Supplement on Elementary Row Operations, Row Spaces, and Reduced Row-Echelon Form (RREF)

Much of the focus in this Linear Algebra course has been on the columns of a matrix, but we actually began with the focus on the rows of a matrix, the three classes of elementary row operations (row swaps, row scalings, and adding a multiple of one row to another), and reduced row-echelon form of a matrix.

There are several important facts that deserve further elaboration. Given an $m \times n$ matrix $\mathbf{A} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}$, we know that $im(\mathbf{A}) = span\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ which is why the image of a matrix is also known as its **column space**, a subspace of its codomain $\mathbf{R}^m$.

We can also think of a matrix in terms of its rows, i.e. $\mathbf{A} = \begin{bmatrix} \leftarrow & \mathbf{w}_1 & \rightarrow \\ & \vdots & \\ \leftarrow & \mathbf{w}_m & \rightarrow \end{bmatrix}$. If we do so, we can consider the span of its rows, i.e. $\text{span}\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$, a subspace of its domain $\mathbf{R}^n$ called its **row space**, denoted by $row(\mathbf{A})$. It's relatively easy to see that even though the elementary row operations will produce matrices with altered rows, they will not change its row space, i.e. the rows of the altered matrices will continue to span the same subspace (see below). Indeed, if we go through a sequence of row operations to obtain the reduced row-echelon form of a given matrix $\mathbf{A}$, then $\boxed{row(RREF(\mathbf{A})) = row(\mathbf{A})}$.

We have shown that $\boxed{[im(\mathbf{A})]^{\perp} = ker(\mathbf{A}^{\mathrm{T}})}$ and it's easy to show from this (by simply considering the transpose) that $\boxed{[im(\mathbf{A}^{\mathrm{T}})]^{\perp} = ker(\mathbf{A})}$ and therefore $\boxed{im(\mathbf{A}^{\mathrm{T}}) = [ker(\mathbf{A})]^{\perp}}$. If we translate these statements in terms of row spaces, we can observe that $\boxed{[im(\mathbf{A}^{\mathrm{T}})]^{\perp} = [row(\mathbf{A})]^{\perp} = ker(\mathbf{A})}$ and $\boxed{im(\mathbf{A}^{\mathrm{T}}) = row(\mathbf{A}) = [ker(\mathbf{A})]^{\perp}}$. Said differently, $row(\mathbf{A})$ and $ker(\mathbf{A})$ are (orthogonally) complementary subspaces of the domain; and $im(\mathbf{A})$ and $ker(\mathbf{A}^{\mathrm{T}})$ are complementary subspaces of the codomain. These four spaces are sometimes referred to as the "***The Four Fundamental Subspaces***" associated with a given matrix $\mathbf{A}$.

Definition: We say that two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are ***row equivalent*** if $\mathbf{B}$ can be obtained from $\mathbf{A}$ via a sequence of elementary row operations.

Note: The elementary row operations are invertible linear transformations from the space $\mathbf{R}^{m \times n}$ of $m \times n$ matrices to itself which transform one matrix to a row equivalent matrix.

A ***row swap*** of the *i*-th and *j*-th rows is executed by multiplying on the left by the invertible $m \times m$ matrix obtained from the Identity matrix by deleting the 1's in the *i*-th and *j*-th rows and inserting 1's in the (*i*,*j*) and (*j*,*i*) entry (a permutation matrix).

A ***row scaling*** by of the *i*-th row by factor *r* is executed by multiplying on the left by the invertible $m \times m$ matrix obtained from the Identity matrix by replacing the 1 in the *i*-th row by *r*.

Adding a multiple of the j-th row **(with factor *r*) to the *i*-th row** is executed by multiplying on the left by the invertible $m \times m$ matrix obtained from the Identity matrix by replacing the 0 in the (*i*,*j*) by *r*.

In other words, each of the elementary row operations can be executed by left-multiplication by one of these elementary matrices, and each can be reversed by left-multiplication by the inverse of the respective elementary matrix.

Thus, transforming a matrix $\mathbf{A}$ to $RREF(\mathbf{A})$ can be described as $RREF(\mathbf{A}) = \mathbf{E}_N \mathbf{E}_{N-1} \cdots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A}$ where each $\mathbf{E}_k$ is an invertible elementary matrix associated with the respective elementary row operation. There are, of course, many possible sequences of row operations that might be used to obtain $RREF(\mathbf{A})$, but why do all roads lead to the same destination?

Note: If two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are ***row equivalent***, then their row vectors span the same subspace of the domain. That is, if $\mathbf{A}$ and $\mathbf{B}$ are row equivalent then $row(\mathbf{A}) = row(\mathbf{B})$.

Row equivalence is an *equivalence relation*, i.e. it is reflexive ($\mathbf{A}$ is row equivalent to itself), symmetric (if $\mathbf{A}$ is equivalent to $\mathbf{B}$, then $\mathbf{B}$ is equivalent to $\mathbf{A}$), and transitive (if $\mathbf{A}$ is equivalent to $\mathbf{B}$ and $\mathbf{B}$ is equivalent to $\mathbf{C}$, then $\mathbf{A}$ is equivalent to $\mathbf{C}$). This enables us to think of the space of all $m \times n$ matrices as being composed of separate *equivalent classes*. The elementary row operations transform matrices in one equivalence class into matrices in the same equivalence class, so the assertion that the reduced row-echelon form of a matrix is unique is really the same thing as saying that each equivalence class has a unique *canonical element* that is the reduced row-echelon form of every element in that equivalence class.

Uniqueness of Reduced Row-Echelon Form of a Matrix (by Omar Antolín Camarena)

In this short note, I'll define reduced row-echelon form explicitly and explain why it is unique, that is, why row-reduction is guaranteed to produce the same answer no matter how you go about it.

A matrix is in RREF if:

1. All the rows consisting entirely of zeros are at the bottom.
2. In each non-zero row, the leftmost non-zero entry is a 1. These are called the leading ones.
3. The column of each leading one is "clean", that is all other entries in the column are 0.

Now, why is there only one reduced row-echelon form for any given matrix?
To explain this we'll need the following simple but important fact about elementary row operations:

- If you can get from **A** to **B** by using elementary row operations, then each row of **B** is a linear combination of rows of **A**.

(This is simply because each elementary row operation replaces a row by a linear combination of some rows.)

Also, each elementary row operation is invertible so that we can "run row-reduction backwards". This means that if we can reduce some matrix **M** to two different RREF matrices **A** and **B**, then we can also start with **A**, perform operations to get to **M** and then reduce to **B**. By the same token we can get from **B** to **A** by using row operations. Therefore it is enough to show that:

- If **A** and **B** are both RREF and each row of one of them is a linear combination of the rows of the other, then necessarily $\mathbf{A} = \mathbf{B}$.

Let's argue that now. First we show that **A** and **B** must have leading ones in the same positions. Start with the first row: say **A**'s first row has a leading one in column number *i*, while **B**'s first row has it in column number *j*. Then *i* can't be bigger than *j*: if it were, no linear combination of rows of **A** can have a non-zero entry in column *j*. Similarly, *i* can't be less than j, and so they must be equal.

Now, we know that each row of **B** is a linear combination of rows of **A**. The linear combination for the first row of **B** definitely uses (by "uses" I mean "uses with a non-zero coefficient") the first row of **A**, otherwise it couldn't have a non-zero entry in column $i = j$; but the linear combinations for all other rows of **B** cannot possibly use the first row of **A**, otherwise they'd have some non-zero entry in column $i = j$, which is impossible since **B** is RREF and thus column $i = j$ is supposed to be clean.

So if we delete the first rows of **A** and **B**, we get two new matrices that (1) are still in RREF, (2) still have the property that each row of one is a linear combination of the rows of the other. Then we can apply the same argument as above to show that these new matrices have the leading one in the first row in the same position. Repeating this until all rows are exhausted, we see that **A** and **B** must have the leading one in the same positions.

And now it is easy to see that **A** and **B** must coincide: take any non-zero row of **A**, say the *k*-th. It must be a linear combination of rows of **B** and we want to show that, in fact, it is just the *k*-th row of **B**. Well, the linear combination must use the *k*-th row of **B** with coefficient 1, since that is the only way to get the leading one in the *k*-th row (the rest of its column is all zeroes). And it can't use any other non-zero row R of **B**, since if it did, it'd be stuck with a non-zero entry in the column of R's leading 1.

Miscellaneous results regarding elementary row operations and determinants

We can rephrase some of the statements in **Lecture Notes #11** regarding the effect of the elementary row operations on the value of the determinant on an $n \times n$ matrix.

Specifically, if $\mathbf{E}_{ij}$ is the matrix that executes a swap of the *i*-th and *j*-th rows of an $n \times n$ matrix, then $\det(\mathbf{E}_{ij}\mathbf{A}) = -\det(\mathbf{A})$. If $\mathbf{E}_{r(i)}$ is the matrix that executes a row scaling of the *i*-th row by fact *r*, then $\det(\mathbf{E}_{r(i)}\mathbf{A}) = r\det(\mathbf{A})$. If $\mathbf{E}_{i+r(j)}$ is the matrix that adds r times the *j*-th row to the *i*-th row, then $\det(\mathbf{E}_{i+r(j)}\mathbf{A}) = \det(\mathbf{A})$. If reducing the matrix $\mathbf{A}$ to $RREF(\mathbf{A}) = \mathbf{E}_N\mathbf{E}_{N-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}$ involves *s* row swaps, *p* row scalings by factors $\frac{1}{k_1}, \frac{1}{k_2}, \cdots, \frac{1}{k_p}$ (where $k_1, k_2, \cdots, k_p \neq 0$), and some number of steps where a multiple of a pivot row is added to another row, then $\det[RREF(\mathbf{A})] = \det(\mathbf{E}_N\mathbf{E}_{N-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = (-1)^s \frac{1}{k_1}\frac{1}{k_2}\cdots\frac{1}{k_p}\det(\mathbf{A})$ and therefore $\det(\mathbf{A}) = (-1)^s k_1 k_2 \cdots k_p \det[RREF(\mathbf{A})]$.

Prop: Suppose that $\mathbf{A}$ is an invertible $n \times n$ matrix and $\mathbf{B}$ is any other $n \times n$ matrix. Then $RREF[\mathbf{A} \mid \mathbf{AB}] = [\mathbf{I}_n \mid \mathbf{B}]$.

Proof: because $\mathbf{A}$ is an invertible it can be reduced to the Identity matrix $\mathbf{I}$ via a sequence of elementary row operations. That is, $RREF(\mathbf{A}) = \mathbf{E}_N\mathbf{E}_{N-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{I}$. Applying the same row operations used to reduce $\mathbf{A}$ to $\mathbf{I}$ identically to the rows of $[\mathbf{A} \mid \mathbf{AB}]$ that , we see that:

$$RREF[\mathbf{A} \mid \mathbf{AB}] = \mathbf{E}_N\mathbf{E}_{N-1}\cdots\mathbf{E}_2\mathbf{E}_1[\mathbf{A} \mid \mathbf{AB}] = [\mathbf{E}_N\mathbf{E}_{N-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A} \mid \mathbf{E}_N\mathbf{E}_{N-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{AB}] = [\mathbf{I} \mid \mathbf{IB}] = [\mathbf{I} \mid \mathbf{B}]$$

This result is used in the **Lecture Notes #11** to show that if **A** and **B** are $n \times n$ matrices, then $\det(\mathbf{AB}) = (\det\mathbf{A})(\det\mathbf{B})$.

Proposition: If **A** and **B** are $n \times n$ matrices, then $\boxed{\det(\mathbf{AB}) = (\det\mathbf{A})(\det\mathbf{B})}$.

Proof: If the matrix **A** is not invertible, then **AB** will also not be invertible and $\det\mathbf{A} = 0$ and $\det(\mathbf{AB}) = 0$, so the result holds in this case. The proposition above shows that in the case where **A** is invertible and **B** is an arbitrary $n \times n$ matrix, then $\mathrm{rref}[\mathbf{A} \mid \mathbf{AB}] = [\mathbf{I}_n \mid \mathbf{B}]$. If the row reduction from **A** to $\mathbf{I}_n$ involves the same row operations as outlined previously, then these same row operations would be applied in reducing **AB** to **B**, so $\det(\mathbf{AB}) = (-1)^s k_1 k_2 \cdots k_r \det(\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})$.