Linear Algebra – Lecture #1 Notes – Addendum

There's a different way to look at row reduction in terms of vectors, the span of a set of vectors, and subspaces – topics that we'll get to relatively soon. In particular, the ***row space*** of a matrix deserves special focus.

For example, when we represent the linear system $\left\{\begin{matrix} x-4y=11 \\ 5x+3y=9 \end{matrix}\right\}$ as an augmented matrix $\left[\begin{array}{cc|c} 1 & -4 & 11 \\ 5 & 3 & 9 \end{array}\right]$, we can identify each row as a vector (in this case a vector in $\mathbf{R}^3$). We can then look at the ***span*** of these row vectors. That is, if the 1st row vector is $\mathbf{w}_1$ and the 2nd row vector $\mathbf{w}_2$, then $Span\{\mathbf{w}_1,\mathbf{w}_2\}=\{c_1\mathbf{w}_1+c_2\mathbf{w}_2 : c_1,c_2\in\mathbf{R}\}$. This is an example of a ***subspace*** (of $\mathbf{R}^3$), i.e. it is closed under scaling and vector addition. This subspace is called the ***Row Space*** $row\left[\mathbf{A}\,|\,\mathbf{b}\right]$ of the given augmented matrix $\left[\mathbf{A}\,|\,\mathbf{b}\right]$.

The ***elementary row operations*** have an especially interesting interpretation in terms of the Row Space.
1) It's OK to scale any row by a nonzero constant. $\Leftrightarrow$ $row\left[\mathbf{A}\,|\,\mathbf{b}\right]$ is closed under scaling.
2) It's OK to interchange any pair of rows. $\Leftrightarrow$ $Span\{\mathbf{w}_1,\mathbf{w}_2\}=Span\{\mathbf{w}_2,\mathbf{w}_1\}$
3) It's OK to add any scalar multiple of one row to any other row. $\Leftrightarrow$ $row\left[\mathbf{A}\,|\,\mathbf{b}\right]$ is a subspace.

That is, the validity of the elementary row operations is essentially the same as saying $row\left[\mathbf{A}\,|\,\mathbf{b}\right]$ is a subspace.

The process of row reduction leading up to the ***reduced row-echelon form*** (RREF) can thus be interpreted simply as finding different basis vectors for the Row Space with the goal of being able to interpret the solution of the corresponding linear system in the simplest possible way.

Elementary matrices
Though we have not yet formally introduced the ideas of a matrix as a linear transformation and matrix multiplication as the composition of linear transformations, the fact is that each of the elementary row operations can be carried out by applying an appropriate (invertible) elementary matrix.

Example #1: Solve the system $\left\{\begin{matrix} x-4y=11 \\ 5x+3y=9 \end{matrix}\right\}$.

The sequence of elementary row operations was:

$$\left[\begin{array}{cc|c} 1 & -4 & 11 \\ 5 & 3 & 9 \end{array}\right]\xrightarrow[R_2-5R_1]{R_1}\left[\begin{array}{cc|c} 1 & -4 & 11 \\ 0 & 23 & -46 \end{array}\right]\xrightarrow[\frac{1}{23}R_2]{R_1}\left[\begin{array}{cc|c} 1 & -4 & 11 \\ 0 & 1 & -2 \end{array}\right]\xrightarrow[R_2]{R_1+4R_2}\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

In terms of elementary matrices:

$$\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}\left[\begin{array}{cc|c} 1 & -4 & 11 \\ 5 & 3 & 9 \end{array}\right]=\left[\begin{array}{cc|c} 1 & -4 & 11 \\ 0 & 23 & -46 \end{array}\right] \qquad \mathbf{E}_1=\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \text{ is an elementary matrix.}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{23} \end{bmatrix}\left[\begin{array}{cc|c} 1 & -4 & 11 \\ 0 & 23 & -46 \end{array}\right]=\left[\begin{array}{cc|c} 1 & -4 & 11 \\ 0 & 1 & -2 \end{array}\right] \qquad \mathbf{E}_2=\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{23} \end{bmatrix} \text{ is an elementary matrix.}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\left[\begin{array}{cc|c} 1 & -4 & 11 \\ 0 & 1 & -2 \end{array}\right]=\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right] \qquad \mathbf{E}_3=\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \text{ is an elementary matrix.}$$

Together, we have $\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\left[\mathbf{A}\,|\,\mathbf{b}\right]=rref\left[\mathbf{A}\,|\,\mathbf{b}\right]$. Each step is invertible, and the composition is also invertible. If we write the augmented matrix simply as $\mathbf{M}$ and the reduced row-echelon form as $\mathbf{R}$, then we have $\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{M}=\mathbf{R}$ or $\mathbf{M}=\mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}\mathbf{R}=\mathbf{CR}$. This is an example of a factorization, i.e. $\mathbf{M}=\mathbf{CR}$ that relates the original matrix $\mathbf{M}$ and its reduced row-echelon form $\mathbf{R}$.

Note: We could also settle for just the ***row-echelon form***, i.e. without any "back-substitution":

$$\mathbf{E}_2\mathbf{E}_1\left[\mathbf{A}\,|\,\mathbf{b}\right]=\left[\begin{array}{cc|c}1 & -4 & 11\\ 0 & 1 & -2\end{array}\right]=ref\left[\mathbf{A}\,|\,\mathbf{b}\right]$$

Example #3: Solve the system $\left\{\begin{array}{r}3x+2y=5\\ -x+y=7\\ 2x+y=1\end{array}\right\}$.

In this example, we had the following sequence of (combined) elementary row operations:

$$\left[\begin{array}{cc|c}3 & 2 & 5\\ -1 & 1 & 7\\ 2 & 1 & 1\end{array}\right]\to\left[\begin{array}{cc|c}\boxed{1} & -1 & -7\\ 3 & 2 & 5\\ 2 & 1 & 1\end{array}\right]\to\left[\begin{array}{cc|c}1 & -1 & -7\\ 0 & 5 & 26\\ 0 & 3 & 15\end{array}\right]\to\left[\begin{array}{cc|c}1 & -1 & -7\\ 0 & \boxed{1} & 5\\ 0 & 5 & 26\end{array}\right]\to\left[\begin{array}{cc|c}1 & 0 & -2\\ 0 & 1 & 5\\ 0 & 0 & 1\end{array}\right]$$

In this case, we can express each of the operations in terms of elementary matrices (or a combination of elementary matrices). Here's the full sequence without any combined steps:

$$\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\left[\begin{array}{cc|c}3 & 2 & 5\\ -1 & 1 & 7\\ 2 & 1 & 1\end{array}\right]=\left[\begin{array}{cc|c}-1 & 1 & 7\\ 3 & 2 & 5\\ 2 & 1 & 1\end{array}\right]\text{ (permute)}$$

$$\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\left[\begin{array}{cc|c}-1 & 1 & 7\\ 3 & 2 & 5\\ 2 & 1 & 1\end{array}\right]=\left[\begin{array}{cc|c}1 & -1 & -7\\ 3 & 2 & 5\\ 2 & 1 & 1\end{array}\right]\text{ (scale, leading 1)}$$

$$\begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\left[\begin{array}{cc|c}\boxed{1} & -1 & -7\\ 3 & 2 & 5\\ 2 & 1 & 1\end{array}\right]=\left[\begin{array}{cc|c}1 & -1 & -7\\ 0 & 5 & 26\\ 2 & 1 & 1\end{array}\right]\text{ (clean)}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}\left[\begin{array}{cc|c}1 & -1 & -7\\ 0 & 5 & 26\\ 2 & 1 & 1\end{array}\right]=\left[\begin{array}{cc|c}1 & -1 & -7\\ 0 & 5 & 26\\ 0 & 3 & 15\end{array}\right]\text{ (clean)}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\left[\begin{array}{cc|c}1 & -1 & -7\\ 0 & 5 & 26\\ 0 & 3 & 15\end{array}\right]=\left[\begin{array}{cc|c}1 & -1 & -7\\ 0 & 3 & 15\\ 0 & 5 & 26\end{array}\right]\text{ (permute)}$$

$$\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\left[\begin{array}{cc|c}1 & -1 & -7\\ 0 & 1 & 5\\ 0 & 5 & 26\end{array}\right]=\left[\begin{array}{cc|c}1 & 0 & -2\\ 0 & 1 & 5\\ 0 & 5 & 26\end{array}\right]\text{ (scale, leading 1)}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -5 & 1\end{bmatrix}\left[\begin{array}{cc|c}1 & 0 & -2\\ 0 & 1 & 5\\ 0 & 5 & 26\end{array}\right]=\left[\begin{array}{cc|c}1 & 0 & -2\\ 0 & 1 & 5\\ 0 & 0 & 1\end{array}\right]\text{ (clean)}$$

$$\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\left[\begin{array}{cc|c}1 & 0 & -2\\ 0 & 1 & 5\\ 0 & 0 & 1\end{array}\right]=\left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 5\\ 0 & 0 & 1\end{array}\right]\text{ (clean)}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -5\\ 0 & 0 & 1\end{bmatrix}\left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 5\\ 0 & 0 & 1\end{array}\right]=\left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]\text{ (clean, RREF)}$$

This can be viewed altogether as: $\mathbf{E}_9\mathbf{E}_8\mathbf{E}_7\mathbf{E}_6\mathbf{E}_5\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\left[\mathbf{A}\,|\,\mathbf{b}\right]=\mathbf{C}\left[\mathbf{A}\,|\,\mathbf{b}\right]=rref\left[\mathbf{A}\,|\,\mathbf{b}\right]=\mathbf{R}$.

Though these examples correspond to augmented matrices, the idea applies generally to row reduction for any matrix. The Row Space remains the same for all matrices in the sequence.

Definition: The **rank** of a matrix is the number of Leading 1's in the RREF of the matrix.

We will soon discuss the idea of the dimension of a subspace. In this case the rank (or ***row rank***) of a matrix is simply the dimension of the Row Space. We will also later discuss the idea of the ***image of a matrix*** (or the image of the linear transformation associated with a matrix). The ***column rank*** will be the dimension of the ***column space***, i.e. the span of the column vectors of a matrix. It is an important theorem that ***the row rank and the column rank are the same for any matrix***, so we can refer to either of these simply as the ***rank*** of the matrix.

Example #2: Solve the system $\left\{\begin{matrix} 3x-2y+2z=12 \\ 4x+4y+z=-4 \end{matrix}\right\}$.

The row reduction steps (combining some steps) were:

$$\left[\begin{array}{ccc|c} 3 & -2 & 2 & 12 \\ 4 & 4 & 1 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} \boxed{1} & 6 & -1 & -16 \\ 4 & 4 & 1 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 6 & -1 & -16 \\ 0 & -20 & 5 & 60 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 6 & -1 & -16 \\ 0 & \boxed{1} & -\frac{1}{4} & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 2 \\ 0 & 1 & -\frac{1}{4} & -3 \end{array}\right]$$

The expanded steps can be carried out using 5 elementary matrices as:

$$\begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{20} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left[\begin{array}{ccc|c} 3 & -2 & 2 & 12 \\ 4 & 4 & 1 & -4 \end{array}\right] = \frac{1}{20}\begin{bmatrix} 4 & 2 \\ -4 & 3 \end{bmatrix}\left[\begin{array}{ccc|c} 3 & -2 & 2 & 12 \\ 4 & 4 & 1 & -4 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 2 \\ 0 & 1 & -\frac{1}{4} & -3 \end{array}\right]$$

Example #4: Solving the system $\left\{\begin{matrix} x+2y-3z=1 \\ 3x+y+z=7 \\ -2x+3y-4z=5 \end{matrix}\right\}$ via row reduction went like:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 3 & 1 & 1 & 7 \\ -2 & 3 & -4 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & -5 & 10 & 4 \\ 0 & 7 & -10 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & -2 & -.8 \\ 0 & 7 & -10 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2.6 \\ 0 & 1 & -2 & -.8 \\ 0 & 0 & 4 & 12.6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2.6 \\ 0 & 1 & -2 & -.8 \\ 0 & 0 & 1 & 3.15 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -.55 \\ 0 & 1 & 0 & 5.5 \\ 0 & 0 & 1 & 3.15 \end{array}\right]$$

In terms of elementary matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 3 & 1 & 1 & 7 \\ -2 & 3 & -4 & 5 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -.55 \\ 0 & 1 & 0 & 5.5 \\ 0 & 0 & 1 & 3.15 \end{array}\right]$$

If we multiply all of these eight elementary matrices, we get:

$$\frac{1}{20}\begin{bmatrix} 7 & 1 & -5 \\ -10 & 10 & 10 \\ -11 & 7 & 5 \end{bmatrix}\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 3 & 1 & 1 & 7 \\ -2 & 3 & -4 & 5 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -.55 \\ 0 & 1 & 0 & 5.5 \\ 0 & 0 & 1 & 3.15 \end{array}\right]$$

Note that if we write the system in matrix form $\mathbf{Ax}=\mathbf{b}$ where $\mathbf{A}=\begin{bmatrix} 1 & 2 & -3 \\ 3 & 1 & 1 \\ -2 & 3 & -4 \end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix} 1 \\ 7 \\ 5 \end{bmatrix}$, then the above relationship can be expressed succinctly as: $\mathbf{A}^{-1}\left[\mathbf{A} \mid \mathbf{b}\right]=\left[\mathbf{I} \mid \mathbf{A}^{-1}\mathbf{b}\right]$, i.e. $\mathbf{A}^{-1}=\frac{1}{20}\begin{bmatrix} 7 & 1 & -5 \\ -10 & 10 & 10 \\ -11 & 7 & 5 \end{bmatrix}$.

There are simpler ways to find the inverse of a matrix, but this illustrates how the inverse can be viewed as a sequence of elementary matrices that essentially "undo" the matrix **A** step-by-step.

Example #5: Solve the system $\left\{\begin{matrix} -2x_1+x_2+3x_3+4x_4=10 \\ x_1+2x_2+7x_3-3x_4=4 \end{matrix}\right\}$.

The row reduction steps are these:

$$\left[\begin{array}{cccc|c} -2 & 1 & 3 & 4 & 10 \\ 1 & 2 & 7 & -3 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 7 & -3 & 4 \\ -2 & 1 & 3 & 4 & 10 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 7 & -3 & 4 \\ 0 & 5 & 17 & -2 & 18 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 7 & -3 & 4 \\ 0 & 1 & \frac{17}{5} & -\frac{2}{5} & \frac{18}{5} \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{5} & -\frac{11}{5} & -\frac{16}{5} \\ 0 & 1 & \frac{17}{5} & -\frac{2}{5} & \frac{18}{5} \end{array}\right]$$

Equivalently, these steps can be carried out via elementary matrices as:

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left[\begin{array}{cccc|c} -2 & 1 & 3 & 4 & 10 \\ 1 & 2 & 7 & -3 & 4 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{5} & -\frac{11}{5} & -\frac{16}{5} \\ 0 & 1 & \frac{17}{5} & -\frac{2}{5} & \frac{18}{5} \end{array}\right]$$

Note that application of each elementary matrix is invertible and preserves rank, and their composition thus also preserves rank.

Addendum by Robert Winters