

## Math S-21b – Lecture #10 Notes

In today's lecture we finished up a few details on Least Square Approximate Solutions (see Lecture #9 notes), and then reviewed inner product spaces and introduced the idea of an orthonormal set of functions. The primary application that we'll focus on is Fourier series. We'll cover this initially in the case of continuous functions on a closed interval, but the ideas can be extended to also represent discontinuous functions that are piecewise continuous. All results can then be used to describe periodic functions more generally. The following notes are adapted from a course in differential equation.

### Inner Products and Orthogonality

We are all familiar with the mutually perpendicular (orthogonal) unit vectors  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  in  $\mathbf{R}^3$  and how we can express any vector  $\mathbf{v} = \langle x, y, z \rangle$  in  $\mathbf{R}^3$  as  $\mathbf{v} = \langle x, y, z \rangle = x\langle 1, 0, 0 \rangle + y\langle 0, 1, 0 \rangle + z\langle 0, 0, 1 \rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ . The components of the vector are just the scalar projections of  $\mathbf{v}$  in the directions of  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ , respectively. We find the scalar projection of a vector in any given direction by calculating its dot product with a unit vector  $\mathbf{u}$  in the given direction, i.e.  $\mathbf{v} \cdot \mathbf{u}$ . So we can also express  $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\mathbf{v} \cdot \mathbf{i})\mathbf{i} + (\mathbf{v} \cdot \mathbf{j})\mathbf{j} + (\mathbf{v} \cdot \mathbf{k})\mathbf{k}$ .

We could do the same thing with any set of three mutually orthogonal unit vectors  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  in  $\mathbf{R}^3$ . That is, if  $\mathbf{v} \in \mathbf{R}^3$  we could write  $\mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{v} \cdot \mathbf{u}_3)\mathbf{u}_3$ .

Just as we can add and scale vectors, we can also do this with functions. We add functions by adding their values and scale them by scaling their values. That is,  $(f + g)(x) = f(x) + g(x)$  and  $(cf)(x) = cf(x)$ . We might speculate that if functions can be combined in a manner analogous to vectors (where we add respective values instead of respective components), perhaps there may be something analogous to the dot product that we could use to define notions such as orthogonality in spaces of functions.

Think about how the dot product of two vectors is calculated: We multiply the respective components of the two vectors and sum these products, i.e. in  $\mathbf{R}^3$  we have  $\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1v_1 + u_2v_2 + u_3v_3 \in \mathbf{R}$  and, more generally, in  $\mathbf{R}^n$  we have  $\mathbf{u} \cdot \mathbf{v} = \langle u_1, \dots, u_n \rangle \cdot \langle v_1, \dots, v_n \rangle = u_1v_1 + \dots + u_nv_n \in \mathbf{R}$ . We also derive using the Law of Cosines that  $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$  where  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ . It is from this fact that we conclude that nonzero vectors  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal if and only if  $\mathbf{u} \cdot \mathbf{v} = 0$ .

Vectors in  $\mathbf{R}^3$  (or any  $\mathbf{R}^n$ ) have just a finite list of components, whereas functions of a real variable have infinitely many values. If we think of the values of a function as analogous to the components of a vector, and if we use integration as analogous to a discrete sum, this suggests the following definition:

**Definition:** If  $f, g$  are functions defined on some interval  $[a, b]$ , we can define an inner product of these functions by  $\langle f, g \rangle = \int_a^b f(t)g(t)dt$ . Such an inner product will then satisfy the following properties (where defined for any functions  $f, g, h$ ):

- (1)  $\langle g, f \rangle = \langle f, g \rangle$  (symmetric)
- (2)  $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$  and  $\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$  (left and right distributive laws)
- (3)  $\langle cf, g \rangle = c\langle f, g \rangle = \langle f, cg \rangle$  for any scalar  $c$

These properties are analogous to the algebraic properties of the dot product. We can use this inner product to define orthogonality.

**Definition:** We say that two (nonzero) functions  $f, g$  are orthogonal if  $\langle f, g \rangle = 0$ .

There is a 4th property of the dot product that doesn't quite work as simply in the context of functions. That is, for any vector  $\mathbf{v}$ , we have  $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \geq 0$  and  $\mathbf{v} \cdot \mathbf{v} = 0$  only if  $\mathbf{v} = \mathbf{0}$ . The corresponding statement for functions is not generally true. It will always be the case that  $\langle f, f \rangle = \int_a^b [f(t)]^2 dt \geq 0$ , but this inner product could be equal to zero for a function that is not identically 0 with isolated discontinuities where the function takes on nonzero values. If we only consider continuous functions, then  $\langle f, f \rangle = \int_a^b [f(t)]^2 dt = 0$  would imply that  $f(t)$  is identically zero on the interval  $[a, b]$ . In any case, we can still define the **norm of a function**  $f$  by  $\|f\|^2 = \langle f, f \rangle = \int_a^b [f(t)]^2 dt$  or  $\|f\| = \sqrt{\langle f, f \rangle}$ .

It's certainly possible that these integrals might not be defined, so we generally restrict the set of functions to those for which  $\|f\|^2 = \langle f, f \rangle = \int_a^b [f(t)]^2 dt$  is finite. These are called "square summable" functions, and the set of all such functions for the interval  $[a, b]$  is denoted by  $L^2([a, b])$ . It can be shown that in this space of functions,  $|\langle f, g \rangle| \leq \|f\| \|g\|$  (Cauchy-Schwartz Inequality) and  $\|f + g\| \leq \|f\| + \|g\|$  (Triangle Inequality).

**Note:** All of the above properties work just as well if we define the inner product as  $\langle f, g \rangle = K \int_a^b f(t)g(t)dt$  for some fixed constant  $K$ . This will alter the way in which the norm of a given function is defined, but using such a normalizing constant is often desirable when working with a particular interval. We'll start by considering functions defined on the interval  $[-\pi, \pi]$  and choose our normalizing constant so that  $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)dt$ .

**Proposition:** In  $L^2([a, b])$ , the finite collection  $\mathcal{B}_n = \left\{ \frac{1}{\sqrt{2}}, \cos t, \sin t, \cos 2t, \sin 2t, \dots, \cos nt, \sin nt \right\}$  is an **orthonormal** set. That is, each function in  $\mathcal{B}_n$  has norm 1 and any distinct pair has inner product equal to 0. We think of this set as consisting of mutually orthogonal unit elements. In linear algebra terminology, we would say that these  $2n+1$  functions **span** a **subspace** (referred to as  $T_n$ ) and that they form a **basis** for this subspace.

**Proof:** This is just a list of integral calculations. We'll calculate a few of them and just quote the rest (though you may want to try some integration techniques or consult an integral table to see why they are true). We have:

$$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \left( \frac{1}{\sqrt{2}} \right)^2 dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} dt = \frac{1}{\pi} \left( \frac{1}{2} \right) 2\pi = 1$$

$$\left\langle \frac{1}{\sqrt{2}}, \cos kt \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \cos kt dt = \frac{1}{\pi\sqrt{2}} \left. \frac{\sin kt}{k} \right|_{-\pi}^{\pi} = 0$$

$$\left\langle \frac{1}{\sqrt{2}}, \sin kt \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \sin kt dt = 0 \quad (\text{integral of an odd function over a symmetric interval})$$

$$\langle \cos kt, \cos kt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 kt dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} [1 + \cos(2kt)] dt = \frac{1}{2\pi} \left[ t + \frac{\sin(2kt)}{2k} \right]_{-\pi}^{\pi} = \frac{1}{2\pi} (2\pi) = 1$$

$$\langle \sin kt, \sin kt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 kt dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} [1 - \cos(2kt)] dt = \frac{1}{2\pi} \left[ t - \frac{\sin(2kt)}{2k} \right]_{-\pi}^{\pi} = \frac{1}{2\pi} (2\pi) = 1$$

$$\langle \cos jt, \cos kt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos jt \cos kt dt = 0 \quad \text{for integers } j \neq k \quad (\text{consult integral table})$$

$$\langle \sin jt, \sin kt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin jt \sin kt dt = 0 \quad \text{for integers } j \neq k \quad (\text{consult integral table})$$

$$\langle \cos jt, \sin kt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos jt \sin kt dt = 0 \quad \text{for integers } j, k \quad (\text{consult integral table})$$

We can define the orthogonal projection of a function  $f$  onto the subspace  $T_n$  analogous to  $\mathbf{R}^n$ , namely:

$$f_n = \text{Proj}_n(f) = \left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \langle f, \cos t \rangle \cos t + \langle f, \sin t \rangle \sin t + \cdots + \langle f, \cos nt \rangle \cos nt + \langle f, \sin nt \rangle \sin nt$$

If we express these in terms of integrals, we get:

$$f_n = \frac{1}{2} \left[ \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt \right] + \left[ \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos t dt \right] \cos t + \left[ \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin t dt \right] \sin t + \cdots \\ \cdots + \left[ \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt \right] \cos nt + \left[ \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt \right] \sin nt$$

This function  $f_n$  is known as the  **$n$ th order Fourier approximation** of the function  $f$ .

This can be expressed more succinctly by defining the **Fourier coefficients** by:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt \quad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt dt, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt dt$$

Then  $f_n = \frac{a_0}{2} + a_1 \cos t + b_1 \sin t + \cdots + a_n \cos nt + b_n \sin nt$  is the  $n$ th order Fourier approximation.

If, for any given  $n$ , we express  $f = (f - f_n) + f_n$ , we can think of  $f_n \in T_n$  and  $(f - f_n) \in T_n^\perp$  (known as the orthogonal complement of  $T_n$ ). There is the analogue of the Pythagorean Theorem in this context that gives that  $\|f\|^2 = \|f - f_n\|^2 + \|f_n\|^2$ . With some careful analysis it can be shown that as  $n$  gets larger, the Fourier approximation converges in the sense that  $\lim_{n \rightarrow \infty} \|f - f_n\|^2 = 0$ , and this implies that  $\boxed{\lim_{n \rightarrow \infty} \|f_n\|^2 = \|f\|^2}$  which can provide some very interesting results.

We also by letting  $n \rightarrow \infty$  produce the **Fourier Series** of  $f$  as  $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$ , and there's this accompanying theorem (proven elsewhere):

**Theorem (Fourier):** Suppose a function  $f(t)$  is periodic with base period  $2\pi$  and continuous except for a finite number of jump discontinuities. Then  $f(t)$  may be represented by a (convergent) Fourier Series:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

where:  $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt$ ,  $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt$ ,  $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt$ .

The numbers  $\{a_0, a_1, b_1, \dots, a_n, b_n, \dots\}$  are called the **Fourier coefficients** of the function  $f(t)$ .

This representation is an equality at all points of continuity of the function  $f(t)$ . At any point of discontinuity  $t = a$ , the series converges to the average of  $f(a^-)$  and  $f(a^+)$ , i.e. the value  $\frac{1}{2}[f(a^-) + f(a^+)]$ .

**Note:** (a) If  $f(t)$  is an even function [ $f(-t) = f(t)$  for all  $t$ ], then  $b_n = 0$  for all  $n$  by basic facts from calculus.

(b) If  $f(t)$  is an odd function [ $f(-t) = -f(t)$  for all  $t$ ], then  $a_0 = 0$  and  $a_n = 0$  for all  $n$  by basic facts from calculus.

**Example (Square wave function):**  $f(t) = sq(t) = \begin{cases} -1 & t \in [-\pi, 0) \\ +1 & t \in [0, \pi) \end{cases}$ , extended periodically for all  $t$ .

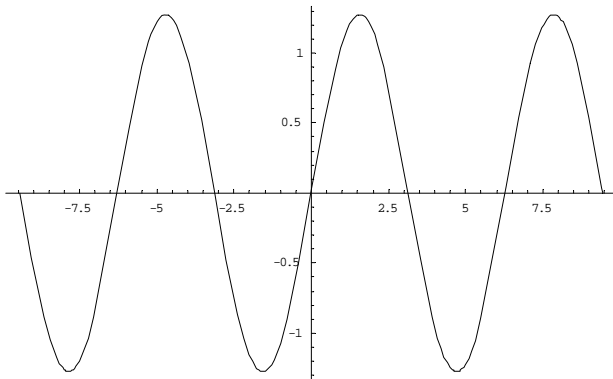
This function is periodic (with period  $2\pi$ ) and antisymmetric, i.e. an odd function. Therefore  $a_0 = 0$  and  $a_n = 0$

for all  $n$ . We calculate  $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt = \frac{1}{\pi} \left[ \int_{-\pi}^0 (-1) \sin nt \, dt + \int_0^{\pi} \sin nt \, dt \right] = \frac{1}{\pi} \left[ \left[ \frac{\cos nt}{n} \right]_{-\pi}^0 - \left[ \frac{\cos nt}{n} \right]_0^{\pi} \right]$

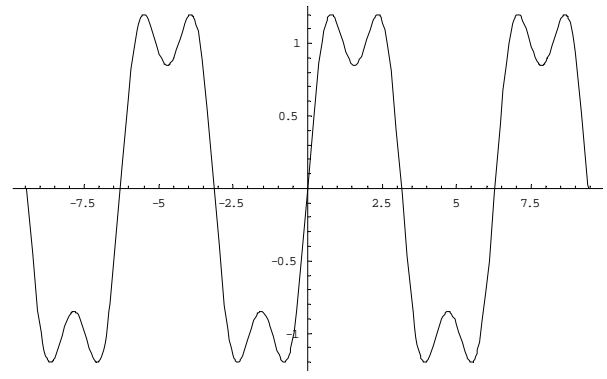
$$= \frac{1}{n\pi} \left[ [1 - (-1)^n] - [(-1)^n - 1] \right] = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}.$$

So  $sq(t) \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n} = \frac{4}{\pi} \left[ \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \dots \right].$

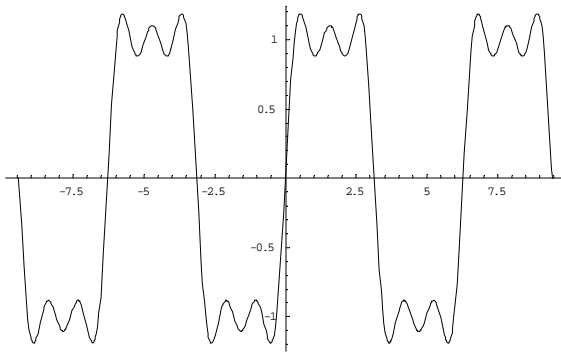
The nature of the convergence of this Fourier series toward the square wave function can be seen by graphing the partial sums:



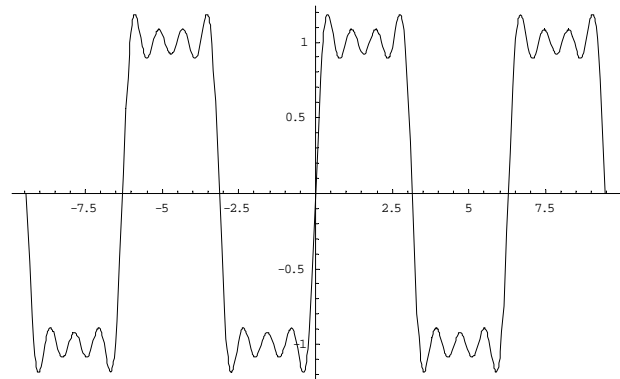
**n = 1**



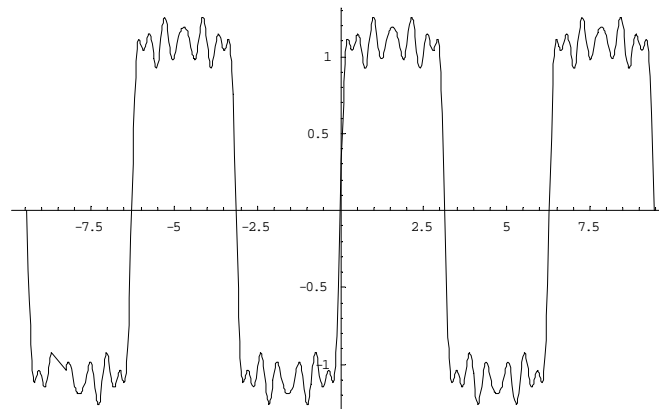
**n = 3**



**n = 5**



**n = 7**



**n = 15**

If we translate  $\lim_{n \rightarrow \infty} \|f_n\|^2 = \|f\|^2$  for this function we get that  $\|f\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 dt = 2$ , and

$\lim_{n \rightarrow \infty} \|f_n\|^2 = \frac{16}{\pi^2} \left[ 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots \right] = \frac{16}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = 2$ . Therefore  $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$ , a curious fact.

We can also apply the last statement in Fourier's Theorem by evaluating the square-wave function at  $\pi/2$ , a point of continuity, to get that  $sq(\pi/2) = 1 = \frac{4}{\pi} \left[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right]$ , so  $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$ , but the convergence is so abysmally slow as to be of no practical consequence – another curiosity.

**Notes by Robert Winters**