Matrix of a linear transformation relative to a preferred basis

(and how you might deal with repeated eigenvalues and complex eigenvalues)

The question of whether a matrix $\mathbf{A}$ can be diagonalized can be described succinctly as follows:

Can a basis of eigenvectors be found for the given matrix $\mathbf{A}$?

If the answer is *Yes*, then there is a basis of eigenvectors $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ with eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ such that $\mathbf{A}\mathbf{v}_k = \lambda_k \mathbf{v}_k$ for k = 1,2,...,n. It is a useful construction to talk about the idea of the <u>matrix of a linear transformation relative to an alternate basis $\mathcal{B}$</u>. Simply put, a matrix $\mathbf{A}$ tells us how to calculate for any given vector expressed in terms of the <u>standard basis</u> $\{e_1, e_2, ..., e_n\}$ its image relative to the <u>standard basis</u>.

If we were to use an alternate basis $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ to express the coordinates of both a vector and its image under the linear transformation corresponding to the matrix $\mathbf{A}$, what would the new matrix $[\mathbf{A}]_{\mathcal{B}}$ look like? To do this, we define the matrix $\mathbf{P} = [\mathbf{v}_1\ \mathbf{v}_2\ ...\ \mathbf{v}_n]$, i.e. we insert the basis vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ as the columns of the matrix $\mathbf{P}$. This is known as a <u>change of basis matrix</u> because of the fact that $\mathbf{P}\mathbf{e}_k = \mathbf{v}_k$ for k = 1,2,...,n.

We have shown that for any vector $\mathbf{x}$, its coordinate vector relative to the basis $\mathcal{B}$ is given by $[\mathbf{x}]_{\mathcal{B}} = \mathbf{P}^{-1}\mathbf{x}$. The coordinate vector of its image is similarly given by $[\mathbf{Ax}]_{\mathcal{B}} = \mathbf{P}^{-1}\mathbf{Ax}$. Since $\mathbf{x} = \mathbf{P}[\mathbf{x}]_{\mathcal{B}}$, we have that $[\mathbf{Ax}]_{\mathcal{B}} = \mathbf{P}^{-1}\mathbf{AP}[\mathbf{x}]_{\mathcal{B}}$.
In other words, $[\mathbf{A}]_{\mathcal{B}} = \mathbf{P}^{-1}\mathbf{AP}$. Two matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$ for some invertible matrix $\mathbf{P}$ are called *similar*. It is generally quite easy, if the basis is chosen in a sensible way, to calculate $[\mathbf{A}]_{\mathcal{B}}$ <u>column-by-column</u> by directly expressing each of the vectors $\mathbf{Av}_1, ..., \mathbf{Av}_n$ in terms of the basis vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$.
As an example, in the case of a diagonalizable matrix $\mathbf{A}$, we were able to find a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ such that $\mathbf{A}\mathbf{v}_k = \lambda_k\mathbf{v}_k$. What this says is:

$$\begin{aligned}
\mathbf{Av}_1 &= \lambda_1\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_{n-1} + 0\mathbf{v}_n \\
\mathbf{Av}_2 &= 0\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + 0\mathbf{v}_{n-1} + 0\mathbf{v}_n \\
&\vdots \\
&\vdots \\
\mathbf{Av}_n &= 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_{n-1} + \lambda_n\mathbf{v}_n
\end{aligned}$$

The k-th equation says that the k-th column of $[\mathbf{A}]_{\mathcal{B}}$ is $\begin{bmatrix} 0 \\ \vdots \\ \lambda_k \\ \vdots \\ 0 \end{bmatrix}$.

We see <u>directly</u> that the matrix $[\mathbf{A}]_{\mathcal{B}} = \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & \vdots & 0 \\ \vdots & 0 & \lambda_k & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix}$,

which coincides with our previous knowledge that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$. This diagonal form allows us to calculate and describe powers of the matrix in a very simple way, a way in which, except for a coordinate change, the eigenvectors and eigenvalues tell the entire story.

The fact is that not all matrices can be diagonalized. The reason is that the eigenvalues of a given matrix, given by the roots of the characteristic polynomial of the matrix, need not be distinct nor must they necessarily be real. We can certainly have repeated roots and complex eigenvalues. Though it sometimes happens that we <u>can</u> find a complete basis of eigenvectors even though some eigenvalues are repeated, this is usually not the case. In the case of complex eigenvalues we certainly cannot find any corresponding *real* eigenvectors. The question in these cases then comes down to this:

Can we find a basis consisting of eigenvectors and other sensibly chosen vectors such that the matrix of **A** *relative to this basis takes on an especially simple (also called a canonical, or normal) form*?

Repeated eigenvalues:

When the algebraic multiplicity k of an eigenvalue λ of **A** (the number of times λ occurs as a root of the characteristic polynomial) is greater than 1, we usually are not able to find k linearly independent eigenvectors corresponding to this eigenvalue. We use the term *geometric multiplicity* for the number of linearly independent eigenvectors corresponding to a given eigenvalue, i.e. the dimension of the kernel of the matrix $(\lambda\mathbf{I} - \mathbf{A})$.

The next best thing to an eigenvector in this case is a vector **v** such that $\mathbf{Av} = \lambda\mathbf{v} + \mathbf{u}$ where **u** *is* an eigenvector with eigenvalue λ. It is easy to show that any such a vector **v** is in the kernel of the matrix $(\lambda\mathbf{I} - \mathbf{A})^2$. If this still leaves us short on basis vectors corresponding to the eigenvalue λ, we can continue by looking for a vector **w** such that $\mathbf{Aw} = \lambda\mathbf{w} + \mathbf{v}$. It will then be the case that **w** is in the kernel of the matrix $(\lambda\mathbf{I} - \mathbf{A})^3$.

It can be shown that this process will always yield k linearly independent vectors corresponding to the eigenvalue λ, the first few vectors of which will be actual eigenvectors of **A**. If a matrix **A** has all real eigenvalues and if we carry out this process for all eigenvalues of **A**, we'll produce a complete basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ where we assume that all vectors corresponding to a given eigenvalue are grouped together and ordered in the way in which they were found.

For example, let **A** be 10×10 matrix with nonrepeating eigenvalues λ_1, λ_2, and λ_3; with eigenvalue λ_4 of multiplicity 3 with only one eigenvector; and with eigenvalue λ_5 with multiplicity 4 with just two linearly independent eigenvectors. It is easy to see that the matrix of **A** relative to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{10}\}$where $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ correspond, respectively, to λ_1, λ_2, and λ_3; $\mathbf{v}_4$, $\mathbf{v}_5$, and $\mathbf{v}_6$ correspond to λ_4; and where $\mathbf{v}_7$, $\mathbf{v}_8$, $\mathbf{v}_9$, and $\mathbf{v}_{10}$ correspond to λ_5; is of the form shown in the frame on the following page, where **P** is the change of basis matrix.

$$[\mathbf{A}]_{\mathcal{B}} = \mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix}$$

If we arrange things so that, for example, the eigenvalues are listed in increasing order, the resulting matrix is called the **Jordan canonical form** of the matrix. It follows that any matrix **A** with all real eigenvalues is similar to a matrix in Jordan canonical form, with **Jordan blocks** (as indicated by the dotted lines) associated with each eigenvalue. It can be further shown that if **A** and **B** are *similar* matrices, they necessarily have the same characteristic polynomials, the same eigenvalues with the same algebraic and geometric multiplicities, and hence the same Jordan canonical forms. In other words, they represent the "same" linear transformation relative to two different bases.

Example: Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & -2 & 3 \end{bmatrix}$. The characteristic polynomial is $\lambda^3 - 5\lambda^2 + 8\lambda - 4$ and the eigenvalues are $\lambda = 1,2,2$. The eigenvalue $\lambda = 1$ yields the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and the repeated eigenvalue $\lambda = 2$ yields the single eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Following the procedure outlined earlier, we can find a third basis vector $\mathbf{v}_3$ such that $\mathbf{A}\mathbf{v}_3 = 2\mathbf{v}_3 + \mathbf{v}_2$. One such vector is the vector $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$. Using the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and the matrix

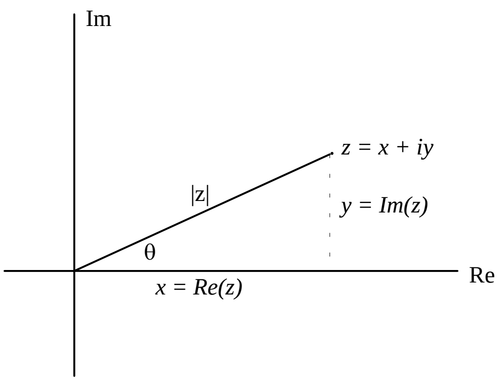
$\mathbf{P} = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$, we have $[\mathbf{A}]_{\mathcal{B}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. If, for example, we need to calculate $\mathbf{A}^n$, we will have that $\mathbf{A} = \mathbf{P}[\mathbf{A}]_{\mathcal{B}}\mathbf{P}^{-1}$ and $\mathbf{A}^n = \mathbf{P}([\mathbf{A}]_{\mathcal{B}})^n\,\mathbf{P}^{-1}$. Since $[\mathbf{A}]_{\mathcal{B}}$ is composed of Jordan blocks and since $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$, it is easy to show that $\mathbf{A}^n = \mathbf{P}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & n2^{n-1} \\ 0 & 0 & 2^n \end{bmatrix}\mathbf{P}^{-1}$.

Complex eigenvalues

This situation is similar to the previous one in the sense that if the matrix **A** has any complex eigenvalues, we will not be able to find a basis of *real* eigenvectors. For all the real eigenvalues, everything is exactly the same as above. What we need to do is to produce additional vectors associated with any complex eigenvalues in order to get a basis relative to which the matrix **A** has a simple, canonical form. In order to do this, we have to temporarily wander off into the world of complex numbers, complex eigenvalues, and complex eigenvectors.

You should not attempt to visualize a vector whose components are complex numbers. This is merely an algebraically consistent extension of the idea of real vectors and real matrices where all the rules of linear algebra are still in effect. This temporary excursion will yield real vectors relative to which the matrix acts in an easy-to-describe fashion, namely as a rotation-dilation, i.e. it rotates vectors in a 2-dimensional (invariant) subspace and scales them by the modulus of the complex eigenvalue.

First, we need a few basic definitions associated with complex numbers. A complex number $z = x + iy$, where $i^2 = -1$ can be viewed in vector-like terms in the complex plane as shown in this diagram to the right. We define:



$$\text{modulus}(z) = \text{mod}(z) = |z| = \sqrt{x^2 + y^2}$$

$$\text{argument}(z) = \arg(z) = \theta = \tan^{-1}\left(\tfrac{y}{x}\right).$$

We add complex numbers by adding their respective real and imaginary parts, in much the same way as vector addition was defined. We multiply complex numbers via the distributive law and the fact that $i^2 = -1$. For example:

$$(3 + 2i)(-1 - 4i) = -3 - 2i - 12i - 8i^2 = -3 - 14i + 8 = 5 - 14i.$$

If we note that $x = |z| \cos \theta$ and $y = |z| \sin \theta$, then we can write $z = |z|(\cos \theta + i \sin \theta)$. A short calculation shows that when we multiply two complex numbers, we multiply their moduli and add their arguments. You may want to try this out with some simple complex numbers to convince yourself of this fact.

The *complex conjugate* of $z = x + iy$ is defined to be $\bar{z} = x - iy$. In the complex plane, z and $\bar{z}$ are reflections of each other across the real axis. It's not hard to show that $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$.

The Fundamental Theorem of Algebra guarantees that, at least in theory, any polynomial of degree n can be factored into n linear factors and will therefore produce n roots. Some roots may have multiplicity greater that 1 and some of the roots may be complex. It is also the case that for a polynomial with all real coefficients, any complex roots will necessarily occur in complex conjugate pairs λ and $\bar{\lambda}$.

Let **A** be a matrix which has a complex conjugate pair of eigenvalues λ and $\bar{\lambda}$. We can proceed just as in the case of real eigenvalues and find a complex vector **v** such that $(\lambda \mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$. The components of such a vector **v** will have complex numbers for its components. If we write $\lambda = a + ib$, and decompose **v** into its real and imaginary vector components as $\mathbf{v} = \mathbf{x} + i\mathbf{y}$, we can calculate that:

(1) $$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} = (a + ib)(\mathbf{x} + i\mathbf{y}) = (a\mathbf{x} - b\mathbf{y}) + i(b\mathbf{x} + a\mathbf{y})$$

If we define the vector $\bar{\mathbf{v}} = \mathbf{x} - i\mathbf{y}$, we see, using the easy to prove fact that for a matrix **A** with all real entries we'll have $\overline{\mathbf{A}\mathbf{v}} = \mathbf{A}\bar{\mathbf{v}}$:

(2) $$\overline{\mathbf{A}\mathbf{v}} = \mathbf{A}\bar{\mathbf{v}} = \bar{\lambda}\,\bar{\mathbf{v}} = (a - ib)(\mathbf{x} - i\mathbf{y}) = (a\mathbf{x} - b\mathbf{y}) - i(b\mathbf{x} + a\mathbf{y})$$

Note that this gives us that the complex vector $\bar{\mathbf{v}}$ is actually also a complex eigenvector corresponding to the complex conjugate eigenvalue $\bar{\lambda}$. The true value of this excursion into the world of complex numbers and complex vectors is seen when we add and subtract equation (1) and (2). We get:

$$\mathbf{A}(\mathbf{v} + \bar{\mathbf{v}}) = 2(a\mathbf{x} - b\mathbf{y})$$

$$\mathbf{A}(\mathbf{v} - \bar{\mathbf{v}}) = 2i(b\mathbf{x} + a\mathbf{y})$$

If we further note that $\mathbf{v} + \bar{\mathbf{v}} = 2\mathbf{x}$ and $\mathbf{v} - \bar{\mathbf{v}} = 2i\mathbf{y}$, we get, after cancellation of the factors of 2 and $2i$ in the respective equations,

$$\mathbf{Ay} = a\mathbf{y} + b\mathbf{x}$$

$$\mathbf{Ax} = -b\mathbf{y} + a\mathbf{x}$$

Note that we are now back in the "real world": all vectors and scalars in the above equations are real. If we use the two vectors **y** and **x** as basis vectors associated with the two complex conjugate eigenvalues, grouped together in the full basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$, we'll produce a 2×2 Jordan block in the matrix $[\mathbf{A}]_{\mathcal{B}}$ of the form:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2}\begin{bmatrix} a/\sqrt{a^2+b^2} & -b/\sqrt{a^2+b^2} \\ b/\sqrt{a^2+b^2} & a/\sqrt{a^2+b^2} \end{bmatrix} = |\lambda|\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = |\lambda|\mathbf{R}_\theta$$

where $\mathbf{R}_\theta$ is the rotation matrix corresponding to the angle $\theta = \arg(\lambda)$.

In other words, the Jordan block associated with the basis vectors {**y**, **x**} is a rotation-dilation matrix where the angle of rotation is the same as the angle of the complex eigenvalue and where the scaling factor is just the modulus (magnitude) of the complex eigenvalue. Again, the very nature of the complex eigenvalues tells us much about the way the matrix acts, at least if we choose the right basis with which to view things.

Example: Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. You'll recognize this as the matrix corresponding to counterclockwise rotation in the plane through an angle of 90°. The characteristic polynomial is $\lambda^2 + 1 = 0$, with complex eigenvalues $\lambda = \pm i$. Note that with $\lambda = i$, we have $\arg(\lambda) = 90°$ and modulus$(\lambda) = 1$. The preceding discussion says that this matrix is similar to a rotation-dilation matrix which does no scaling and which rotates by an angle of 90°. But this should come as no surprise at all. The given matrix is already in the form of exactly this rotation-dilation matrix, i.e. Jordan form.

Example: Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$. We have $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda-2 & 1 \\ -3 & \lambda-2 \end{bmatrix}$ and the characteristic polynomial is $\lambda^2 - 4\lambda + 7 = 0$. This yield the two eigenvalues $\lambda = 2 + i\sqrt{3}$ and $\bar{\lambda} = 2 - i\sqrt{3}$.

If we substitute λ into $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda-2 & 1 \\ -3 & \lambda-2 \end{bmatrix}$, we get that if $\mathbf{v} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is to be an eigenvector, we must have $\begin{bmatrix} i\sqrt{3} & 1 \\ -3 & i\sqrt{3} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This means that $(i\sqrt{3})\alpha + \beta = 0$. (The second equation is redundant, even though this might not immediately appear to be the case.) One choice for α and β is $\alpha = 1$, $\beta = -i\sqrt{3}$. This gives us the complex eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ -i\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ -\sqrt{3} \end{bmatrix} = \mathbf{x} + i\mathbf{y}$. Using $\mathcal{B} = \{\mathbf{y}, \mathbf{x}\}$ as a basis, and calling $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -\sqrt{3} & 0 \end{bmatrix}$, we have that

$$[\mathbf{A}]_{\mathcal{B}} = \mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 2 & -\sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix} = \sqrt{7}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = |\lambda|\mathbf{R}_\theta$$

where $\mathbf{R}_\theta$ is the rotation matrix corresponding to the angle $\theta = \arg(\lambda) = \tan^{-1}(\sqrt{3}/2) \cong 40.89°$.

If we have need to consider the powers $\mathbf{A}^n$, we'll have that $\mathbf{A} = \mathbf{P}[\mathbf{A}]_{\mathcal{B}}\mathbf{P}^{-1}$ and $\mathbf{A}^n = \mathbf{P}([\mathbf{A}]_{\mathcal{B}})^n\,\mathbf{P}^{-1}$.

Since $([\mathbf{A}]_{\mathcal{B}})^n = (|\lambda|\mathbf{R}_\theta)^n = |\lambda|^n\,\mathbf{R}_{n\theta}$, we have that $\mathbf{A}^n = \mathbf{P}(|\lambda|^n\,\mathbf{R}_{n\theta})\mathbf{P}^{-1}$. In other words, except for the change of basis, $\mathbf{A}^n$ corresponds to rotation through the angle $n\theta$ and scaling by the factor $|\lambda|^n$.

Example: Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. This matrix permutes the standard basis vectors (and hence the coordinate axes) taking the *x*-axis to the *y*-axis, the *y*-axis to the *z*-axis, and the *z*-axis to the *x*-axis.

The characteristic polynomial for this matrix is $\lambda^3 - 1 = (\lambda - 1)(\lambda^2 + \lambda + 1) = 0$. This gives the three eigenvalues $\mu = 1$, $\lambda = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $\overline{\lambda} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. These eigenvalues all have modulus = 1, one real eigenvalue and a complex conjugate pair with arguments ±120°. They are equally spaced on the unit circle in the complex plane.

The eigenvalue $\mu = 1$ gives the eigenvector $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, which is, in fact, the axis of rotation for this linear transformation.

The eigenvalue $\lambda = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ gives:

$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 & -1 \\ -1 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 \\ 0 & -1 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix}$ and the eigenvector $\mathbf{v} = \begin{bmatrix} 2 \\ -1 - i\sqrt{3} \\ -1 + i\sqrt{3} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ -\sqrt{3} \\ \sqrt{3} \end{bmatrix} = \mathbf{x} + i\mathbf{y}$.

If we use the basis $\mathcal{B} = \{\mathbf{u}, \mathbf{y}, \mathbf{x}\}$ and let $\mathbf{P} = [\mathbf{u}\ \ \mathbf{y}\ \ \mathbf{x}]$, we get:

$$[\mathbf{A}]_{\mathcal{B}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \left[\begin{array}{c:cc} 1 & 0 & 0 \\ \hdashline 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{array}\right] = \left[\begin{array}{c:cc} 1 & 0 & 0 \\ \hdashline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{R}_{120°}$}} \\ 0 & & \end{array}\right]$$

In this form, we see exactly the 120° rotation associated with this matrix. Furthermore, the vectors $\{\mathbf{y}, \mathbf{x}\}$ are a basis for the plane perpendicular to the axis of rotation, a plane that remains invariant under this transformation. This is much like the subspace spanned by a real eigenvector, which is a fixed direction. For a typical 3 by 3 matrix with one real eigenvalue and a pair of complex conjugate eigenvalues, the invariant direction corresponding to the real eigenvalue need not be perpendicular to the rotational plane associated with a complex conjugate pair of eigenvalues.