## Math S-21b - Lecture \#18 Notes

## Linear Differential Operators

## Higher order linear ordinary differential equations with constant coefficients

In general, an $n$th order linear ordinary differential equation is a differential equation of the form
$\frac{d^{n} x}{d t^{n}}+p_{n-1}(t) \frac{d^{n-1} x}{d t^{n-1}}+\cdots+p_{1}(t) \frac{d x}{d t}+p_{0}(t) x(t)=q(t)$, where $p_{n-1}(t), \ldots, p_{1}(t), p_{0}(t), q(t)$ are functions of the independent variable $t$. We solve this by (1) finding an expression for all homogeneous solutions $x_{h}(t)$, (2) using some productive method to find one particular solution $x_{p}(t)$ to the inhomogeneous equation, and then (3) adding these to get the general solution $x(t)=x_{h}(t)+x_{p}(t)$. If we are solving an initial value problem, we would then use the initial conditions to determine any unknown constants in the expression for $x(t)$.

One case of special interest is the case where all of the coefficient functions $p_{i}(t)=a_{i}$ are constant. In this case the differential equation simplifies to $\frac{d^{n} x}{d t^{n}}+a_{n-1} \frac{d^{n-1} x}{d t^{n-1}}+\cdots+a_{1} \frac{d x}{d t}+a_{0} x(t)=q(t)$.

A linear ODE of the form $x^{(n)}(t)+p_{n-1}(t) x^{(n-1)}(t)+\cdots+p_{1}(t) x^{\prime}(t)+p_{0}(t) x(t)=q(t)$ where $p_{n-1}(t), \ldots, p_{1}(t), p_{0}(t), q(t)$ are functions of the independent variable $t$ can be expressed in the form $T(x(t))=g(t)$ where $T$ is a linear operator of the form $T=\frac{d^{n}}{d t^{n}}+p_{n-1}(t) \frac{d^{n-1}}{d t^{n-1}}+\cdots+p_{1}(t) \frac{d}{d t}+p_{0}(t) \cdot$. The last term refers to multiplication by $p_{0}(t)$. A useful way of formulating such an ODE is to thing of the left-handside as corresponding to "the system" and the inhomogeneity $q(t)$ on the right-hand-side as corresponding to the "input signal" or, more simply, the "input." The general solution of the ODE is then referred to as the "output signal" or "response." Some motivating examples are in order.

## Differential operators

We have previously defined a linear differential operator as a transformation which acts linearly (preserves scaling and addition) and that takes functions to other functions, i.e. $T\left(c_{1} f_{1}+c_{1} f_{2}\right)=c_{1} T\left(f_{1}\right)+c_{1} T\left(f_{2}\right)$. There are many such operators, but some basic operators are of special interest to us as building blocks for more general linear differential operators. In particular:
(1) $D f=f^{\prime}$ acts linearly. That is, $\frac{d}{d t}\left[c_{1} f_{1}(t)+c_{2} f_{2}(t)\right]=c_{1} f_{1}^{\prime}(t)+c_{2} f_{2}^{\prime}(t)$.
(2) $M_{h(t)}[f(t)]=h(t) \cdot f(t)$, i.e. multiplication by the function $h(t)$ also acts linearly. [This is just the Distributive Law.
(3) The "shift-by-a" operator defined by $\left[L_{a}(f)\right](t)=f(t-a)$ also acts linearly. This is also called a "time shift" if the independent variable $t$ represents time.
(4) The Identity operator $I(f)=f$ is a linear operator.

It's routine to prove the following facts:
Fact 1: Any composition of linear operators is also a linear operator.
Fact 2: Any linear combination of linear operators is also a linear operator.
These facts enable us to express a linear ODE with constant coefficients in a simple and useful way. For example, in the case of a mass-spring-dashpot system with ODE $m \ddot{x}+c \dot{x}+k x=f(t)$, we can write this as $\left[m D^{2}+c D+k I\right] x(t)=f(t)$ and, if we let $L=m D^{2}+c D+k I$ we could write simply $L[x(t)]=f(t)$. The linear operator $L$ represents the system, and the function $f$ represents the input signal. For any linear ODE with constant coefficients of the form $\frac{d^{n} x}{d t^{n}}+a_{n-1} \frac{d^{n-1} x}{d t^{n-1}}+\cdots+a_{1} \frac{d x}{d t}+a_{0} x(t)=q(t)$, we can express this simply as
$\left[D^{n}+a_{n-1} D^{n-1}+\cdots+a_{1} D+a_{0} I\right] x(t)=q(t)$ and as $\left[D^{n}+a_{n-1} D^{n-1}+\cdots+a_{1} D+a_{0} I\right] x(t)=0$ for a homogeneous system. The fact that the "system" $D^{n}+a_{n-1} D^{n-1}+\cdots+a_{1} D+a_{0} I$ is independent of $t$ is why we refer to such an operator as a Linear, Time-Invariant (LTI) operator.

## Characteristic polynomial, exponential solutions

Given the operator $D^{n}+a_{n-1} D^{n-1}+\cdots+a_{1} D+a_{0} I$ and the fact that $D\left[e^{r t}\right]=r e^{r t}, D^{2}\left[e^{r t}\right]=r^{2} e^{r t}$, etc., it follows that $\left[D^{n}+a_{n-1} D^{n-1}+\cdots+a_{1} D+a_{0} I\right] e^{r t}=r^{n} e^{r t}+a_{n-1} r^{n-1} e^{r t}+\cdots+a_{1} r e^{r t}+a_{0} e^{r t}=\left(r^{n}+a_{n-1} r^{n-1}+\cdots+a_{1} r+a_{0}\right) e^{r t}$.

We define $p(r)=r^{n}+a_{n-1} r^{n-1}+\cdots+a_{1} r+a_{0}$ as the characteristic polynomial. We can also formally write $p(D)=D^{n}+a_{n-1} D^{n-1}+\cdots+a_{1} D+a_{0} I$, and write simply $[p(D)] e^{r t}=p(r) e^{r t}$. [This can be interpreted as saying that the function $e^{r t}$ is an eigenfunction of the operator $p(D)$ with eigenvalue $p(r)$.] In the case of a homogeneous system, this means we would have $[p(D)] e^{r t}=p(r) e^{r t}=0$ for all $t$, and this is only possible when $p(r)=0$, i.e. when $r$ is a root of the characteristic polynomial (called a characteristic root). According to the Fundamental Theorem of Algebra, we should be able to fact $p(r)$ into a product of linear and irreducible quadratic factors and produce $n$ roots, possible with multiplicity, and possibly including complex conjugate pairs.

In other words, if we seek exponential solutions of the form $e^{r t}$ for the homogeneous equation $\frac{d^{n} x}{d t^{n}}+a_{n-1} \frac{d^{n-1} x}{d t^{n-1}}+\cdots+a_{1} \frac{d x}{d t}+a_{0} x(t)=0$, we calculate $\frac{d x}{d t}=r e^{r t}, \frac{d^{2} x}{d t^{2}}=r^{2} e^{r t}, \ldots, \frac{d^{n} x}{d t^{n}}=r^{n} e^{r t}$, and substitution gives $r^{n} e^{r t}+a_{n-1} r^{n-1} e^{r t}+\cdots+a_{2} r^{2} e^{r t}+a_{1} r e^{r t}+a_{0} e^{r t}=\left(r^{n}+a_{n-1} r^{n-1}+\cdots+a_{2} r^{2}+a_{1} r+a_{0}\right) e^{r t}=0$. This yields a solution only when the characteristic polynomial $p(r)=r^{n}+a_{n-1} r^{n-1}+\cdots+a_{2} r^{2}+a_{1} r+a_{0}=0$. So, for any root $r_{i}$ of the characteristic polynomial, $e^{r i t}$ will be a homogeneous solution. As long as there are no repeated roots, and since we can use the quadratic formula to produce a complex conjugate pair of roots for each irreducible quadratic factor, we will be able to produce $n$ distinct roots and a corresponding set of exponential solutions $\left\{e^{r_{1} t}, e^{r_{2} t}, \ldots, e^{r_{t} t}\right\}$. In the case of repeated roots, this will yield fewer solutions of this form.

By linearity, any function of the form $x_{h}(t)=c_{1} e^{r_{t} t}+c_{2} e^{r_{2} t}+\ldots+c_{n} e^{r_{n} t}$ will solve the homogeneous equation.
Question: Does this yield all solutions?
A second order example should explain why the answer is YES. Suppose we wish to solve the ODE $\ddot{x}-3 \dot{x}+2 x=0$. Any exponential solution $e^{r t}$ would give $p(r)=r^{2}-3 r+2=(r-2)(r-1)=0$. Its characteristic roots are $r_{1}=2$ and $r_{2}=1$, and these yield solutions $e^{2 t}$ and $e^{t}$. Why are ALL homogeneous solutions of the form $x(t)=c_{1} e^{2 t}+c_{2} e^{t}$ ?

If we write the differential equation in terms of linear differential operators with $D=\frac{d}{d t}$, we might write this as $[D-2 I] \circ[D-I] x(t)=0$, i.e. as a composition of two 1 st order linear differential operators. If we let $[D-I] x(t)=y(t)$, this gives two 1st order equations: $\frac{d x}{d t}-x=y(t)$ and $\frac{d y}{d t}-2 y=0$. The latter equation is easily solved to give all solutions $y(t)=c_{1} e^{2 t}$ where $c_{1}$ is a constant. We then substitute this into the former equation to get $\frac{d x}{d t}-x=c_{1} e^{2 t}$. This is an inhomogeneous equation with integrating factor $e^{-t}$. Multiplication by this gives $e^{-t} \frac{d x}{d t}-x e^{-t}=\frac{d}{d t}\left(x e^{-t}\right)=c_{1} e^{t}$, so $x e^{-t}=c_{1} e^{t}+c_{2}$. Finally, multiplying both sides by $e^{t}$ gives $x(t)=c_{1} e^{2 t}+c_{2} e^{t}$.

This approach can be generalized to the $n$th order case as long as the characteristic polynomial can be factored into distinct linear factors. (We write the differential equation as a composition of $n 1$ st order linear operators and iterate the above process.) This even works in the case of complex roots as long as they are not repeated. The more difficult case is when there are repeated roots of the characteristic polynomial, but, as we'll soon see, this case also yields a relatively simple solution.

In Linear Algebra terms, we say that $\left\{e^{r_{t} t}, e^{r_{2} t}, \ldots, e^{r_{n} t}\right\}$ span all solutions in the above case. It is a valid question to ask whether all of these solutions are necessary, i.e. if we could span all solutions with a subset of these exponential solutions. In Linear Algebra terms, we would ask: Are these solutions are linearly independent? In other words, is it possible to express any of these solutions as a linear combination of the other solutions?

Definition: A set of functions $\left\{f_{1}, f_{2}, \ldots, f_{n}\right\}$ is called linearly independent if the equation $c_{1} f_{1}(t)+c_{2} f_{2}(t)+\ldots+c_{n} f_{n}(t)=0($ for all $t)$ implies that $c_{1}=c_{2}=\ldots=c_{n}=0$.

When seeking solutions to an $\boldsymbol{n}$ th order linear differential equation, we actually want more than this. In order to guarantee unique solutions to any well-posed initial value problem with initial conditions given for the function and its derivatives up to order ( $n-1$ ), we would also want that:

$$
\left\{\begin{array}{c}
c_{1} f_{1}\left(t_{0}\right)+c_{2} f_{2}\left(t_{0}\right)+\cdots+c_{n} f_{n}\left(t_{0}\right)=x\left(t_{0}\right) \\
c_{1} f_{1}^{\prime}\left(t_{0}\right)+c_{2} f_{2}^{\prime}\left(t_{0}\right)+\cdots+c_{n} f_{n}^{\prime}\left(t_{0}\right)=\dot{x}\left(t_{0}\right) \\
\vdots \\
c_{1} f_{1}^{(n-1)}\left(t_{0}\right)+c_{2} f_{2}^{(n-1)}\left(t_{0}\right)+\cdots+c_{n} f_{n}^{(n-1)}\left(t_{0}\right)=x^{(n-1)}\left(t_{0}\right)
\end{array}\right\} \Rightarrow \text { unique }\left\{c_{1}, c_{2}, \ldots, c_{n}\right\}
$$

In terms of matrices, we can express this as:

$$
\left[\begin{array}{cccc}
f_{1}\left(t_{0}\right) & f_{2}\left(t_{0}\right) & \cdots & f_{n}\left(t_{0}\right) \\
f_{1}^{\prime}\left(t_{0}\right) & f_{2}^{\prime}\left(t_{0}\right) & \cdots & f_{n}^{\prime}\left(t_{0}\right) \\
\vdots & \vdots & \cdots & \vdots \\
f_{1}^{(n-1)}\left(t_{0}\right) & f_{2}^{(n-1)}\left(t_{0}\right) & \cdots & f_{n}^{(n-1)}\left(t_{0}\right)
\end{array}\right]\left[\begin{array}{c}
c_{1} \\
c_{2} \\
\vdots \\
c_{n}
\end{array}\right]=\left[\begin{array}{c}
x\left(t_{0}\right) \\
\dot{x}\left(t_{0}\right) \\
\vdots \\
x^{(n-1)}\left(t_{0}\right)
\end{array}\right] \Rightarrow \text { unique }\left[\begin{array}{c}
c_{1} \\
c_{2} \\
\vdots \\
c_{n}
\end{array}\right]
$$

Two fundamental results in linear algebra say that this will only be the case when the above matrix is invertible, and this will only be the case when its determinant is never equal to 0 .

Definition: $\operatorname{det}\left[\begin{array}{cccc}f_{1} & f_{2} & \cdots & f_{n} \\ f_{1}^{\prime} & f_{2}^{\prime} & \cdots & f_{n}^{\prime} \\ \vdots & \vdots & \cdots & \vdots \\ f_{1}^{(n-1)} & f_{2}^{(n-1)} & \cdots & f_{n}^{(n-1)}\end{array}\right]=\left|\begin{array}{cccc}f_{1} & f_{2} & \cdots & f_{n} \\ f_{1}^{\prime} & f_{2}^{\prime} & \cdots & f_{n}^{\prime} \\ \vdots & \vdots & \cdots & \vdots \\ f_{1}^{(n-1)} & f_{2}^{(n-1)} & \cdots & f_{n}^{(n-1)}\end{array}\right|$ is called the Wronskian determinant.
Corollary: If the Wronskian determinant is never 0 , the given ODE will yield unique solutions in the form $x_{h}(t)=c_{1} f_{1}(t)+c_{2} f_{2}(t)+\ldots+c_{n} f_{n}(t)$ for any given initial conditions given for the function and its derivatives up to order ( $n-1$ ).

Though not routinely used to ensure a linearly independent set of solutions (there are arguments with less tedious calculations that can be made), the Wronskian is one tool for ensuring that a set of solutions to a homogeneous linear ODE is valid for uniquely expressing all homogeneous solutions.

Example \#1 (diffusion): Suppose a closed container has an initial interior temperature of $32^{\circ} \mathrm{F}$ at 10 am and that the outside temperature (also in ${ }^{\circ} \mathrm{F}$ ) rises steadily according to $y(t)=60+6 t$ where time $t$ is measured in hours.

According to Newton's Law of Cooling, $\frac{d x}{d t}=k(y-x)$ where $k$ is known as the coupling constant. If $k=\frac{1}{3}$, (a) how will the interior temperature vary in time, and (b) at what time will the interior temperature reach $60^{\circ} \mathrm{F}$ ?

Solution: The temperature will be governed by $\frac{d x}{d t}=\frac{1}{3}(y-x)$ or $\frac{d x}{d t}+\frac{1}{3} x=\frac{1}{3} y(t)=\frac{1}{3}(60+6 t)=20+2 t$, so the inhomogeneous ODE is $\frac{d x}{d t}+\frac{1}{3} x=20+2 t$.
(1) The homogeneous equation $\frac{d x}{d t}+\frac{1}{3} x=0$ easily yields the solutions of the form $x_{h}(t)=c e^{-\frac{1}{3} t}$. It's worth noting that over time any such homogeneous solution will tend toward 0 and become negligible. For this reason we often refer to this as a transient. In the short term it may be relevant, but in the long term it is not.
(2) We can use the Method of Undetermined Coefficients to find a particular solution. The nature of the inhomogeneity $q(t)=20+2 t$ suggests that we seek a solution of the form $x_{p}(t)=A+B t$. We have $\frac{d x_{p}}{d t}(t)=B$, so we must have $B+\frac{1}{3}(A+B t)=\left(B+\frac{1}{3} A\right)+\frac{1}{3} B t=20+2 t$, so $B+\frac{1}{3} A=20$ and $\frac{1}{3} B=2$. This gives $B=6$ and $A=42$, so $x_{p}(t)=42+6 t$. Once the transients have become negligible, this is all that will remain. For this reason we might refer to this as the "steady state" solution.
(3) The general solution is $x(t)=x_{h}(t)+x_{p}(t)=c e^{-\frac{1}{3} t}+42+6 t$. If we substitute the initial condition $x(0)=32$, we have $x(0)=c+42=32$, so $c=-10$ and $x(t)=42+6 t-10 e^{-\frac{1}{3} t}$. Note that eventually the interior temperature will be rising at the same rate as the outside temperature but always $18^{\circ} \mathrm{F}$ cooler.
The interior temperature will reach $60^{\circ} \mathrm{F}$ at a time $T$ when $42+6 T-10 e^{-\frac{1}{3} T}=60$ or $6 T-10 e^{-\frac{1}{3} T}=18$. This cannot be solved algebraically, but it's easy to get a numerical solution using a graphing calculator and the trace function. It gives a time $T \approx 3.33 \approx 3 \mathrm{hrs}, 20 \mathrm{~min}$, i.e. about $1: 20 \mathrm{pm}$.

Example \#2 (exponential input): Solve the initial value problem $\frac{d^{2} x}{d t^{2}}+3 \frac{d x}{d t}+2 x=e^{t}, x(0)=4, x^{\prime}(0)=2$.
Solution: This ODE is of the type we might expect from a mass-spring system, though the external driving force is not especially realistic (relentlessly exponential in a single direction). It is nonetheless good for illustrating the methods, and the exponential input will be very relevant in the days and weeks to come. For simplicity, let's write the ODE as $x^{\prime \prime}+3 x^{\prime}+2 x=e^{t}$.
(1) For the homogeneous solutions, look for exponential solutions $x=e^{r t}$ to the equation $x^{\prime \prime}+3 x^{\prime}+2 x=0$. This gives $r^{2} e^{r t}+3 r e^{r t}+2 e^{r t}=\left(r^{2}+3 r+2\right) e^{r t}=0$, so $r^{2}+3 r+2=(r+1)(r+2)=0 \Rightarrow r=-1, r=-2$. Individual homogeneous solutions are $x_{1}(t)=e^{-t}$ and $x_{2}(t)=e^{-2 t}$. By linearity any solution $x_{h}(t)=c_{1} e^{-t}+c_{2} e^{-2 t}$ will satisfy the homogeneous ODE. We previously showed that these give all homogeneous solutions (by thinking of a 2nd order homogenous linear operator as a composition of two 1st order linear operators).
Note that, in this case, the homogeneous solutions are transient, i.e., they rapidly decay in time.
(2) The Method of Undetermined Coefficients provides the simplest way to find a particular solution in this case. The obvious choice is to try a solution of the form $x=A e^{t}$. This gives $x^{\prime}=A e^{t}, x^{\prime \prime}=A e^{t}$, and we get that $A e^{t}+3 A e^{t}+2 A e^{t}=6 A e^{t}=e^{t} \Rightarrow A=\frac{1}{6}$, so our particular solution is $x_{p}(t)=\frac{1}{6} e^{t}$.
(3) Our general solution is then $x(t)=c_{1} e^{-t}+c_{2} e^{-2 t}+\frac{1}{6} e^{t}$. We compute $x^{\prime}(t)=-c_{1} e^{-t}-2 c_{2} e^{-2 t}+\frac{1}{6} e^{t}$, and the initial conditions give $\left\{\begin{array}{l}x(0)=c_{1}+c_{2}+\frac{1}{6}=4 \\ x^{\prime}(0)=-c_{1}-2 c_{2}+\frac{1}{6}=2\end{array}\right\} \Rightarrow\left\{\begin{array}{c}c_{1}+c_{2}=\frac{23}{6} \\ -c_{1}-2 c_{2}=\frac{11}{6}\end{array}\right\} \Rightarrow c_{1}=\frac{19}{2}, c_{2}=-\frac{17}{3}$.

So, the unique solution to the initial value problem is $x(t)=\underbrace{\frac{19}{2} e^{-t}-\frac{17}{3} e^{-2 t}}_{\text {transient }}+\underset{\text { steady-state }}{\frac{1}{6} e^{t}}$.
Example \#3 (sinusoidal input): Find the general solution to the ODE $\frac{d x}{d t}+2 x=\cos 3 t$
Solution: As with all 1st order linear equations, solving using an integrating factor is always an option, though it could lead to some difficult integration. In this example, the integrating factor is $e^{2 t}$ which gives $e^{2 t} \frac{d x}{d t}+2 e^{2 t} x=\frac{d}{d t}\left(e^{2 t} x\right)=e^{2 t} \cos 3 t$. Integration gives $e^{2 t} x(t)=\int e^{2 t} \cos 3 t+C$ and $x(t)=e^{-2 t}\left[\int e^{2 t} \cos 3 t+C\right]$. The integration can be done using integration by parts (twice) and some additional algebra.

If we solve this using linearity:
(1) $\frac{d x}{d t}+2 x=0$ gives the homogeneous solutions $x_{h}(t)=c e^{-2 t}$
(2) For a particular solution, try $x=a \cos 3 t+b \sin 3 t$. We calculate $x^{\prime}=3 b \cos 3 t-3 a \sin 3 t$, and substitution gives $x^{\prime}+2 x=(2 a+3 b) \cos 3 t+(-3 a+2 b) \sin 3 t=\cos 3 t$, so

$$
\begin{aligned}
& \left\{\begin{array}{r}
2 a+3 b=1 \\
-3 a+2 b=0
\end{array}\right\} \Rightarrow\left[\begin{array}{cc}
2 & 3 \\
-3 & 2
\end{array}\right]\left[\begin{array}{l}
a \\
b
\end{array}\right]=\left[\begin{array}{l}
1 \\
0
\end{array}\right] \Rightarrow\left[\begin{array}{l}
a \\
b
\end{array}\right]=\frac{1}{13}\left[\begin{array}{cc}
2 & -3 \\
3 & 2
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right]=\left[\begin{array}{l}
\frac{2}{13} \\
\frac{3}{13}
\end{array}\right], \text { so } x_{p}(t)=\frac{2}{13} \cos 3 t+\frac{3}{13} \sin 3 t \text { or } \\
& x_{p}(t)=\frac{1}{13}(2 \cos 3 t+3 \sin 3 t) .
\end{aligned}
$$



## A little trigonometry

Any expression of the form $a \cos \omega t+b \sin \omega t$ actually represents a single sinusoidal curve with frequency $\omega$ and an appropriate translation (phase shift), i.e. a function of the form $A \cos \left(\omega t-\phi_{0}\right)$. We can see this quickly using the sum of angle formula for cosine:

$$
A \cos \left(\omega t-\phi_{0}\right)=A \cos \omega t \cos \phi_{0}+A \sin \omega t \sin \phi_{0}=a \cos \omega t+b \sin \omega t
$$

We must therefore have $\left\{\begin{array}{l}A \cos \phi_{0}=a \\ A \sin \phi_{0}=b\end{array}\right\}$.
This is most easily understood with a right triangle as shown.
From this we see that $A=\sqrt{a^{2}+b^{2}}$ and $\tan \phi_{0}=\frac{b}{a}$.


In our example with $x_{p}(t)=\frac{1}{13}(2 \cos 3 t+3 \sin 3 t)$ we would get $A=\frac{1}{13} \sqrt{2^{2}+3^{2}}=\frac{\sqrt{13}}{13}=\frac{1}{\sqrt{13}}$ and $\tan \phi_{0}=\frac{3}{2}$. This gives $\phi_{0} \cong 56.31^{\circ}$ or $\phi_{0} \cong 0.9828$ radians. The period of the oscillation would be $\frac{2 \pi}{\omega}=\frac{2 \pi}{3}$.

Example \#4: Solve the initial value problem $\ddot{x}+5 \dot{x}+4 x=3 \sin 2 t$ with initial conditions $x(0)=3, \dot{x}(0)=2$.
Solution: We first solve the homogeneous equation $\ddot{x}+5 \dot{x}+4 x=0$. Its characteristic polynomial is $p(r)=r^{2}+5 r+4=(r+4)(r+1)$ and this yields two distinct roots $r=-4$ and $r=-1$. The corresponding exponential solutions are $e^{-4 t}$ and $e^{-t}$. We can check that these are, in fact, linearly independent by calculating the Wronskian determinant: $\left|\begin{array}{cc}e^{-4 t} & e^{-t} \\ -4 e^{-4 t} & -e^{-t}\end{array}\right|=-e^{-5 t}+4 e^{-5 t}=3 e^{-5 t} \neq 0$. From our previous arguments, we know that all homogeneous solutions are of the form $x_{h}(t)=c_{1} e^{-4 t}+c_{2} e^{-t}$.

Next, we seek a particular solution. There are at least two good ways to do this. We could do "complex replacement" and simultaneously solve $\ddot{x}+5 \dot{x}+4 x=3 \cos 2 t$ and $\ddot{y}+5 \dot{y}+4 y=3 \sin 2 t$ by solving the inhomogeneous equation $\ddot{z}+5 \dot{z}+4 z=3 e^{2 i t}$ and then taking the "imaginary" part. It's perhaps easier to solve using undetermined coefficients.
If we let $x=a \cos 2 t+b \sin 2 t$, we get $\left\{\begin{array}{l}x=a \cos 2 t+b \sin 2 t \\ \dot{x}=2 b \cos 2 t-2 a \sin 2 t \\ \ddot{x}=-4 a \cos 2 t-4 b \sin 2 t\end{array}\right\} \Rightarrow \ddot{x}+5 \dot{x}+4 x=(10 b) \cos 2 t+(-10 a) \sin 2 t$
We must therefore have $10 b=0$ and $-10 a=3$, so $a=-\frac{3}{10}$ and $b=0$. So $x_{p}(t)=-\frac{3}{10} \cos 2 t$.
The general solution is therefore $x(t)=c_{1} e^{-4 t}+c_{2} e^{-t}-\frac{3}{10} \cos 2 t$, and we have $\dot{x}(t)=-4 c_{1} e^{-4 t}-c_{2} e^{-t}+\frac{3}{5} \cos 2 t$.
If we substitute the initial conditions $x(0)=3, \dot{x}(0)=2$, we have:
$\left\{\begin{array}{c}x(0)=c_{1}+c_{2}-\frac{3}{10}=3 \\ \dot{x}(0)=-4 c_{1}-c_{2}=2\end{array}\right\} \Rightarrow\left\{\begin{array}{r}c_{1}+c_{2}=\frac{33}{10} \\ -4 c_{1}-c_{2}=2\end{array}\right\} \Rightarrow\left[\begin{array}{cc}1 & 1 \\ -4 & -1\end{array}\right]\left[\begin{array}{l}c_{1} \\ c_{2}\end{array}\right]=\left[\begin{array}{c}\frac{33}{10} \\ 2\end{array}\right] \Rightarrow\left[\begin{array}{c}c_{1} \\ c_{2}\end{array}\right]=\left[\begin{array}{c}-\frac{53}{30} \\ \frac{76}{15}\end{array}\right]$.
We certainly don't have to use matrices to solve these two equations, but it's worth noting that the nonvanishing of the Wronskian determinant is precisely why there is a unique solution for these constants. The unique solution to this initial value problem is therefore $x(t)=-\frac{53}{30} e^{-4 t}+\frac{76}{15} e^{-t}-\frac{3}{10} \cos 2 t$.
Note: In this example, the exponential terms are transients (they decay quickly) and the "steady state" solution is the particular solution that we calculated.

## Mass-Spring-Dashpot systems

Of particular interest to us (for a variety of reasons) are mass-spring-dashpot systems in which a spring is governed by Hooke's Law but also subject to friction that is proportional to the velocity. The simplest case is where this system is confined with the spring attached to one fixed wall, the dashpot on the other side of the mass attached to another fixed wall, and the mass moving relative to its equilibrium position. In this case, we would express the force acting on the mass as $F=-k x-c v$ where $v=\dot{x}$ and $F=m a=m \ddot{x}$. This gives the system $m \ddot{x}+c \dot{x}+k x=0$ or $\ddot{x}+\frac{c}{m} \dot{x}+\frac{k}{m} x=0$.

We could also imagine a system that is "driven" by moving either the fixed end of the spring or by moving the fixed end of the dashpot. If we incorporate this additional acceleration, the resulting system would be governed by an inhomogeneous ODE of the form $\ddot{x}+\frac{c}{m} \dot{x}+\frac{k}{m} x=q(t)$.

Note: We get similar equations in the case of an electric circuit with inductance $(L)$, resistance $(R)$, and capacitance ( $C$ ), i.e. and $\boldsymbol{L R C}$ circuit.

## Spring only case

The simplest case is a pure spring with no friction and no external driving force. In this case, the differential equation governing the motion would be simply $\ddot{x}+\frac{k}{m} x=0$. In anticipation of what will follow, it's useful to let $\omega^{2}=\frac{k}{m}$ or $\omega=\sqrt{\frac{k}{m}}$. This gives the differential equation $\ddot{x}+\omega^{2} x=0$. Its characteristic polynomial is $p(r)=r^{2}+\omega^{2}=0 \Rightarrow r= \pm i \omega$. So all solutions to this homogeneous equation can be expressed as the span of $\left\{e^{i \omega t}, e^{-i \omega t}\right\}$, i.e. in the form $x(t)=c_{1} e^{i \omega t}+c_{2} e^{-i \omega t}$ where $c_{1}, c_{2}$ are complex constants. We would, of course, prefer to express solutions as real-valued functions. Using Euler's Formula, we could rewrite the solutions as $x(t)=c_{1}(\cos \omega t+i \sin \omega t)+c_{2}(\cos \omega t-i \sin \omega t)=\left(c_{1}+c_{2}\right) \cos \omega t+i\left(c_{1}-c_{2}\right) \sin \omega t$ and then hope that any given initial condition will produce real coefficients (they will). Another way to look at this is to note that since
$e^{i \omega t}=\cos \omega t+i \sin \omega t$ and $e^{-i \omega t}=\cos \omega t-i \sin \omega t$, and we can solve for $\cos \omega t=\frac{e^{i \omega t}+e^{-i \omega t}}{2}$ and $\sin \omega t=\frac{e^{i \omega t}-e^{-i \omega t}}{2 i}$. So $\operatorname{Span}\left\{e^{i \omega t}, e^{-i \omega t}\right\}=\operatorname{Span}\{\cos \omega t, \sin \omega t\}$. That is, all solutions must be of the form $x(t)=a \cos \omega t+b \sin \omega t$. We also have the option of expressing this as $x(t)=A \cos (\omega t-\phi)$ where $A=\sqrt{a^{2}+b^{2}}$ and $\tan \phi=\frac{b}{a}$.

Note: If we felt the urge to inquire whether the set $\left\{e^{i \omega t}, e^{-i \omega t}\right\}$ or the set $\{\cos \omega t, \sin \omega t\}$ were linearly independent solutions, the corresponding Wronskians would give either $\left|\begin{array}{cc}e^{i \omega t} & e^{-i \omega t} \\ i \omega e^{i \omega t} & -i \omega e^{-i \omega t}\end{array}\right|=-2 i \omega \neq 0$ or $\left|\begin{array}{cc}\cos \omega t & \sin \omega t \\ -\omega \sin \omega t & \omega \cos \omega t\end{array}\right|=\omega\left(\cos ^{2} \omega t+\sin ^{2} \omega t\right)=\omega \neq 0$. They both provide a linearly independent spanning set for the solutions, i.e. a basis for the solutions (in linear algebra terms).

If the ODE is not homogeneous but is in the simple form $[p(D)] x(t)=a e^{r t}$ for some (possibly complex) numbers $a$ and $r$, we can use the method of undetermined coefficients to produce a particular solution. That is, if we let $x(t)=A e^{r t}$, this will be a particular solution if:

$$
[p(D)] A e^{r t}=\left[D^{n}+a_{n-1} D^{n-1}+\cdots+a_{1} D+a_{0} I\right] A e^{r t}=A\left(r^{n}+a_{n-1} r^{n-1}+\cdots+a_{1} r+a_{0}\right) e^{r t}=A p(r) e^{r t}=a e^{r t}
$$

Cancellation of the exponential factors and division by $p(r)$ gives $A=\frac{a}{p(r)}$, so $x_{p}(t)=\frac{a e^{r t}}{p(r)}$. This gives:
Exponential Response Formula (ERF): Suppose the ODE $[p(D)] x(t)=a e^{r t}$ has characteristic polynomial $p(s)$ and that $r$ is not a characteristic root, then a particular solution will be $x_{p}(t)=\frac{a e^{r t}}{p(r)}$. This result can make easy work of solving constant coefficient linear ODE's in this form.

Example \#5: Solve the ODE $\ddot{x}+3 \dot{x}+2 x=5 e^{3 t}$ with $x(0)=2, \dot{x}(0)=3$.
Solution: The characteristic polynomial is $p(s)=s^{2}+3 s+2=(s+2)(s+1)$. This gives roots $s_{1}=-2, s_{2}=-1$, and the homogeneous solutions are of the form $x_{h}(t)=c_{1} e^{-2 t}+c_{2} e^{-t}$. If we use the Exponential Response Formula, we calculate $p(3)=9+9+2=20$, so a particular solution is $x_{p}(t)=\frac{5 e^{3 t}}{p(3)}=\frac{5 e^{3 t}}{20}=\frac{1}{4} e^{3 t}$. The general solution is therefore $x(t)=c_{1} e^{-2 t}+c_{2} e^{-t}+\frac{1}{4} e^{3 t}$. Differentiation gives $\dot{x}(t)=-2 c_{1} e^{-2 t}-c_{2} e^{-t}+\frac{3}{4} e^{3 t}$. Evaluating these at $t=0$ gives $\left\{\begin{array}{c}x(0)=c_{1}+c_{2}+\frac{1}{4}=2 \\ \dot{x}(0)=-2 c_{1}-c_{2}+\frac{3}{4}=3\end{array}\right\} \Rightarrow c_{1}=-4, c_{2}=\frac{23}{4}$, so $x(t)=-4 e^{-2 t}+\frac{23}{4} e^{-t}+\frac{1}{4} e^{3 t}$.

## Complex roots

In the case where the characteristic polynomial yields a complex conjugate pair of roots, $\lambda=a+i b$ and $\bar{\lambda}=a+i b$, we formally have solutions $e^{\lambda t}=e^{(a+i b) t}=e^{a t} e^{i b t}=e^{a t}(\cos b t+i \sin b t)$ and $e^{\bar{\lambda} t}=e^{(a-i b) t}=e^{a t} e^{-i b t}=e^{a t}(\cos b t-i \sin b t)$. Though we could express any solutions in $\operatorname{Span}\left\{e^{\lambda t}, e^{\bar{\lambda} t}\right\}$ in the form $x(t)=c_{1} e^{\lambda t}+c_{2} e^{\bar{\lambda} t}$ where $c_{1}$ and $c_{2}$ would necessarily have to be complex constants, these solutions can also be characterized as being in $\operatorname{Span}\left\{e^{a t} \cos b t, e^{a t} \sin b t\right)$, i.e. they must be of the form $x(t)=c_{1} e^{a t} \cos b t+c_{2} e^{a t} \sin b t$ with real constants $c_{1}$ and $c_{2}$.

## Repeated roots

Suppose we have a 2nd order (homogeneous) linear ODE with characteristic polynomial $p(r)=(r-a)^{2}$ that yields the repeated root $r=a$ (in this case with multiplicity 2 ). We know that one solution is $e^{a t}$, but this does not span all homogeneous solutions. Note that in the case the ODE can be expressed as
$\left(D^{2}+2 a D+a^{2} I\right) x(t)=(D-a I)^{2} x(t)=(D-a I)[(D-a I) x(t)]=0$. If we give the name $y(t)=(D-a I) x(t)$, then the ODE becomes $(D-a I) y(t)=\frac{d y}{d t}-a y=0$ or simply $\frac{d y}{d t}=a y$. This easily yields all solutions of the form $y(t)=c_{1} e^{a t}$. Therefore $(D-a I) x(t)=\frac{d x}{d t}-a x=c_{1} e^{a t}$. This is now an inhomogeneous 1st order ODE that can be solved by multiplying both sides by the integrating factor $e^{-a t}$. This gives
$e^{-a t} \frac{d x}{d t}-a e^{-a t} x=\frac{d}{d t}\left(e^{-a t} x\right)=c_{1} e^{a t} e^{-a t}=c_{1}$. Integrating both sides gives $e^{-a t} x(t)=c_{1} t+c_{2}$ where $c_{2}$ is another arbitrary constant. Multiplying both sides by $e^{a t}$ then gives $e^{a t} x(t)=c_{1} t e^{a t}+c_{2} e^{a t}$. That is, all solutions are in $\operatorname{Span}\left\{e^{a t}, t e^{a t}\right\}$.
This can be generalized to the case of higher multiplicities to give solutions in $\operatorname{Span}\left\{e^{a t}, t e^{a t}, t^{2} e^{a t}, \ldots, t^{k} e^{a t}\right\}$ where $k$ is one less than the multiplicity.

Example \#6: Solve the 3 rd order homogeneous ODE $\dddot{x}+5 \ddot{x}+8 \dot{x}+4 x=0$ with initial conditions $x(0)=2$, $\dot{x}(0)=3, \ddot{x}(0)=1$.
Solution: We'll solve this two ways - first using the operator approach, and second using reduction of order and matrix methods.
Via Operators: We can express this ODE as $\left(D^{3}+5 D^{2}+8 D+4 I\right) x(t)=0$. This gives the characteristic polynomial $p(r)=r^{3}+5 r^{2}+8 r+4=(r+1)(r+2)^{2}=0$. The characteristic roots are $r_{1}=-1$ (with multiplicity 1) and $r_{2}=-2$ (with multiplicity 2). All solutions are therefore in $\operatorname{Span}\left\{e^{-t}, e^{-2 t}, t e^{-2 t}\right\}$ and must be of the form $x(t)=c_{1} e^{-t}+c_{2} e^{-2 t}+c_{3} t e^{-2 t}$. Differentiation gives:

$$
\begin{aligned}
& x(t)=c_{1} e^{-t}+c_{2} e^{-2 t}+c_{3} t e^{-2 t} \quad x(0)=c_{1}+c_{2}=2 \\
& \dot{x}(t)=-c_{1} e^{-t}-2 c_{2} e^{-2 t}+c_{3} e^{-2 t}-2 c_{3} t e^{-2 t} \Rightarrow \dot{x}(0)=-c_{1}-2 c_{2}+c_{3}=3 \\
& \ddot{x}(t)=c_{1} e^{-t}+4 c_{2} e^{-2 t}-4 c_{3} e^{-2 t}+4 c_{3} t e^{-2 t} \quad \ddot{x}(0)=c_{1}+4 c_{2}-4 c_{3}=1
\end{aligned}
$$

These equations can then be solved using either row reduction or matrix inversion to get $c_{1}=21, c_{2}=-19$, and $c_{3}=-14$. So the unique solution to this initial value problem is $x(t)=21 e^{-t}-19 e^{-2 t}-14 t e^{-2 t}$.
 $\frac{d z}{d t}=\dddot{x}$, we get $\left\{\begin{array}{lr}\frac{d x}{d t}= & y \\ \frac{d y}{d t}= & z \\ \frac{d z}{d t}=-4 x-8 y-5 z\end{array}\right\}$ or $\frac{d \mathbf{x}}{d t}=\mathbf{A x}$ with $\mathbf{A}=\left[\begin{array}{ccc}0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5\end{array}\right]$ and $\mathbf{x}(0)=\left[\begin{array}{l}2 \\ 3 \\ 1\end{array}\right]$.

This gives $\lambda \mathbf{I}-\mathbf{A}=\left[\begin{array}{ccc}\lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 4 & 8 & \lambda=5\end{array}\right]$. Its characteristic polynomial is $p(\lambda)=\lambda^{3}+5 \lambda^{2}+8 \lambda+4=(\lambda+1)(\lambda+2)^{2}$ which yields the two eigenvalues $\lambda_{1}=-1$ (with algebraic and geometric multiplicity 1 ) and $\lambda_{2}=-2$ (with algebraic multiplicity 2 but geometric multiplicity 1 ). The first of these eigenvalues gives the eigenvector $\mathbf{v}_{1}=\left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]$. The other eigenvalue gives the eigenvector $\mathbf{v}_{2}=\left[\begin{array}{c}1 \\ -2 \\ 4\end{array}\right]$ and generalized eigenvector $\mathbf{v}_{3}=\left[\begin{array}{c}1 \\ -1 \\ 0\end{array}\right]$.

Taking these vectors as a basis, we have the change-of-basis matrix $\mathbf{S}=\left[\begin{array}{ccc}1 & 1 & 1 \\ -1 & -2 & -1 \\ 1 & 4 & 0\end{array}\right]$ and $\mathbf{S}^{-1}=\left[\begin{array}{ccc}4 & 4 & 1 \\ -1 & -1 & 0 \\ -2 & -3 & -1\end{array}\right]$. We know that $\mathbf{S}^{-1} \mathbf{A S}=\mathbf{B}=\left[\begin{array}{c:cc}-1 & 0 & 0 \\ \hdashline 0 & -2 & -1 \\ 0 & 0 & -2\end{array}\right]$ and $\mathbf{A}=\mathbf{S B S}^{-1}$ and $\left[e^{t \mathbf{A}}\right]=\mathbf{S}\left[e^{i \mathbf{B}}\right] \mathbf{S}^{-1}$ where $\left[e^{i \mathbf{B}}\right]=\left[\begin{array}{c:cc}e^{-t} & 0 & 0 \\ \hdashline 0 & e^{-2 t} & t e^{-2 t} \\ 0 & 0 & e^{-2 t}\end{array}\right]$.

$$
\begin{aligned}
\mathbf{x}(t) & =\left[e^{t \mathbf{A}}\right] \mathbf{x}(0)=\mathbf{S}\left[e^{i \mathbf{B}}\right] \mathbf{S}^{-1} \mathbf{x}(0)=\mathbf{S}=\left[\begin{array}{ccc}
1 & 1 & 1 \\
-1 & -2 & -1 \\
1 & 4 & 0
\end{array}\right]\left[\begin{array}{ccc}
e^{-t} & 0 & 0 \\
0 & e^{-2 t} & t e^{-2 t} \\
0 & 0 & e^{-2 t}
\end{array}\right]\left[\begin{array}{ccc}
4 & 4 & 1 \\
-1 & -1 & 0 \\
-2 & -3 & -1
\end{array}\right]\left[\begin{array}{l}
2 \\
3 \\
1
\end{array}\right] \\
& =\left[\begin{array}{ccc}
e^{-t} & e^{-2 t} & (t+1) e^{-2 t} \\
-e^{-t} & -2 e^{-2 t} & (-2 t-1) e^{-2 t} \\
e^{-t} & 4 e^{-2 t} & 4 t e^{-2 t}
\end{array}\right]\left[\begin{array}{c}
21 \\
-5 \\
-14
\end{array}\right]=\left[\begin{array}{c}
21 e^{-t}-5 e^{-2 t}-14(t+1) e^{-2 t} \\
\cdots \\
\cdots
\end{array}\right]
\end{aligned}
$$

We're only interested in $x(t)$, so the solution is $x(t)=21 e^{-t}-5 e^{-2 t}-14(t+1) e^{-2 t}=21 e^{-t}-19 e^{-2 t}-14 t e^{-2 t}$.
Suffice to say that in the case the Operator Method is much simpler.

Notes by Robert Winters

