Math S-21b – Summer 2024 – Homework #9

Problems due Mon, Aug 5

Problem 1. Consider a linear system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ of arbitrary size. Suppose $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions of this system. Show that the sum $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ is also a solution for arbitrary scalars c_1 and c_2.

In Problems 2-5, (a) solve the system with the given initial value and (b) sketch rough phase portraits for the dynamical systems (or use the Java tool to sketch the underlying vector fields and some trajectories).

Problem 2. $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

Problem 3. $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & 3 \\ 4 & 8 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 4. $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

Problem 5. $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. [Sketching the 3D phase portrait is optional for this one.]

Problem 6. Consider the system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ with $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(a) Sketch a direction field for $\mathbf{A}\mathbf{x}$ (or use the Java tool) and, based on your sketch, describe the trajectories geometrically.

(b) Find the solutions analytically, i.e. give a formula for the general solution.

Problem 7. Let $\mathbf{A}$ be an $n \times n$ matrix and k a scalar.

Consider the following two systems: $\left\{ \begin{matrix} \frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} & \text{(I)} \\ \frac{d\mathbf{c}}{dt} = (\mathbf{A} + k\,\mathbf{I}_n)\mathbf{c} & \text{(II)} \end{matrix} \right\}$

Show that if $\mathbf{x}(t)$ is a solution of system (I), then $\mathbf{c}(t) = e^{kt}\mathbf{x}(t)$ is a solution of system (II).

Problem 8. Find all solutions of the system $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\mathbf{x}$, where λ is an arbitrary constant.

Hint: Problems 6 and 7 are helpful. Sketch a phase portrait. For which choices of λ is the zero state a stable equilibrium solution?

Problem 9. Find the general solution to the system: $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}\mathbf{x}$.

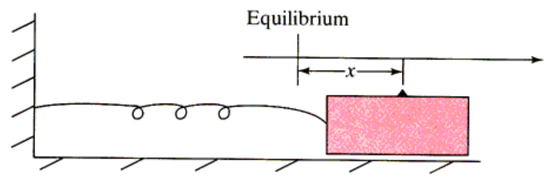
Compare this with Problem 8 above. When is the zero state a stable equilibrium solution?

Problem 10. Solve the system $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Give the solution in real form. Sketch the solution.

Problem 11. Solve the system $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Sketch the solution.

Problem 12. Consider the following mass-spring system:



Let $x(t)$ be the deviation of the block from the equilibrium position at time t. Consider the velocity $v(t) = \frac{dx}{dt}$ of the block. There are two forces acting on the mass: The spring force F_s, which is assumed to be proportional to the displacement x, and the force F_f of friction, which is assumed to be proportional to the velocity

$$F_s = -px \ , \quad F_f == -qv$$

where $p > 0$ and $q \geq 0$. (q is 0 if the oscillation is frictionless.) The total force acting on the mass is:

$$F = F_s + F_f = -px - qv \, .$$

By Newton's second law of motion, we have

$$F = ma = m\frac{dv}{dt} ,$$

where a represents acceleration and m the mass of the block. Combining the last two equations, we find that

$$m\frac{dv}{dt} = -px - qv \quad \text{or} \quad \frac{dv}{dt} = -\frac{p}{m}x - \frac{q}{m}v \, .$$

Let $b = \frac{p}{m}$ and $c = \frac{q}{m}$ for simplicity. Then the dynamics of this mass-spring system are described by the system

$$\left\{ \begin{aligned} \frac{dx}{dt} &= \quad\quad\quad v \\ \frac{dv}{dt} &= -bx - cv \end{aligned} \right\} \quad (b > 0, \ c \geq 0).$$

Sketch a phase portrait for this system in each of the following cases and describe briefly the significance of your trajectories in terms of the movement of the block. Comment on the stability in each case.

a) $c = 0$ (frictionless). Find the period. b) $c^2 < 4b$ (underdamped). c) $c^2 > 4b$ (overdamped).

Problem 13. Find all real solutions of the differential equation $\frac{dx}{dt} - 2x = \cos 3t$.

Problem 14. Solve the initial value problem $\frac{dx}{dt} + 3x = 7$; $x(0) = 0$.

Problem 15. Solve the initial value problem $f''(t) + f'(t) - 12f(t) = 0$; $f(0) = f'(0) = 0$.

Problem 16. The displacement of a certain forced oscillator can be modeled by the differential equation:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = \cos 3t \, .$$

a. Find all solutions of this differential equation.

b. Describe the long-term behavior of the oscillator.

For additional practice and inspiration (not to be turned in):

Section 9.1:

42. Consider the interaction of two species of animals in a habitat. We are told that the change of the populations **x**(t) and **y**(t) can be modeled by the equations

$$\begin{cases} \dfrac{dx}{dt} = 1.4x - 1.2y \\ \dfrac{dy}{dt} = 0.8x - 1.4y \end{cases}$$

where time t is measured in years.

a. What kind of interaction do we observe (symbiosis, competition, or predator-prey)?

b. Sketch a phase portrait for this system. From the nature of the problem, we are interested only in the first quadrant.

c. What will happen in the long term? Does the outcome depend on the initial populations? If so, how?

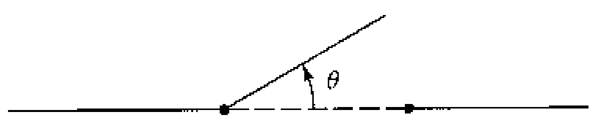
49. Here is a continuous model of a person's glucose regulatory system. (Compare this with Exercise 7.1.52.)

Let $g(t)$ and $h(t)$ be the excess glucose and insulin concentrations in a person's blood. We are told that

$$\begin{cases} \dfrac{dg}{dt} = -g - 0.2h \\ \dfrac{dh}{dt} = 0.6g - 0.2h \end{cases}$$

where time t is measured in hours. After a heavy holiday dinner, we measure $g(0) = 30$ and $h(0) = 0$. Find closed formulas for $g(t)$ and $h(t)$. Sketch the trajectory.

54. Consider a door that opens to only one side (as most doors do). A spring mechanism closes the door automatically. The state of the door at any given time t (measured in seconds) is determined by the angular displacement $\theta(t)$ (measured in radians) and the angular velocity $\omega(t) = \dfrac{d\theta}{dt}$. Note that θ is always positive or zero (since the door opens to only one side), but ω can be positive or negative (depending on whether the door is opening or closing).



When the door is moving freely (nobody is pushing or pulling), its movement is subject to the following differential equations:

$$\begin{cases} \dfrac{d\theta}{dt} = \omega & \text{(the definition of } \omega) \\ \dfrac{d\omega}{dt} = -2\theta - 3\omega & \text{(-}2\theta \text{ reflects the force of the spring, and -}3\omega \text{ models friction)} \end{cases}$$

a. Sketch the phase portrait of this system.

b. Discuss the movement of the door represented by the qualitatively different trajectories. For which initial states does the door slam (i,e,, reach $\theta = 0$ with velocity $\omega < 0$)?

55. Answer the questions posed in Exercise 54 for the system $\begin{cases} \dfrac{d\theta}{dt} = \omega \\ \dfrac{d\omega}{dt} = -p\theta - q\omega \end{cases}$

where p and q are positive, and $q^2 > 4p$.

Section 9.2:

6. Find all complex solutions of the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}\mathbf{x}$ in the form given in Fact 9.2.3.

7. Determine the stability of the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}\mathbf{x}$.

12. Determine the stability of the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{bmatrix}\mathbf{x}$.

For each of the linear systems in Exercises 22 through 26, find the matching phase portrait. (See right.)

22. $\mathbf{x}(t+1) = \begin{bmatrix} 3 & 0 \\ -2.5 & 0.5 \end{bmatrix}\mathbf{x}(t)$

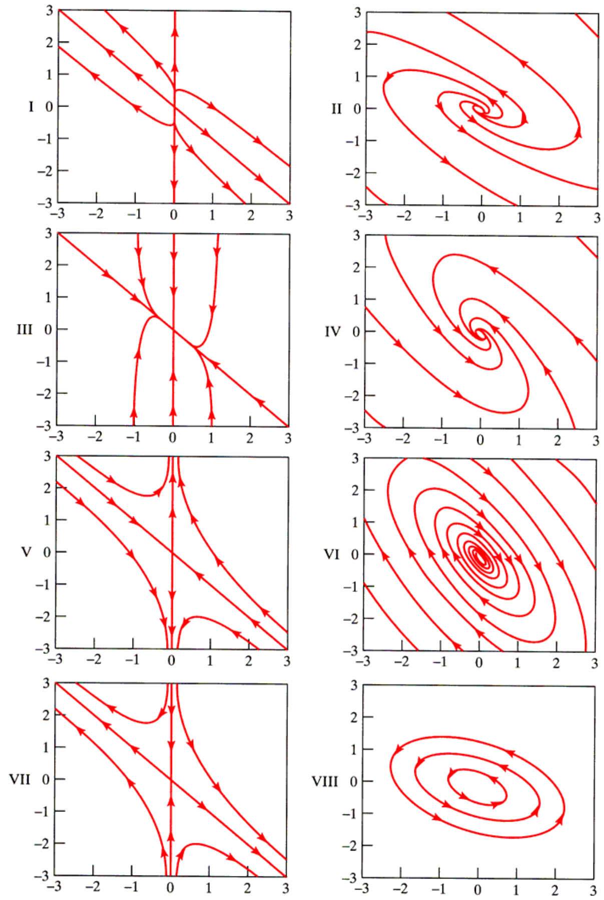
23. $\mathbf{x}(t+1) = \begin{bmatrix} -1.5 & -1 \\ 2 & 0.5 \end{bmatrix}\mathbf{x}(t)$

24. $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & 0 \\ -2.5 & 0.5 \end{bmatrix}\mathbf{x}$

25. $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} -1.5 & -1 \\ 2 & 0.5 \end{bmatrix}\mathbf{x}$

26. $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}\mathbf{x}$

Phase Portraits for Exercises 22-26



34. Solve the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 7 & 10 \\ -4 & -5 \end{bmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Give the solution in real form. Sketch the solution.

Section 9.3:

2. Find all real solutions of the differential equation $\dfrac{dx}{dt} + 3x = 7$.

7. Find all real solutions of the differential equation $f''(t) + f'(t) - 12f(t) = 0$.

13. Find all real solutions of the differential equation $f''(t) + 2f'(t) + f(t) = 0$.

17. Find all real solutions of the differential equation $f''(t) + 2f'(t) + f(t) = \sin t$.

43. The displacement of a certain forced oscillator can be modeled by the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = \cos t.$$

a. Find all solutions of this differential equation.

b. Describe the long-term behavior of the oscillator.