{10}=2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} c_1+c_2=\frac{7}{10} \\ c_1+2c_2=\frac{19}{10} \end{Bmatrix} \Rightarrow c_2=\frac{6}{5}, c_1=-\frac{1}{2} \Rightarrow \boxed{x(t)=-\frac{1}{2}e^t+\frac{6}{5}e^{2t}+\frac{1}{10}\sin t+\frac{3}{10}\cos t}$$

Input-Response formulation for linear ODEs

A linear ODE of the form $x^{(n)}(t)+p_{n-1}(t)x^{(n-1)}(t)+\cdots+p_1(t)x'(t)+p_0(t)x(t)=q(t)$ where $p_{n-1}(t),\ldots,p_1(t),p_0(t),q(t)$ are functions of the independent variable *t* can be expressed in the form $T(x(t))=g(t)$ where *T* is a **linear operator** of the form $T=\frac{d^n}{dt^n}+p_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}+\cdots+p_1(t)\frac{d}{dt}+p_0(t)\cdot$. The last term refers to multiplication by $p_0(t)$. A useful way of formulating such an ODE is to think of the left-hand-side as corresponding to "the system" and the inhomogeneity $q(t)$ on the right-hand-side as corresponding to the "input signal" or, more simply, the "input." The general solution of the ODE is then referred to as the "output signal" or "response." Some motivating examples are in order.

Banking: If we let $x=x(t)$ represent how much money (in dollars) we have in a bank account after *t* years with a fixed interest rate I, the simple model for this is $\frac{dx}{dt}=Ix$ and we have solved this to get $x(t)=Ae^{It}$. If we have only the initial deposit $x(0)=x_0$, then we'll have $x(t)=x_0e^{It}$. Note, however, that we can write the ODE as $\frac{dx}{dt}-Ix=0$, a homogeneous 1st order linear ODE. This corresponds to the situation where you make your deposit and then go home and let the system grow your money without further intervention.
Now let's suppose that you make deposits and withdrawals according to some function $q(t)$ (in dollars/year). If we add this rate into our model, we have $\frac{dx}{dt}=Ix+q(t)$, or $\boxed{\frac{dx}{dt}-Ix=q(t)}$. Note how this intervention (or input) corresponds to the inhomogeneity of this linear ODE. The "system" will carry on as before but will be subject to the input associated with the deposits and withdrawals. The "response" to all this internal and external activity will be the output $x(t)$, i.e. how much money you'll have in the bank at any given time.

Newton's Law of Cooling (diffusion): Suppose we have an enclosed space such as a building or a cooler chest and that the temperature at any given time *t* in the interior space is measured as some function $x(t)$ and that the initial temperature is $x(0)=x_0$. If the outside temperature is given by some function $y(t)$ (possibly constant or possibly variable), then we might expect the interior temperature to change depending on the quality of the insulation and on the difference between outside and inside temperatures. That is, the rate of change of temperature might be modeled as $\frac{dx}{dt}=F(y-x)$ for some function *F*. We would expect that when $y>x$ the temperature would increase, i.e. that $\frac{dx}{dt}>0$; when $y<x$ the temperature would decrease, i.e. that $\frac{dx}{dt}<0$; and that when the outside temperatures are the same there would be no change in temperature, i.e. $\frac{dx}{dt}=0$. The simplest model for this would be $\boxed{\frac{dx}{dt}=k(y-x)}$ for some positive constant *k* (called the coupling constant) that depends on the level of insulation. We can rewrite this as $\boxed{\frac{dx}{dt}+kx=ky}$. In this form, we can think of the homogeneous equation $\frac{dx}{dt}+kx=0$ as representing how this system would be governed if the outside temperature remained constant (at 0) and the interior temperature gradually rose or fell to that level. The inhomogeneous equation $\frac{dx}{dt}+k\,x(t)=k\,y(t)$ would govern how the interior temperature would respond to the input $y(t)$ (or $ky(t)$) in the case where the outside temperature varied according to some known pattern, e.g. the sinusoidal temperature change that might be associated with either a day/night cycle or seasonal cycle, or perhaps some other temperature variation.

Hooke's Law: A simple model for a frictionless mass-spring system is given by Hooke's Law $F=-kx$ where *F* represents an applied force, *x* represents the displacement of the mass from the equilibrium position, and where *k* is the spring constant that corresponds to the stiffness of the spring. If we combine this with Newton's

2nd Law that $F = ma$ where m is the mass, $v = \frac{dx}{dt}$ is the velocity, and $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ is the acceleration of the mass, we have $ma = -kx$ or $m\frac{d^2x}{dt^2} = -kx$. We can rewrite this as $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$. If these is some friction in the system, a simple model suggests that this friction would grow proportionally to the velocity, i.e. there would be an additional force $F_f = -cv$ opposing the motion. The revised equation becomes $m\frac{d^2x}{dt^2} = -kx - cv$ or $\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$. Physicists often favor the "dot notation" for time derivatives with $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$, so the equation may also be expressed as $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$.

Now imagine that you mess with this spring system by "driving" the system with an additional external force $E(t)$. The model might then look like $m\frac{d^2x}{dt^2} = -kx - cv + E(t)$ and if we write $E(t) = mq(t)$ for simplicity, the ODE becomes $\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{E(t)}{m} = q(t)$ or $\boxed{\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = q(t)}$. Once again, the inhomogeneity corresponds to the external input imposed on the system, and the homogeneous ODE would govern how the system would evolve without this intervention.

As we've seen previously, a good approach to solving all of these linear ODEs is to use linear methods that involve finding all homogeneous solutions (the system), finding one particular solution, and combining these to determine the overall response $x(t)$.

Example (diffusion): Suppose a closed container has an initial interior temperature of 32°F at 10:00am and that the outside temperature (also in °F) rises steadily according to $y(t) = 60 + 6t$ where time t is measured in hours. Further suppose that Newton's Law of Cooling applies where the coupling constant is $k = \frac{1}{3}$. (a) How will the interior temperature vary in time, and (b) at what time will the interior temperature reach 60°F?

Solution: The temperature will be governed by $\frac{dx}{dt} = \frac{1}{3}(y - x)$ or $\frac{dx}{dt} + \frac{1}{3}x = \frac{1}{3}y(t) = \frac{1}{3}(60 + 6t) = 20 + 2t$, so the inhomogeneous ODE is $\frac{dx}{dt} + \frac{1}{3}x = 20 + 2t$.

(1) The homogeneous equation $\frac{dx}{dt} + \frac{1}{3}x = 0$ easily yields the solutions of the form $\boxed{x_h(t) = ce^{-\frac{1}{3}t}}$. It's worth noting that over time any such homogeneous solution will tend toward 0 and become negligible. For this reason we often refer to this as a transient. In the short term it may be relevant, but in the long term it is not.

(2) We can use undetermined coefficients to find a particular solution. The nature of the inhomogeneity $q(t) = 20 + 2t$ suggests that we seek a solution of the form $x_p(t) = A + Bt$. We have $\frac{dx_p}{dt}(t) = B$, so we must have $B + \frac{1}{3}(A + Bt) = (B + \frac{1}{3}A) + \frac{1}{3}Bt = 20 + 2t$, so $B + \frac{1}{3}A = 20$ and $\frac{1}{3}B = 2$. This gives $B = 6$ and $A = 42$, so $x_p(t) = 42 + 6t$. Once the transients have become negligible, this is all that will remain. For this reason we might refer to this as the "steady state" solution.

(3) The general solution is $x(t) = x_h(t) + x_p(t) = ce^{-\frac{1}{3}t} + 42 + 6t$. If we substitute the initial condition $x(0) = 32$, we have $x(0) = c + 42 = 32$, so $c = -10$ and $\boxed{x(t) = 42 + 6t - 10e^{-\frac{1}{3}t}}$. Note that eventually the interior temperature will be rising at the same rate as the outside temperature but always 18°F cooler.

The interior temperature will reach 60°F at a time T when $42 + 6T - 10e^{-\frac{1}{3}T} = 60$ or $6T - 10e^{-\frac{1}{3}T} = 18$. This cannot be solved algebraically, but it's easy to get a numerical solution using a graphing calculator and the *trace* function. It gives a time $T \approx 3.33 \approx 3$ hrs, 20 min, i.e. about 1:20pm.

Example (exponential input): Solve the initial value problem $\frac{d^2x}{dt^2}+3\frac{dx}{dt}+2x=e^t$, $x(0)=4$, $x'(0)=2$.

Solution: This ODE is of the type we might expect from a mass-spring system, though the external driving force is not especially realistic (relentlessly exponential in a single direction). It is nonetheless good for illustrating the methods. For simplicity, let's write the ODE as $x''+3x'+2x=e^t$.

(1) For the homogeneous solutions, look for exponential solutions $x=e^{rt}$ to the equation $x''+3x'+2x=0$. This gives $r^2e^{rt}+3re^{rt}+2e^{rt}=(r^2+3r+2)e^{rt}=0$, so $r^2+3r+2=(r+1)(r+2)=0 \Rightarrow r=-1, r=-2$. Individual homogeneous solutions are $x_1(t)=e^{-t}$ and $x_2(t)=e^{-2t}$, and by linearity any solution of the form $x_h(t)=c_1e^{-t}+c_2e^{-2t}$ will satisfy the homogeneous ODE. It's not hard to prove that these give all homogeneous solutions if we think of the 2nd order homogenous linear ODE as a composition of two 1st order linear ODEs and use the fact that we can always solve such equations. [See if you can complete the argument.]

Note that, in this case, the homogeneous solutions are transient (they decay quickly).

(2) Once again, undetermined coefficients provide the simplest way to find a particular solution in this case. The obvious choice is to try a solution of the form $x=Ae^t$. This gives $x'=Ae^t, x''=Ae^t$, and we get that $Ae^t+3Ae^t+2Ae^t=6Ae^t=e^t \Rightarrow A=\frac{1}{6}$, so our particular solution is $x_p(t)=\frac{1}{6}e^t$.

(3) Our general solution is then $\boxed{x(t)=c_1e^{-t}+c_2e^{-2t}+\frac{1}{6}e^t}$. We compute $x'(t)=-c_1e^{-t}-2c_2e^{-2t}+\frac{1}{6}e^t$, and the initial conditions give $\begin{Bmatrix} x(0)=c_1+c_2+\frac{1}{6}=4 \\ x'(0)=-c_1-2c_2+\frac{1}{6}=2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} c_1+c_2=\frac{23}{6} \\ -c_1-2c_2=\frac{11}{6} \end{Bmatrix} \Rightarrow c_1=\frac{19}{2}, c_2=-\frac{17}{3}$. So the unique solution to the initial value problem is $\boxed{x(t)=\underbrace{\frac{19}{2}e^{-t}-\frac{17}{3}e^{-2t}}_{transient}+\underbrace{\frac{1}{6}e^t}_{steady-state}}$.

Example (sinusoidal input): Find the general solution to the ODE $\frac{dx}{dt}+2x=\cos 3t$

Solution: If we solve this using linearity:

(1) $\frac{dx}{dt}+2x=0$ gives the homogeneous solutions $x_h(t)=ce^{-2t}$

(2) For a particular solution, try $x=a\cos 3t+b\sin 3t$. We calculate $x'=3b\cos 3t-3a\sin 3t$, and substitution gives $x'+2x=(2a+3b)\cos 3t+(-3a+2b)\sin 3t=\cos 3t$, so

$$\begin{Bmatrix} 2a+3b=1 \\ -3a+2b=0 \end{Bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix}=\frac{1}{13}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} \frac{2}{13} \\ \frac{3}{13} \end{bmatrix}$$, so $x_p(t)=\frac{2}{13}\cos 3t+\frac{3}{13}\sin 3t$ or $x_p(t)=\frac{1}{13}(2\cos 3t+3\sin 3t)$.

(3) The general solution is therefore $\boxed{x(t)=\underbrace{ce^{-2t}}_{transient}+\underbrace{\frac{1}{13}(2\cos 3t+3\sin 3t)}_{steady-state}}$.

Notes by Robert Winters