Linear Algebra – Lecture #13 Notes

Vector fields, Continuous Dynamical Systems, and Systems of 1st Order Linear Differential Equations

Definition: A **vector field** in $\mathbf{R}^n$ is an assignment of a vector to every point in $\mathbf{R}^n$ (with the possible exception of some singular points). This can be viewed as a function $\mathbf{F}(x_1,\cdots,x_n)=\begin{bmatrix} f_1(x_1,\cdots,x_n) \\ \vdots \\ f_n(x_1,\cdots,x_n)\end{bmatrix}$ where $f_i(x_1,\cdots,x_n)$ is the *i*-th component of the vector assigned to the point $(x_1,\cdots,x_n)$. We can also write this more succinctly as $\mathbf{F}(\mathbf{x})=\begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x})\end{bmatrix}$. In practice we usually assume some reasonable properties such as that the component functions are continuous or differentiable except perhaps at a finite number of *singular points*.

If we view the vector assigned to each point as a ***velocity vector*** associated with some smoothly varying system, a reasonable question to ask is this: Given a starting point $\mathbf{x}_0$ (the initial condition), can we find a parameterized curve $\mathbf{x}(t)$ such that $\mathbf{x}(0)=\mathbf{x}_0$ and the velocity vector at any point on this parameterized curve matches the underlying vector field, i.e. $\frac{d\mathbf{x}}{dt}=\mathbf{F}(\mathbf{x}(t))$. This is equivalent to a system of (time-independent) first-order differential equations, i.e. $\left\{\begin{matrix} \frac{dx_1}{dt} = f_1(x_1,\cdots,x_n) \\ \vdots \\ \frac{dx_n}{dt} = f_n(x_1,\cdots,x_n)\end{matrix}\right\}$. We are interested in knowing how a system defined in this way evolves over time for any given initial condition. This describes what we call a **continuous dynamical system**. We call the set of all such solution curves the **flow** of the dynamical system.

If you imagine a vector field as describing a flowing liquid, then these parameterized curves simply describe what happens if you drop a particle into the flow and see where it goes as it carried by the flow. This is a good way to think about a continuous dynamical system even when the variables are describing such things as populations or economic variables rather than geometric coordinates. We'll still refer to the solutions as the flow of the system even though there's nothing physical about this flow.

We are typically interested in the long-term behavior of such a system, but we often would also like to predict exactly where the particle will be after a specified time *t*, i.e. formulas for how the component functions evolve in time. In general, if the component functions of the underlying vector field are nonlinear, it's very difficult to find a tidy formula for how the system evolves over time. The linear case, on the other hand, is completely solvable using matrix methods we've recently discussed.

Definition: A linear continuous dynamical system is a system of first-order differential equations of the form $\left\{\begin{matrix} \frac{dx_1}{dt} = a_{11}x_1+\cdots+a_{1n}x_n \\ \vdots \\ \frac{dx_n}{dt} = a_{n1}x_1+\cdots+a_{nn}x_n\end{matrix}\right\}$. If $\mathbf{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix}$, then $\frac{d\mathbf{x}}{dt}=\begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt}\end{bmatrix}=\begin{bmatrix} a_{11}x_1+\cdots+a_{1n}x_n \\ \vdots \\ a_{n1}x_1+\cdots+a_{nn}x_n\end{bmatrix}=\begin{bmatrix} a_{11} & \cdots & a_{11} \\ \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{11}\end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix}=\mathbf{A}\mathbf{x}$ where $\mathbf{A}=\begin{bmatrix} a_{11} & \cdots & a_{11} \\ \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{11}\end{bmatrix}$ is the matrix of coefficients. That is, $\frac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is an $n\times n$ real matrix.

Situation: You want to solve a system of first-order linear differential equations of the form $\frac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x}$ given some initial condition $\mathbf{x}(0)=\mathbf{x}_0$. How is this most efficiently accomplished?

Example 1: The simplest linear continuous dynamical system is the single equation $\frac{dx}{dt} = kx$ with initial condition $x(0) = x_0$. This is something we solved in basic calculus and yields exponential growth or decay (depending on whether $k > 0$ or $k < 0$). Specifically, we write $\frac{1}{x}\frac{dx}{dt} = k$ and integrate both sides to get $\ln|x(t)| = kt + c$ for some arbitrary constant *c*. [Many people choose to do this calculation as $\frac{dx}{x} = kdt$ and integrate both sides to get $\int \frac{dx}{x} = \int kdt \quad \Rightarrow \quad \ln|x| = kt + c$.] In any case, exponentiating both sides gives $|x(t)| = e^{kt+c} = e^c e^{kt} = ae^{kt}$, and we can remove the absolute value by allowing the constant *a* to be either positive or negative, so we get $x(t) = ae^{kt}$. Using the initial condition $x(0) = x_0$ we see that $x(0) = a = x_0$, so the solution is $\boxed{x(t) = x_0 e^{kt}}$.

Uncoupled systems: We call a system uncoupled (or unlinked) if the rates of change of each of the variables do not depend on any of the other variables. In the linear case, this would mean a system of the form $\left\{ \begin{array}{c} \frac{dx_1}{dt} = k_1 x_1 \\ \vdots \\ \frac{dx_n}{dt} = k_n x_n \end{array} \right\}$ with initial conditions $x_1(0), \ldots, x_n(0)$. Note that such a system can be expressed in matrix form as $\frac{d\mathbf{x}}{dt} = \mathbf{Dx}$ where $\mathbf{D}$ is the diagonal matrix $\mathbf{D} = \begin{bmatrix} k_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & k_n \end{bmatrix}$. Solving this system is nothing more than solving the previous problem repeatedly with different rate constants and corresponding initial conditions. We get the solution $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} x_1(0)e^{k_1 t} \\ \vdots \\ x_n(0)e^{k_n t} \end{bmatrix} = \begin{bmatrix} e^{k_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{k_n t} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$. Note that when $t = 0$ this matrix is just the identity matrix which simply reflects the fact that $t = 0$ corresponds to the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$. Of greater interest is the fact that this time-varying matrix evolves over time to produce the flow emanating from any given initial condition. It is for this reason that we refer to this matrix as the **evolution matrix** for this uncoupled system. If we refer to this matrix as $[e^{t\mathbf{D}}]$, a notation that is perhaps best not taken too literally, then the system $\frac{d\mathbf{x}}{dt} = \mathbf{Dx}$ with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ has solution $\mathbf{x}(t) = [e^{t\mathbf{D}}]\mathbf{x}(0)$.

A **coupled system**, i.e. a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$ where the matrix A is not diagonal, can often be solved by changing coordinates so that relative to some new basis (of eigenvectors) the system has a diagonal matrix. The tool at the heart of these methods is diagonalization or, in the case where a matrix cannot be diagonalized, finding an appropriate change of basis relative to which the underlying linear transformation has the simplest possible matrix representation, i.e. Jordan Canonical Form. The introduction of corresponding "evolution matrices" is a useful formalism for handling these general cases.

Solving systems using diagonalization and evolution matrices

Given an $n \times n$ matrix **A**, suppose **S** is a change of basis matrix corresponding to either diagonalization or reduction to Jordan Canonical Form. We will have $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{B}$ in this case, where **B** is diagonal or otherwise in simplest form. We then calculate $\mathbf{A} = \mathbf{SBS}^{-1}$, and substitution gives $\frac{d\mathbf{x}}{dt} = \mathbf{SBS}^{-1}\mathbf{x}$.

Multiplying on the left by $\mathbf{S}^{-1}$ and using the basic calculus fact that $\frac{d}{dt}(\mathbf{Mx}) = \mathbf{M}\frac{d\mathbf{x}}{dt}$ for any (constant) matrix **M**, we have $\mathbf{S}^{-1}\frac{d\mathbf{x}}{dt} = \frac{d(\mathbf{S}^{-1}\mathbf{x})}{dt} = \mathbf{B}(\mathbf{S}^{-1}\mathbf{x})$. If we write $\mathbf{u} = \mathbf{S}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathscr{B}}$, where $\mathscr{B}$ is the new, preferred basis, then in these new coordinates the system becomes $\frac{d\mathbf{u}}{dt} = \mathbf{Bu}$, but now the system will be much more straightforward to solve.

The diagonalizable case

In the case where **B** is a diagonal matrix with the eigenvalues of **A** on the diagonal, the system is just

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\mathbf{u} \text{ or } \left\{ \begin{matrix} \frac{du_1}{dt} = \lambda_1 u_1 \\ \vdots \\ \frac{du_n}{dt} = \lambda_n u_n \end{matrix} \right\}.$$

This has the solution $\left\{ \begin{matrix} u_1(t) = e^{\lambda_1 t}u_1(0) \\ \vdots \\ u_n(t) = e^{\lambda_n t}u_n(0) \end{matrix} \right\}$ or $\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} u_1(0) \\ \vdots \\ u_n(0) \end{bmatrix} = [e^{t\mathbf{B}}]\mathbf{u}(0)$.

To revert back to the original coordinates, we write $\mathbf{x} = \mathbf{Su}$, so $\mathbf{x}(t) = \mathbf{Su}(t) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{u}(0) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. If we denote the evolution matrix for the system in its original coordinates as $[e^{t\mathbf{A}}]$ where $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$, then the previous calculation gives the simple relation $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$.

In other words, the evolution matrices for the solution are in the same relationship as the matrices **A** and **B**, namely $\mathbf{A} = \mathbf{SBS}^{-1}$. This pattern is very easy to remember, and this same pattern will again be the case where **B** is not diagonal but where the corresponding evolution matrix is still relatively easy to calculate.

$\mathbf{A} = \mathbf{SBS}^{-1} \quad \Rightarrow \quad [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$, and the solution of the original system will be $\boxed{\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)}$.

The complex eigenvalue case

Suppose we want to solve a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$ where **A** is an 2×2 real matrix with a complex conjugate pair of eigenvalues $\lambda = a + ib$ and $\bar{\lambda} = a - ib$. There are several reasonable ways to proceed, but they all come down to determining the evolution matrix $[e^{t\mathbf{A}}]$ so that we can solve for $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$.

<u>First, put the system into (real) normal form</u>. [**Standardize**]

Use the complex eigenvalue $\lambda = a + ib$ to find a complex eigenvector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$. If we change to the basis $\{\mathbf{v}, \mathbf{u}\}$ then, using the change of basis matrix $\mathbf{S} = [\mathbf{v} \;\; \mathbf{u}]$, we'll get $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, a rotation-dilation matrix. Noting, as before, that $\mathbf{A} = \mathbf{SBS}^{-1} \quad \Rightarrow \quad [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$, we need only to determine $[e^{t\mathbf{B}}]$.

<u>Second, find the evolution matrix for the (real) normal form</u>. [**Solve**]

In fact, if $\mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ then $[e^{t\mathbf{B}}] = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$, a time-varying rotation matrix with exponential scaling. For any initial condition (except the zero vector), this yields a trajectory that spirals out in the case where $\text{Re}(\lambda) = a > 0$ (look to the original vector field to see whether it's clockwise or counterclockwise), or a trajectory that spirals inward toward **0** in the case where $\text{Re}(\lambda) = a < 0$.

To derive this expression for $[e^{t\mathbf{B}}]$, make another coordinate change with complex eigenvectors starting with $\mathbf{B}=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. We know this has the same eigenvalues of **A**, namely $\lambda = a+ib$ and $\bar{\lambda} = a-ib$. Use $\lambda = a+ib$ to get the complex eigenvector $\mathbf{w}=\begin{bmatrix} 1 \\ -i \end{bmatrix}$. The eigenvalue $\bar{\lambda} = a-ib$ will then give the eigenvector $\hat{\mathbf{w}}=\begin{bmatrix} 1 \\ i \end{bmatrix}$. Using the (complex) change of basis matrix $\mathbf{P}=\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$, we have that $\mathbf{P}^{-1}\mathbf{B}\mathbf{P}=\mathbf{D}=\begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$. It follows that (using Euler's Formula as needed):

$$[e^{t\mathbf{B}}]=\mathbf{P}[e^{t\mathbf{D}}]\mathbf{P}^{-1}=\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{(a+ib)t} & 0 \\ 0 & e^{(a-ib)t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}=e^{at}\begin{bmatrix} \frac{e^{ibt}+e^{-ibt}}{2} & -\frac{e^{ibt}-e^{-ibt}}{2i} \\ \frac{e^{ibt}-e^{-ibt}}{2i} & \frac{e^{ibt}+e^{-ibt}}{2} \end{bmatrix}=e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}.$$

<u>Express the solution in terms of the original coordinates</u>. [**Switch Back**]

These calculations enable us to write down a closed form expression for the solution of this linear system, namely $\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)$ where $[e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}=e^{at}\mathbf{S}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}\mathbf{S}^{-1}$. However, the more important result is the ability to qualitatively describe the trajectories for this system by knowing only the real part of the eigenvalues of the matrix **A** and the direction of the corresponding vector field (clockwise vs. counterclockwise).

Repeated eigenvalues (with geometric multiplicity less than the algebraic multiplicity)

Suppose we want to solve a system of the form $\frac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x}$ where **A** is a non-diagonalizable 2×2 real matrix with a repeated eigenvalue λ. We've seen that in this case, we can always find a change of basis matrix **S** such that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\mathbf{B}=\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. As in the previous two cases, $\mathbf{A}=\mathbf{S}\mathbf{B}\mathbf{S}^{-1} \quad \Rightarrow \quad [e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$ and it comes down to finding $[e^{t\mathbf{B}}]$. This is perhaps most easily done by explicitly solving the corresponding differential equations. In the new coordinates, this system translates into $\left\{ \begin{aligned} \frac{du_1}{dt} &= \lambda u_1 + u_2 \\ \frac{du_2}{dt} &= \lambda u_2 \end{aligned} \right\}$. The second equation is easily solved to get $u_2(t)=e^{\lambda t}u_2(0)$. We can guess a solution for the first equation of the form $u_1(t)=c_1te^{\lambda t}+c_2e^{\lambda t}$. Differentiating this and substituting into the first equation, we get $c_1(e^{\lambda t}+\lambda te^{\lambda t})+c_2\lambda e^{\lambda t}=\lambda(c_1te^{\lambda t}+c_2e^{\lambda t})+e^{\lambda t}u_2(0)$. Comparing like terms, we conclude that $c_1=u_2(0)$. Substituting t = 0, we further conclude that $u_1(0)=c_2$. Putting these results together, we get $u_1(t)=u_2(0)te^{\lambda t}+u_1(0)e^{\lambda t}=e^{\lambda t}u_1(0)+te^{\lambda t}u_2(0)$. We therefore have that

$$\mathbf{u}(t)=\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}=\begin{bmatrix} e^{\lambda t}u_1(0)+te^{\lambda t}u_2(0) \\ e^{\lambda t}u_2(0) \end{bmatrix}=\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}=\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\mathbf{u}(0)$$

So, $[e^{t\mathbf{B}}]=\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$ in this case and the solution is given by $\mathbf{x}(t)=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)=\mathbf{S}\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\mathbf{S}^{-1}\mathbf{x}(0)$.

An alternate method of deriving this result may be found in the homework exercises.

Similar calculations enable us to deal with cases such as a repeated eigenvalue where the geometric multiplicity is 1 and the algebraic multiplicity is 3 (or even worse).

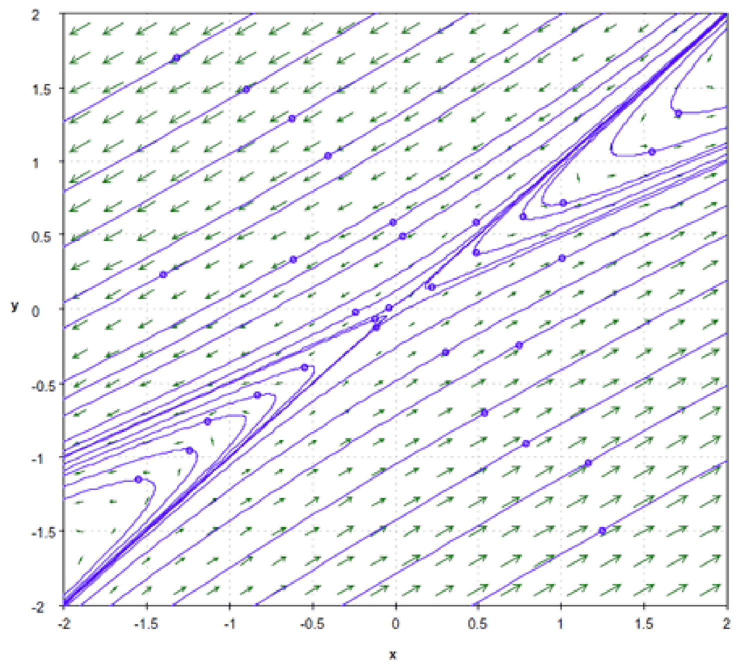
Finally, an actual system may exhibit several of these qualities – one or more complex pairs of eigenvalues, repeated eigenvalues, and distinct real eigenvalues. The Jordan Canonical Form of the matrix for such a system can be analyzed block by block and each of the above solutions applied within each block to determine the evolution matrix for the entire system.

The Main Idea:

Given a system of 1st order linear differential equations $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ with initial conditions $\mathbf{x}(0)$, we use eigenvalue-eigenvector analysis to find an appropriate basis $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ for $\mathbf{R}^n$ and a change of basis matrix $\mathbf{S} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}$ such that in coordinates relative to this basis ($\mathbf{u} = \mathbf{S}^{-1}\mathbf{x}$) the system is in a standard form with a known solution. Specifically, we find a standard matrix $\mathbf{B} = [\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, transform the system into $\frac{d\mathbf{u}}{dt} = \mathbf{B}\mathbf{u}$, solve it as $\mathbf{u}(t) = [e^{t\mathbf{B}}]\mathbf{u}(0)$ where $[e^{t\mathbf{B}}]$ is the ***evolution matrix*** for **B**, then transform back to the original coordinates to get $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$ where $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$ is the ***evolution matrix*** for **B**. That is $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. This is easier to do than it is to explain, so here are a few illustrative examples:

The diagonalizable case

Problem: Solve the system $\left\{ \begin{aligned} \frac{dx}{dt} &= 5x - 6y \\ \frac{dy}{dt} &= 3x - 4y \end{aligned} \right\}$ with initial conditions $x(0) = 3,\ y(0) = 1$.



Solution: In matrix form, we have $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. We start by finding the eigenvalues of the matrix:

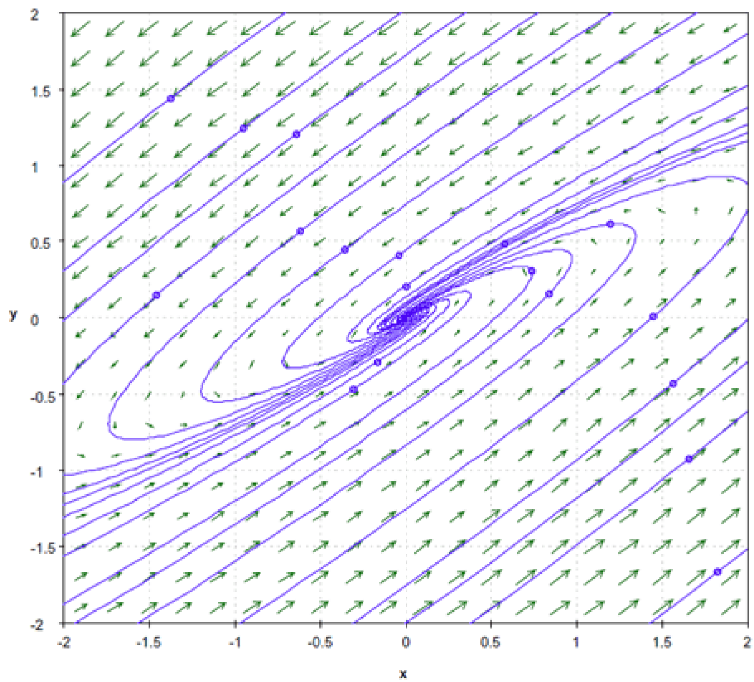
$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 5 & 6 \\ -3 & \lambda + 4 \end{bmatrix}$, and the characteristic polynomial is $p_{\mathbf{A}}(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$. This gives the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. The first of these gives the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and the second gives the eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So we have $\left\{ \begin{aligned} \mathbf{A}\mathbf{v}_1 &= \lambda_1\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 &= \lambda_2\mathbf{v}_2 \end{aligned} \right\}$. The change of basis matrix is $\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and with the new basis (of eigenvectors) $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ we have $[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \mathbf{D}$, a diagonal matrix. [There is no need to carry out the multiplication of the matrices if $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is known to be is a basis of eigenvectors. It will always yield a diagonal matrix with the eigenvalues on the diagonal.]

The evolution matrix for this diagonal matrix is $[e^{t\mathbf{D}}] = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}$, and the solution of the system is:

$$\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)=\mathbf{S}[e^{t\mathbf{D}}]\mathbf{S}^{-1}\mathbf{x}(0)=\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}e^{2t}&0\\0&e^{-t}\end{bmatrix}\begin{bmatrix}1&-1\\-1&2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}2e^{2t}&e^{-t}\\e^{2t}&e^{-t}\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}$$

$$=\begin{bmatrix}4e^{2t}-e^{-t}\\2e^{2t}-e^{-t}\end{bmatrix}=2e^{2t}\begin{bmatrix}2\\1\end{bmatrix}-e^{-t}\begin{bmatrix}1\\1\end{bmatrix}=2e^{2t}\mathbf{v}_1-e^{-t}\mathbf{v}_2$$

The complex eigenvalue case



Problem: Solve the system $\left\{\begin{array}{l}\dfrac{dx}{dt}=2x-5y\\ \dfrac{dy}{dt}=2x-4y\end{array}\right\}$ with initial conditions $x(0)=0,\ y(0)=1$.

Solution: In matrix form, we have $\dfrac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x}$ where $\mathbf{A}=\begin{bmatrix}2&-5\\2&-4\end{bmatrix}$ and $\mathbf{x}(0)=\begin{bmatrix}0\\1\end{bmatrix}$. We again start by finding the eigenvalues of the matrix: $\lambda\mathbf{I}-\mathbf{A}=\begin{bmatrix}\lambda-2&5\\-2&\lambda+4\end{bmatrix}$, and the characteristic polynomial is $p_{\mathbf{A}}(\lambda)=\lambda^2+2\lambda+2=(\lambda+1)^2+1$.

This gives the complex eigenvalue pair $\lambda=-1+i$ and $\bar{\lambda}=-1-i$. We seek a complex eigenvector for the first of these: $\begin{bmatrix}-3+i&5\\-2&3+i\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ gives the (redundant) equations $(-3+i)\alpha+5\beta=0$ and $-2\alpha+(3+i)\beta=0$. The first of these can be written as $5\beta=(3-i)\alpha$, and an easy solution to this is where $\alpha=5,\ \beta=3-i$. (We could also have used the second equation – which is a scalar multiple of the first. The eigenvector might then have been different, but ultimately we'll get the same result.) This gives the complex eigenvector $\mathbf{w}=\begin{bmatrix}5\\3-i\end{bmatrix}=\begin{bmatrix}5\\3\end{bmatrix}+i\begin{bmatrix}0\\-1\end{bmatrix}=\mathbf{u}+i\mathbf{v}$.

We have shown that with the specially chosen basis $\mathcal{B}=\{\mathbf{v},\mathbf{u}\}$, the new system will have standard matrix $[\mathbf{A}]_{\mathcal{B}}=\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\begin{bmatrix}a&-b\\b&a\end{bmatrix}=\mathbf{B}$ where a is the real part of the complex eigenvalue and b is its imaginary part.

We also showed that $[e^{t\mathbf{B}}]=e^{at}\begin{bmatrix}\cos bt&-\sin bt\\\sin bt&\cos bt\end{bmatrix}$. In this example, $a=-1$ and $b=1$, $\mathbf{S}=\begin{bmatrix}\mathbf{v}&\mathbf{u}\end{bmatrix}=\begin{bmatrix}0&5\\-1&3\end{bmatrix}$, $\mathbf{S}^{-1}=\frac{1}{5}\begin{bmatrix}3&-5\\1&0\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}-1&-1\\1&-1\end{bmatrix}$, and $[e^{t\mathbf{B}}]=e^{-t}\begin{bmatrix}\cos t&-\sin t\\\sin t&\cos t\end{bmatrix}$. The solution to the system is therefore:

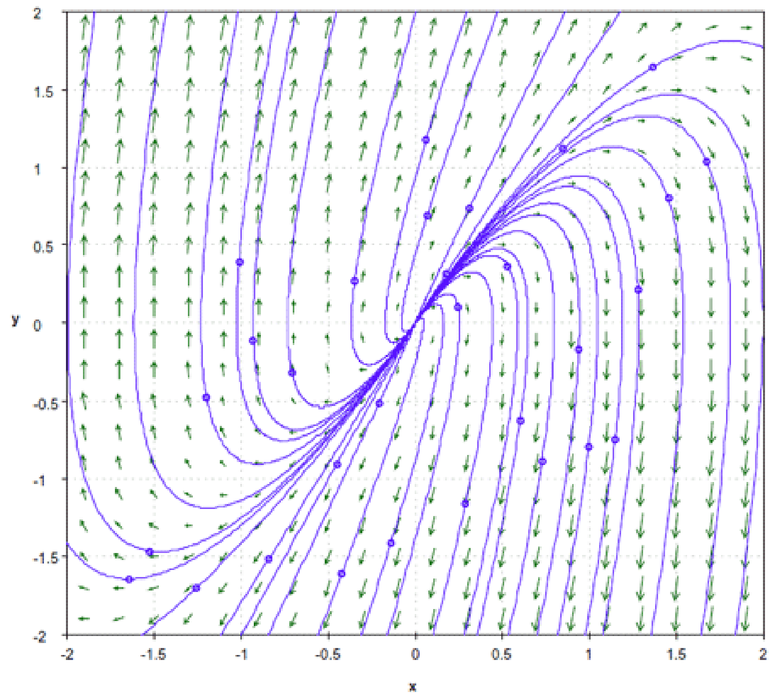
$$\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)=\frac{e^{-t}}{5}\begin{bmatrix}0&5\\-1&3\end{bmatrix}\begin{bmatrix}\cos t&-\sin t\\\sin t&\cos t\end{bmatrix}\begin{bmatrix}3&-5\\1&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$

$$=\frac{e^{-t}}{5}\begin{bmatrix}5\sin t&5\cos t\\-\cos t+3\sin t&\sin t+3\cos t\end{bmatrix}\begin{bmatrix}-5\\0\end{bmatrix}=e^{-t}\begin{bmatrix}-5\sin t\\\cos t-3\sin t\end{bmatrix}.\ \text{That is, }\left\{\begin{array}{l}x(t)=-5e^{-t}\sin t\\y(t)=e^{-t}\cos t-3e^{-t}\sin t\end{array}\right\}.$$

<u>Repeated eigenvalue case</u> [with geometric multiplicity (GM) less than the algebraic multiplicity (AM)]:

Problem: Solve the system $\left\{ \begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -4x + 4y \end{aligned} \right\}$

with initial conditions $x(0) = 3,\ y(0) = 2$.



Solution: In matrix form, we have $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$

where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. We again start by finding the eigenvalues of the matrix: $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & -1 \\ 4 & \lambda - 4 \end{bmatrix}$, and the characteristic polynomial is $p_{\mathbf{A}}(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$.

This gives the repeated eigenvalue $\lambda = 2$ with (algebraic) multiplicity 2. We seek eigenvectors: $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives the (redundant) equations $2\alpha - \beta = 0$ and $4\alpha - 2\beta = 0$. Therefore $\beta = 2\alpha$, so we can choose $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or any scalar multiple of this as an eigenvector, but we are <u>unable to find a second linearly independent eigenvector</u>. (We say that the geometric multiplicity of the $\lambda = 2$ eigenvalue is 1.)

The standard procedure in this case is to seek a ***generalized eigenvector*** for this repeated eigenvalue, i.e. a vector $\mathbf{v}_2$ such that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_2$ is not zero, but rather a multiple of the eigenvector $\mathbf{v}_1$. Specifically, we seek a vector such that $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2$. This translates into seeking $\mathbf{v}_2$ such that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_2 = -\mathbf{v}_1$. That is, $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. This gives redundant equations the first of which is $2\alpha - \beta = -1$ or $\beta = 2\alpha + 1$.

If we (arbitrarily) choose $\alpha = 0$, then $\beta = 1$, so $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The fact that $\left\{ \begin{aligned} \mathbf{A}\mathbf{v}_1 &= 2\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 &= \mathbf{v}_1 + 2\mathbf{v}_2 \end{aligned} \right\}$ tells us that with the change of basis matrix $\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, we will have $[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \mathbf{B}$.

The standard form in this repeated eigenvalue case is a matrix of the form $\mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. (There are analogous forms in cases larger than 2×2 matrices.) Note that we can write $\mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \lambda\mathbf{I} + \mathbf{P}$ where $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

There is a simple relationship between the solutions of the systems $\frac{d\mathbf{x}}{dt} = \mathbf{B}\mathbf{x}$ and $\frac{d\mathbf{u}}{dt} = \mathbf{P}\mathbf{u}$, namely $\mathbf{x}(t) = e^{\lambda t}\mathbf{u}(t)$. This is easily seen by differentiation:

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}[e^{\lambda t}\mathbf{u}(t)] = e^{\lambda t}\frac{d\mathbf{u}}{dt} + \lambda e^{\lambda t}\mathbf{u} = e^{\lambda t}\mathbf{P}\mathbf{u} + \lambda e^{\lambda t}\mathbf{u} = e^{\lambda t}(\mathbf{P}\mathbf{u} + \lambda\mathbf{I}\mathbf{u}) = e^{\lambda t}(\lambda\mathbf{I} + \mathbf{P})\mathbf{u} = (\lambda\mathbf{I} + \mathbf{P})e^{\lambda t}\mathbf{u} = \mathbf{B}\mathbf{x}$$

together with the fact that $\mathbf{x}(0) = \mathbf{u}(0)$.

Furthermore, solving $\frac{d\mathbf{u}}{dt} = \mathbf{Pu}$ is simple. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, then with $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have $\begin{Bmatrix} u_1'(t) = u_2 \\ u_2'(t) = 0 \end{Bmatrix}$.

The second equation gives that $u_2(t) = c_2 = u_2(0)$, a constant. The first equation is then $u_1'(t) = u_2(0)$, so $u_1(t) = u_2(0)\cdot t + c_1$. At $t = 0$ this gives $u_1(0) = c_1$, so $u_1(t) = u_1(0) + u_2(0)\cdot t$. Together this gives:

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} u_1(0) + u_2(0)\cdot t \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\mathbf{u}(0) = \left[e^{t\mathbf{P}}\right]\mathbf{u}(0)$$

Therefore $\mathbf{x}(t) = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\mathbf{x}(0) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\mathbf{x}(0)$, so $\left[e^{t\mathbf{B}}\right] = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$ for $\mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

If we apply this to the problem at hand, we get $[e^{t\mathbf{B}}] = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$. The solution to the system is therefore

$$\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & 2te^{2t} + e^{2t} \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{2t} - 4te^{2t} \\ 6e^{2t} - 8te^{2t} - 4e^{2t} \end{bmatrix} = \begin{bmatrix} 3e^{2t} - 4te^{2t} \\ 2e^{2t} - 8te^{2t} \end{bmatrix} = e^{2t}\begin{bmatrix} 3 - 4t \\ 2 - 8t \end{bmatrix}$$. That is, $\begin{Bmatrix} x(t) = e^{2t}(3 - 4t) \\ y(t) = e^{2t}(2 - 8t) \end{Bmatrix}$.

It's worth noting that this can also be expressed as $\mathbf{x}(t) = e^{2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} - 4te^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The phase portrait in this case has just one invariant (eigenvector) direction. It gives an unstable **node** which can be viewed as a degenerate case of a (clockwise) outward spiral that cannot get past the eigenvector direction.

Moral of the Story: It's always possible to find a special basis relative to which a given linear system is in its simplest possible form. The new basis provides a way to decompose the given problem into several simple, standard problems which can be easily solved. Any complication in the algebraic expressions for the solution is the result of changing back to the original coordinates.

The standard 2×2 cases are:

Diagonalizable with eigenvalues λ_1, λ_2: $\mathbf{B} = \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $[e^{t\mathbf{B}}] = [e^{t\mathbf{D}}] = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$

Complex pair of eigenvalues $\lambda = a \pm ib$: $\mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $[e^{t\mathbf{B}}] = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$

Repeated eigenvalue λ **with** $\mathrm{GM} < \mathrm{AM}$: $\mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $[e^{t\mathbf{B}}] = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

In general, you should expect to encounter systems more complicated than these 2×2 examples. To illustrate the line of reasoning in a significantly more complicated case, here is a **Big Problem**.

Big Problem: a) Find the general solution for the following system of differential equations:

$$\begin{cases} \dfrac{dx_1}{dt} = 2x_1 - 4x_4 + 3x_5 \\ \dfrac{dx_2}{dt} = 2x_2 - 2x_3 + 2x_4 \\ \dfrac{dx_3}{dt} = x_2 - x_4 \\ \dfrac{dx_4}{dt} = -x_4 \\ \dfrac{dx_5}{dt} = -3x_4 + 2x_5 \end{cases}$$

b) Find the solution in the case where $\mathbf{x}(0) = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$.

Solution: This is a continuous dynamical system of the form $\dfrac{d\mathbf{x}}{dt} = \mathbf{Ax}$ where $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & -4 & 3 \\ 0 & 2 & -2 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 2 \end{bmatrix}$.

We start by seeking the eigenvalues. We have $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda-2 & 0 & 0 & 4 & -3 \\ 0 & \lambda-2 & 2 & -2 & 0 \\ 0 & -1 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda+1 & 0 \\ 0 & 0 & 0 & 3 & \lambda-2 \end{bmatrix}$.

The characteristic polynomial is $p_{\mathbf{A}}(\lambda) = (\lambda-2)^2(\lambda+1)(\lambda^2 - 2\lambda + 2)$ which yields the repeated eigenvalue $\lambda_1 = \lambda_2 = 2$ (with algebraic multiplicity 2), the distinct eigenvalue $\lambda_3 = -1$, and the complex pair $\lambda_4 = 1 + i$ and $\lambda_5 = \overline{\lambda}_4 = 1 - i$.

The repeated eigenvalue $\lambda_1 = \lambda_2 = 2$ yields just one eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, so its geometric multiplicity if just 1.

We then seek a "generalized eigenvector" $\mathbf{v}_2$ such that $\mathbf{Av}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2$ where $\lambda = 2$. That is, we seek a vector $\mathbf{v}_2$ such that $\lambda\mathbf{v}_2 - \mathbf{Av}_2 = (\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_2 = -\mathbf{v}_1$. This is just an inhomogeneous system which yields solutions of the form $\mathbf{v}_2 = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}$. For simplicity, take the solution with $t = 0$, i.e. $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}$.

The eigenvalue $\lambda_3 = -1$ yields the eigenvector $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \\ 3 \end{bmatrix}$. A straightforward calculation with the complex eigenvalue $\lambda_4 = 1+i$ yields the complex eigenvector $\mathbf{v} = \begin{bmatrix} 0 \\ 1+i \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{v}_5 + i\,\mathbf{v}_4$ in accordance with the method previously derived.

Using the basis $\mathcal{B} = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ and change of basis matrix

$\mathbf{S} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 1/3 & 3 & 0 & 0 \end{bmatrix}$, we compute the inverse matrix $\mathbf{S}^{-1} = \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$.

We know that $\begin{Bmatrix} \mathbf{A}\mathbf{v}_1 = 2\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + 2\mathbf{v}_2 \\ \mathbf{A}\mathbf{v}_3 = -\mathbf{v}_3 \\ \mathbf{A}\mathbf{v}_4 = \mathbf{v}_4 + \mathbf{v}_5 \\ \mathbf{A}\mathbf{v}_5 = -\mathbf{v}_4 + \mathbf{v}_5 \end{Bmatrix}$, so the matrix of **A** relative to the basis $\mathcal{B}$ is

$$\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \left[\begin{array}{cc:c:cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & -1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

Since $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$, it will be the case that the evolution matrices are related via $\left[e^{t\mathbf{A}}\right] = \mathbf{S}\left[e^{t\mathbf{B}}\right]\mathbf{S}^{-1}$ where

$$\left[e^{t\mathbf{B}}\right] = \left[\begin{array}{cc:c:cc} e^{2t} & te^{2t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 & 0 \\ \hdashline 0 & 0 & e^{-t} & 0 & 0 \\ \hdashline 0 & 0 & 0 & e^t\cos t & -e^t\sin t \\ 0 & 0 & 0 & e^t\sin t & e^t\cos t \end{array}\right].$$

The solution is then

$$\mathbf{x}(t) = \left[e^{t\mathbf{A}}\right]\mathbf{x}(0) = \mathbf{S}\left[e^{t\mathbf{B}}\right]\mathbf{S}^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 1/3 & 3 & 0 & 0 \end{bmatrix} \left[\begin{array}{cc:c:cc} e^{2t} & te^{2t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 & 0 \\ \hdashline 0 & 0 & e^{-t} & 0 & 0 \\ \hdashline 0 & 0 & 0 & e^t\cos t & -e^t\sin t \\ 0 & 0 & 0 & e^t\sin t & e^t\cos t \end{array}\right] \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \mathbf{x}(0).$$

If we multiply the leftmost matrices and write $\mathbf{S}^{-1}\mathbf{x}(0)=\begin{bmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{bmatrix}$, this yields the general solution:

$$\mathbf{x}(t)=\left[e^{t\mathbf{A}}\right]\mathbf{x}(0)=\mathbf{S}\left[e^{t\mathbf{B}}\right]\mathbf{S}^{-1}\mathbf{x}(0)=\begin{bmatrix}e^{2t} & te^{2t} & e^{-t} & 0 & 0\\ 0 & 0 & 0 & e^t(\cos t+\sin t) & e^t(\cos t-\sin t)\\ 0 & 0 & 3e^{-t} & e^t\sin t & e^t\cos t\\ 0 & 0 & 3e^{-t} & 0 & 0\\ 0 & \frac{1}{3}e^{2t} & 3e^{-t} & 0 & 0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{bmatrix}$$

$$\text{or }\left\{\begin{array}{l}x_1(t)=c_1e^{2t}+c_2te^{2t}+c_3e^{-t}\\ x_2(t)=c_4e^t(\cos t+\sin t)+c_5e^t(\cos t-\sin t)\\ x_3(t)=3c_3e^{-t}+c_4e^t\sin t+c_5e^t\cos t\\ x_4(t)=3c_3e^{-t}\\ x_5(t)=\frac{1}{3}c_2e^{2t}+3c_3e^{-t}\end{array}\right\}.$$

If, on the other hand, we use the initial condition $\mathbf{x}(0)=\begin{bmatrix}5\\4\\3\\2\\1\end{bmatrix}$, we get the specific solution:

$$\mathbf{x}(t)=\begin{bmatrix}e^{2t} & te^{2t} & e^{-t} & 0 & 0\\ 0 & 0 & 0 & e^t(\cos t+\sin t) & e^t(\cos t-\sin t)\\ 0 & 0 & 3e^{-t} & e^t\sin t & e^t\cos t\\ 0 & 0 & 3e^{-t} & 0 & 0\\ 0 & \frac{1}{3}e^{2t} & 3e^{-t} & 0 & 0\end{bmatrix}\begin{bmatrix}13/3\\-3\\2/3\\3\\1\end{bmatrix}$$

$$\text{or }\left\{\begin{array}{l}x_1(t)=\frac{13}{3}e^{2t}-3te^{2t}+\frac{2}{3}e^{-t}\\ x_2(t)=e^t(4\cos t+2\sin t)\\ x_3(t)=2e^{-t}+e^t(3\sin t+\cos t)\\ x_4(t)=2e^{-t}\\ x_5(t)=-e^{2t}+2e^{-t}\end{array}\right\}.$$

Notes by Robert Winters