Math S-21b – Lecture #2 Notes

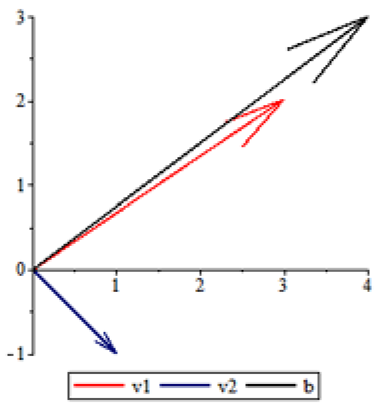
Today's lecture focuses on the vector and matrix formulations for a system of linear equations, linear transformations defined by matrices, the meaning of the columns of a matrix, and how to find matrices for several important geometrically defined linear transformations.

Vector form of a system of linear equations

Any system of *m* linear equations in *n* unknowns is of the form $\left\{\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{matrix}\right\}$. If we choose to represent vectors in $\mathbf{R}^m$ as columns and use only the definitions of scalar multiplication of a vector and vector addition, i.e. $t\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} tx_1 \\ \vdots \\ tx_m \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_m + y_m \end{bmatrix}$, we can express these linear equations in the form:

$$x_1\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$ (**vector form** of the linear system}

If we denote the column vectors as $\mathbf{v}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \cdots, \mathbf{v}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$, we can then rewrite this more succinctly as $\boxed{x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{b}}$. This can be understood geometrically. What this says is that this system will have a solution (or many solutions) if the vector **b** on the right-hand-side can be expressed as a **linear combination** of the vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, i.e. the vector **b** can be "built" out of these vectors by appropriate scaling and vector addition.



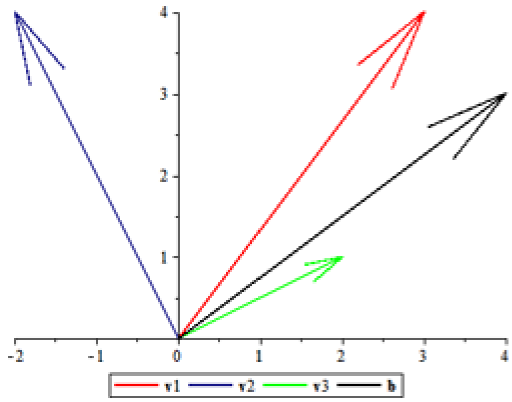
Example: The linear system $\left\{\begin{matrix} 3x + y = 4 \\ 2x - y = 3 \end{matrix}\right\}$ can be written in vector form as $x\begin{bmatrix} 3 \\ 2 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. With $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (in red) and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (in blue), and $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (in black), we want to know what *x* and *y* must be so that $x\mathbf{v}_1 + y\mathbf{v}_2 = \mathbf{b}$. Visually, we might guess that this can be done with *x* between perhaps 1 and 1.5, and *y* a small negative number. We solve for these values using row reduction methods:

$$\left[\begin{array}{cc|c} 3 & 1 & 4 \\ 2 & -1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & -1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -5 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -0.2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1.4 \\ 0 & 1 & -0.2 \end{array}\right]$$

So $x = 1.4$ and $y = -0.2$, and this agrees with our expectations.

Example #2: If we write the system $\left\{\begin{matrix} 3x - 2y + 2z = 4 \\ 4x + 4y + z = 3 \end{matrix}\right\}$ in vector form, we have $x\begin{bmatrix} 3 \\ 4 \end{bmatrix} + y\begin{bmatrix} -2 \\ 4 \end{bmatrix} + z\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. If we write $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (in red) and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ (in blue) and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (in green), and $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (in black), we are then seeking values for x, y, z so that $x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = b$. There are (infinitely) many ways to do this.

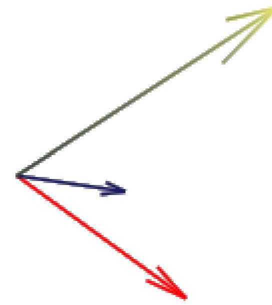
This agrees with what we found when we solved a similar system last week using row reduction:



$$\left[\begin{array}{ccc|c} 3 & -2 & 2 & 4 \\ 4 & 4 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{11}{10} \\ 0 & 1 & -\frac{1}{4} & -\frac{7}{20} \end{array}\right] \Rightarrow \left\{\begin{array}{l} x = \frac{11}{10} - 2t \\ y = -\frac{7}{20} + t \\ z = 4t \end{array}\right\}_{t\in\mathbf{R}}$$

Every choice of t gives a different way to construct the vector $\mathbf{b}$ out of these three spanning vectors (see picture at right).

Example #3: The system $\left\{\begin{array}{c} 3x + 2y = 5 \\ -x + y = 7 \\ 2x + y = 1 \end{array}\right\}$ can be written in vector form as $x\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}$. Writing $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}$ as vectors in $\mathbf{R}^3$, there is some doubt as to whether it's possible to do this, and this agrees with the fact that we previously found this system to be inconsistent. This situation is illustrated in the diagram at right (where the axes have been rotated for a better view). The red and blue vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, span a plane, and the third vector, $\mathbf{b}$, does not lie in this plane. We will soon express this by saying $\mathbf{b} \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.



Matrix form of a linear system

If we take the vector form above and assemble the vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ side-by-side to form an $m \times n$ matrix $\mathbf{A} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$, and if we write $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, we can define the product of this matrix and the vector as $\mathbf{Ax} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \equiv x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$. Using this definition, we can express the linear system $\left\{\begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{array}\right\}$ simply as $\mathbf{Ax} = \mathbf{b}$. This is called the **matrix form of the linear system**.

This can also be understood in terms of (linear) functions. Note that if we write $T(\mathbf{x}) = \mathbf{Ax}$, we have the input vector $\mathbf{x} \in \mathbf{R}^n$ and the output vector $T(\mathbf{x}) = \mathbf{Ax} = \mathbf{b} \in \mathbf{R}^m$. We can therefore understand such a system of linear equations in terms of the function $T : \mathbf{R}^n \to \mathbf{R}^m$. We also sometimes represent this by writing either:

$$\mathbf{R}^n \xrightarrow{T} \mathbf{R}^m \quad \text{or} \quad \mathbf{R}^n \xrightarrow{\mathbf{A}} \mathbf{R}^m \quad \text{or} \quad \mathbf{x} \in \mathbf{R}^n \xrightarrow{\mathbf{A}} \mathbf{Ax} = \mathbf{b} \in \mathbf{R}^m$$

A function defined in this manner is called a linear transformation.

<u>Definition</u>: A function $T : \mathbf{R}^n \to \mathbf{R}^m$ is called a **linear transformation** if for all vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{R}^n$ and for all scalars $c_1, c_2 \in \mathbf{R}$, T satisfies the <u>linearity property</u> $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$. This can also be expressed more geometrically by saying that <u>*T* preserves vector addition</u>, i.e. $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$, and <u>*T* preserves scalar multiplication</u>, i.e. $T(c\,\mathbf{v}) = cT(\mathbf{v})$.

We call the input space $\mathbf{R}^n$ the **domain** (as expected), and we refer to the output space $\mathbf{R}^m$ as the **codomain**.

Note 1: One thing worth mentioning here is that this notion of a linear function may not entirely agree with previous usage of the term "linear" as seen in calculus courses and before. Specifically, any function of the form $L(x) = ax + b$ is first-order, but it is not linear unless b = 0. Note that $L(x+y) = a(x+y) + b \neq L(x) + L(y)$. Also, preservation of scalar multiplication means that it would have to be the case that $L(0) = b = 0$, so in order to be linear it must be the case that $L(x) = ax$ (the graph of this line would have to pass through the origin). In particular, not that a function like $f(x) = 2x + 3$ is not a linear function!

More generally, a linear function $T : \mathbf{R}^n \to \mathbf{R}^1$ would have to be of the form $T(x_1, \cdots, x_n) = c_1 x_1 + \cdots + c_n x_n$, i.e. a pure first-order expression without constant term. For a linear function $T : \mathbf{R}^n \to \mathbf{R}^m$, all m (output) components of the value of this function would have to be of this form.

Note 2: In the case of a function defined by $T(\mathbf{x}) = \mathbf{Ax}$ for an $m \times n$ matrix **A**, the linearity property simply becomes the distributive law: $\mathbf{A}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1\mathbf{A}(\mathbf{x}_1) + c_2\mathbf{A}(\mathbf{x}_2)$.

Proposition: $T(\mathbf{x}) = \mathbf{Ax}$ (for an $m \times n$ matrix **A**) is a linear transformation.

Proof: If we write the matrix **A** in terms of its columns, $\mathbf{A} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}$ and let $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ and let $\alpha, \beta \in \mathbf{R}$, then $\alpha\mathbf{x} + \beta\mathbf{y} = \alpha\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \beta\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix} + \begin{bmatrix} \beta y_1 \\ \vdots \\ \beta y_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \vdots \\ \alpha x_n + \beta y_n \end{bmatrix}$ using basic facts about scaling and adding vectors. Using our definition of the product of a matrix and a vector, we have:

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \mathbf{A}(\alpha\mathbf{x} + \beta\mathbf{y}) = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}\begin{bmatrix} \alpha x_1 + \beta y_1 \\ \vdots \\ \alpha x_n + \beta y_n \end{bmatrix} = (\alpha x_1 + \beta y_1)\mathbf{v}_1 + \cdots + (\alpha x_n + \beta y_n)\mathbf{v}_n$$

$$= \alpha x_1\mathbf{v}_1 + \beta y_1\mathbf{v}_1 + \cdots + \alpha x_n\mathbf{v}_n + \beta y_n\mathbf{v}_n = \alpha x_1\mathbf{v}_1 + \cdots + \alpha x_n\mathbf{v}_n + \beta y_1\mathbf{v}_1 + \cdots + \beta y_n\mathbf{v}_n$$

$$= \alpha(x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n) + \beta(y_1\mathbf{v}_1 + \cdots + y_n\mathbf{v}_n) = \alpha\mathbf{Ax} + \beta\mathbf{Ay} = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$$

As you can see, the linearity property ultimately flows from the distributive law for vector addition.

Important Note: We began by looking at systems of linear equations and introduced matrices initially as a convenient way of keeping track of the manipulation of equations en route to a solution of the system. A matrix was effectively just "a box of numbers". We now have a very different and extremely important new view of a **matrix as a linear function**. This functional view of an $m \times n$ matrix as a linear function from $\mathbf{R}^n$ to $\mathbf{R}^m$ will be with us from now on.

Meaning of the columns of a matrix

Now that we are able to think of a matrix as a function, it's possible to provide a simple interpretation of the columns of a matrix that will allow us to construct matrices based on information about how they act on vectors.

In $\mathbf{R}^n$ we introduce the standard or elementary basis vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, …, $\mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$. You have probably seen these vectors before under different names. For example, in $\mathbf{R}^2$, we have $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{i}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{j}$, and we can write any vector in $\mathbf{R}^2$ as $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x\,\mathbf{i} + y\,\mathbf{j}$.

Similarly, in $\mathbf{R}^3$, we have $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{i}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{j}$, and $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{k}$, and we can write any vector in $\mathbf{R}^3$ as

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}.$$

This same decomposition can be done in $\mathbf{R}^n$ as $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$. Using our definition of the product of a matrix and a vector, we see that:

$$\mathbf{A}\mathbf{e}_1 = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 1\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{v}_1 = \{\text{1st column of the matrix } \mathbf{A}\}$$

$$\mathbf{A}\mathbf{e}_2 = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = 0\mathbf{v}_1 + 1\mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{v}_2 = \{\text{2nd column of the matrix } \mathbf{A}\}$$

$$\vdots$$

$$\mathbf{A}\mathbf{e}_n = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 1\mathbf{v}_n = \mathbf{v}_n = \{n\text{th column of the matrix } \mathbf{A}\}$$

In other words, the columns of a matrix tell us how the corresponding linear function acts on the basic vectors $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ and, quite significantly, <u>these completely determine the matrix</u>. In fact, for any vector $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$, we have $\mathbf{A}\mathbf{x} = \mathbf{A}(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) = x_1\mathbf{A}\mathbf{e}_1 + \cdots + x_n\mathbf{A}\mathbf{e}_n = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$.

We can now begin writing down some important examples of matrices.

Identity in $\mathbf{R}^n$: The identity function $Id : \mathbf{R}^n \to \mathbf{R}^n$ is simply $Id(\mathbf{x}) = \mathbf{x}$. This is clearly linear (it preserves everything, including scaling and addition of vectors) and we have all we need to determine its corresponding $n \times n$ (square) matrix, denoted by $\mathbf{I}_n$ (often just as **I**), and called the $n \times n$ **Identity matrix**.

$$\begin{aligned} Id(\mathbf{e}_1) &= \mathbf{e}_1 = \{\text{1st column of the matrix}\} \\ Id(\mathbf{e}_2) &= \mathbf{e}_2 = \{\text{2nd column of the matrix}\} \\ &\vdots \\ Id(\mathbf{e}_n) &= \mathbf{e}_n = \{n\text{th column of the matrix}\} \end{aligned} \quad \Rightarrow \quad \mathbf{I}_n = \begin{bmatrix} \uparrow & \cdots & \uparrow \\ \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ \downarrow & \cdots & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

This matrix has 0's everywhere except on the **main diagonal**, and all of the diagonal entries are equal to 1.

Dilation (scaling) in $\mathbf{R}^n$: This is a transformation of the form $T(\mathbf{x}) = r\,\mathbf{x}$ for some fixed scalar *r*. We have:

$$\begin{aligned} T(\mathbf{e}_1) &= r\,\mathbf{e}_1 = \{\text{1st column of the matrix}\} \\ T(\mathbf{e}_2) &= r\,\mathbf{e}_2 = \{\text{2nd column of the matrix}\} \\ &\vdots \\ T(\mathbf{e}_n) &= r\,\mathbf{e}_n = \{n\text{th column of the matrix}\} \end{aligned} \quad \Rightarrow \quad \mathbf{A} = \begin{bmatrix} \uparrow & \cdots & \uparrow \\ r\,\mathbf{e}_1 & \cdots & r\,\mathbf{e}_n \\ \downarrow & \cdots & \downarrow \end{bmatrix} = \begin{bmatrix} r & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & r \end{bmatrix}.$$

This also yields another **diagonal** matrix, once again with equal entries on the main diagonal.

Counterclockwise rotation in $\mathbf{R}^2$: The transformation that rotates any vector in $\mathbf{R}^2$ counterclockwise through a fixed angle θ is, in fact, a linear transformation (think about it in terms of preserving vector addition and scalar multiplication). We can determine its 2×2 (square) matrix by drawing a picture and using basic trigonometry. (It's best to draw the angle relatively small to most easily see things.) We see that rotation of the basic vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ give: $\mathbf{e}_1 \to \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\mathbf{e}_2 \to \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, so its matrix is $\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Rotation-dilation in $\mathbf{R}^2$: If we combine counterclockwise rotation through a fixed angle θ and scaling by a fixed scalar *r*, we have $\mathbf{e}_1 \to \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$ and $\mathbf{e}_2 \to \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix}$, so its matrix is $\mathbf{A} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$. If we let $a = r\cos\theta$ and $b = r\sin\theta$ w see that any matrix of the form $\mathbf{A} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ will represent a rotation-dilation where the scaling is by $r = \sqrt{a^2+b^2}$ and the angle of rotation is determined by $\tan\theta = b/a$ (in the appropriate quadrant as determined by the signs of the entries).

For example, the matrix $\mathbf{A} = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}$ represents a rotation-dilation with scalar $r = \sqrt{13}$ with angle of rotation determined by $\tan\theta = -3/2$ in the 2nd quadrant. This gives $\theta \cong 123.69°$.

Notes by Robert Winters