Math S-21b – Summer 2024 – Homework #8

Problems due Wed, July 31

Problem 1. Find all the complex eigenvalues of the matrix $\mathbf{A}=\begin{bmatrix}0&1&0\\0&0&1\\5&-7&3\end{bmatrix}$ (including the real ones, of course). Do not use technology. Show all your work.

Problem 2. Suppose a 3×3 matrix **A** has the real eigenvalue 2 and two complex eigenvalues. Also suppose that $\det(\mathbf{A})=50$ and $\operatorname{tr}(\mathbf{A})=8$ .Find the complex eigenvalues.

Problem 3. A real $n\times n$ matrix **A** is called a *regular transition matrix* if all entries of **A** are positive, and the entries in each column add up to 1. (See Exercises 24 through 31 of Section 7.2.)

An example is $\mathbf{A}=\begin{bmatrix}0.4&0.3&0.1\\0.5&0.1&0.2\\0.1&0.6&0.7\end{bmatrix}$. You may take the following properties of a regular transition matrix for granted (a partial proof is outlined in Exercise 7.2.31.):

- 1 is an eigenvalue of **A**, with $\dim(E_1)=1$.
- If λ is a complex eigenvalue of **A** other than 1, then $|\lambda|<1$.

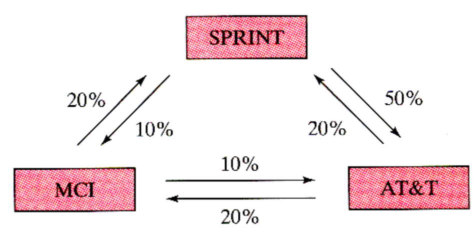
a. Consider a regular $n\times n$ transition matrix **A** and a vector **x** in $\mathbf{R}^n$ whose entries add up to 1. Show that the entries of **Ax** will also add up to 1.

b. Pick a regular transition matrix **A**, and compute some powers of **A** (using technology): $\mathbf{A}^2,\ldots,\mathbf{A}^{10},\ldots,\mathbf{A}^{100},\ldots$. What do you observe? Explain your observation. [You may assume there is a complex eigenbasis for **A**.]

Problem 4. Most long-distance telephone service in the United States was once upon a time provided by three companies: AT&T, MCI, and Sprint. The three companies were in fierce competition, offering discounts or even cash to those who switched. If the figures advertised by the companies were to be believed, people switched their provider from one month to the next according to the diagram shown. For example, 20% of the people who use AT&T go to Sprint one month later.

a. We introduce the state vector

$$\mathbf{x}(t)=\begin{bmatrix}a(t)\\m(t)\\s(t)\end{bmatrix}\begin{matrix}\text{fraction using AT\&T}\\\text{fraction using MCI}\\\text{fraction using Sprint}\end{matrix}$$



Find the matrix **A** such that $\mathbf{x}(t+1)=\mathbf{A}\mathbf{x}(t)$, assuming that the customer base remains unchanged. Note that **A** is a regular transition matrix.

b. Which fraction of the customers will be with each company in the long term? Do you have to know the current market shares to answer this question? Use the power method introduced in Exercise 30.

Problem 5. For the matrix $\mathbf{A}=\begin{bmatrix}1&-3\\1.2&-2.6\end{bmatrix}$, find real closed formulas for the trajectory $\mathbf{x}(t+1)=\mathbf{A}\mathbf{x}(t)$, where $\mathbf{x}(0)=\begin{bmatrix}0\\1\end{bmatrix}$. Draw a rough sketch.

Problem 6. Consider an <u>affine transformation</u> $T(\mathbf{x})=\mathbf{A}\mathbf{x}+\mathbf{b}$, where **A** is an $n\times n$ matrix and **b** is a vector in $\mathbf{R}^n$. Suppose that 1 is not an eigenvalue of **A**.

a. Find the vector **v** in $\mathbf{R}^n$ such that $T(\mathbf{v})=\mathbf{v}$; this vector is called the equilibrium state of the dynamical system $\mathbf{x}(t+1)=T(\mathbf{x}(t))$.

b. When is the equilibrium **v** in part (a) stable (meaning that $\lim_{t\to\infty}\mathbf{x}(t)=\mathbf{v}$ for all trajectories)?

Problem 7. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}$, find an orthogonal matrix **S** and a diagonal matrix **D** such that $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{D}$. Do not use technology.

Problem 8. Let L from $\mathbf{R}^3$ to $\mathbf{R}^3$ be the reflection about the line spanned by $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

a. Find an orthonormal eigenbasis $\mathscr{B}$ for L.
b. Find the matrix **B** of L with respect to $\mathscr{B}$.
c. Find the matrix **A** of L with respect to the standard basis of $\mathbf{R}^3$.

Problem 9.

a. Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$ with their multiplicities.

Note that the algebraic multiplicity agrees with the geometric multiplicity. (Why?)
Hint: What is the kernel of **A**?

b. Find the eigenvalues of the matrix

$$\mathbf{B} = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}$$ with their multiplicities.

Do not use technology.

c. Use your result in part (b) to find det(**B**).

Problem 10. Consider a symmetric $n \times n$ matrix **A** with $\mathbf{A}^2 = \mathbf{A}$. Is the linear transformation $T(\mathbf{x}) = \mathbf{Ax}$ necessarily the orthogonal projection onto a subspace of $\mathbf{R}^n$?

Problem 11. If **A** is a symmetric matrix, what can you say about the definiteness of $\mathbf{A}^2$? When is $\mathbf{A}^2$ positive definite?

Problem 12. Sketch the curve $q(x, y) = xy = 1$. Show the principal axes, label the intercepts of the curve with the principal axes, and give the formula of the curve in the coordinates defined by the principal axes.

Problem 13. Sketch the curve $q(x, y) = 9x^2 - 4xy + 6y^2 = 1$. Show the principal axes, label the intercepts of the curve with the principal axes, and give the formula of the curve in the coordinates defined by the principal axes.

Problem 14. On the surface $-x^2 + y^2 - z^2 + 10xz = 1$, find the two points closest to the origin.

Problem 15 (SVD). a) Given the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, find an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for the domain and an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ for the codomain such that the images of the basis vectors of the domain $\{\mathbf{Au}_1, \mathbf{Au}_2, \mathbf{Au}_3\}$ are scalar multiples of the orthonormal basis vectors $\{\mathbf{w}_1, \mathbf{w}_2\}$ (or the zero vector **0**). [Reference: exercises 8.1/19,20].

b) Use these to find the singular value decomposition $\mathbf{A} = \mathbf{Q\Sigma P}^{\mathrm{T}}$ where **Q** is a 2×2 orthogonal matrix, **P** is a 3×3 orthogonal matrix, and $\boldsymbol{\Sigma}$ is a 2×3 matrix that contains the "singular values" associated with the given matrix (as described in the text and notes).

c) Repeat the above exercise for the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. This is the transpose of the previous matrix.

[Note: There is a very simple way to answer (c) without additional calculation!]

For additional practice:

Section 7.5:

2. Find all complex numbers z such that $z^4 = 1$. Represent your answers graphically in the complex plane.

3. For an arbitrary positive integer n, find all complex numbers z such that $z^n = 1$ (in polar form). Represent your answers graphically.

8. Use de Moivre's formula to express $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\cos\theta$ and $\sin\theta$.

   [*Note*: de Moivre's formula states that $e^{in\theta} = \left(e^{i\theta}\right)^n = \left(\cos\theta + i\sin\theta\right)^n = \cos(n\theta) + i\sin(n\theta)$]

Find all complex eigenvalues of the matrices in Exercises 21, 23, and 25 (including the real ones, of course). Do not use technology. Show all your work.

21. $\mathbf{A} = \begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}$ 23. $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ 25. $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

27. Suppose a 3×3 matrix **A** has only two distinct eigenvalues. Suppose that tr(**A**) = 1 and det(**A**) = 3. Find the eigenvalues of **A** with their algebraic multiplicities.

36. In 1990, the population of the African country Benin was about 4.6 million people. Its composition by age was as follows:

Age Bracket	0-15	15-30	30-45	45-60	60-75	75-90
Percent of Population	46.6	25.7	14.7	8.4	3.8	0.8

We represent these data in a state vector whose components are the populations in the various age brackets, in millions:

$$\mathbf{x}(0) = 4.6\begin{bmatrix} 0.466 \\ 0.257 \\ 0.147 \\ 0.084 \\ 0.038 \\ 0.008 \end{bmatrix} \approx \begin{bmatrix} 2.14 \\ 1.18 \\ 0.68 \\ 0.39 \\ 0.17 \\ 0.04 \end{bmatrix}$$

We measure time in increments of 15 years, with $t = 0$ in 1990. For example, $\mathbf{x}(3)$ gives the age composition in the year 2035 (1990 + 3 · 15). If current age-dependent birth and death rates are extrapolated, we have the following model:

$$\mathbf{x}(t+1) = \begin{bmatrix} 1.1 & 1.6 & 0.6 & 0 & 0 & 0 \\ 0.82 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.89 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.81 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.53 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.29 & 0 \end{bmatrix}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)$$

a. Explain the significance of all the entries in the matrix **A** in terms of population dynamics.

b. Find the eigenvalue of **A** with largest modulus and associated eigenvalue. (Use technology.) What is the significance of these quantities in terms of population dynamics? (For a summary on matrix techniques used in the study of age-structured populations, see Dmitrii O. Logofet, *Matrices and Graphs: Stability Problems in Mathematical Ecology*, Chapters 2 and 3, CRC Press, 1993.)

Section 7.6:

For the matrices in Exercises 1-4, determine whether the zero state is a stable equilibrium of the dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$.

1. $\mathbf{A} = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.8 \end{bmatrix}$ 2. $\mathbf{A} = \begin{bmatrix} -1.1 & 0 \\ 0 & 0.9 \end{bmatrix}$ 3. $\mathbf{A} = \begin{bmatrix} 0.8 & 0.7 \\ -0.7 & 0.8 \end{bmatrix}$ 4. $\mathbf{A} = \begin{bmatrix} -0.9 & -0.4 \\ 0.4 & -0.9 \end{bmatrix}$

10. Determine whether the zero state is a stable equilibrium of the dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$, where

$$\mathbf{A} = \begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{bmatrix}.$$

12. Given the matrix $\mathbf{A} = \begin{bmatrix} 0.6 & k \\ -k & 0.6 \end{bmatrix}$, for which real numbers k is the zero state a stable equilibrium of the dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$?

15. Given the matrix $\mathbf{A} = \begin{bmatrix} 1 & k \\ 0.01 & 1 \end{bmatrix}$, for which real numbers k is the zero state a stable equilibrium of the dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$?

Section 8.1:

3. Without using technology, find an orthonormal eigenbasis for the matrix $\mathbf{A} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$.

5. Without using technology, find an orthonormal eigenbasis for the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

6. Without using technology, find an orthonormal eigenbasis for the matrix $\mathbf{A} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$.

15. If **A** is invertible and orthogonally diagonalizable, is $\mathbf{A}^{-1}$ orthogonally diagonalizable as well?

19. Consider a linear transformation L from $\mathbf{R}^m$ to $\mathbf{R}^n$. Show that there is an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ of $\mathbf{R}^m$ such that the vectors $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_m)\}$ are orthogonal. Note that some of the vectors $L(\mathbf{v}_i)$ may be zero. (*Hint*: Consider an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ for the symmetric matrix $\mathbf{A}^{\mathrm{T}}\mathbf{A}$.)

20. Consider a linear transformation T from $\mathbf{R}^m$ to $\mathbf{R}^n$, where $m \le n$. Show that there is an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ of $\mathbf{R}^m$ and an orthonormal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ of $\mathbf{R}^n$ such that $T(\mathbf{v}_i)$ is a scalar multiple of $\mathbf{w}_i$, for $i = 1, \ldots, m$. *Hint*: Exercise 19 is helpful.

24. Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Find an orthonormal eigenbasis for **A**.

29. Consider a symmetric matrix **A**. If the vector **v** is in the image of **A** and **w** is in the kernel of **A**, is **v** necessarily orthogonal to **w**? Justify your answer.

Section 8.2:

3. For the quadratic form $q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 6x_1x_3 + 7x_2x_3$, find a symmetric matrix **A** such that $q(\mathbf{x}) = \mathbf{x} \cdot \mathbf{A}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$.

4. Determine the definiteness of the quadratic form $q(x_1, x_2) = 6x_1^2 + 4x_1x_2 + 3x_2^2$.

6. Determine the definiteness of the quadratic form $q(x_1, x_2) = 2x_1^2 + 6x_1x_2 + 4x_2^2$.

9. Recall that a real square matrix **A** is called *skew-symmetric* if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$.
   a. If **A** is skew-symmetric, is $\mathbf{A}^2$ skew-symmetric as well? Or is $\mathbf{A}^2$ symmetric?
   b. If **A** is skew-symmetric, what can you say about the definiteness of $\mathbf{A}^2$? What about the eigenvalues of $\mathbf{A}^2$?
   c. What can you say about the complex eigenvalues of a skew-symmetric matrix? Which skew-symmetric matrices are diagonalizable over **R** (the real numbers)?

11. If A is an invertible symmetric matrix, what is the relationship between the definiteness of **A** and $\mathbf{A}^{-1}$?

Sketch the curves in Exercises 15 and 19. In each case, draw and label the principal axes, label the intercepts of the curve with the principal axes, and give the formula of the curve in the coordinate system defined by the principal axes.

15. $q(x_1, x_2) = 6x_1^2 + 4x_1x_2 + 3x_2^2 = 1$

19. $q(x_1, x_2) = x_1^2 + 4x_1x_2 + 4x_2^2 = 1$

Chapter 7 True/False

1. The algebraic multiplicity of an eigenvalue cannot exceed its geometric multiplicity.

2. If an $n \times n$ matrix A is diagonalizable (over $\mathbb{R}$), then there must be a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

3. If the standard vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are eigenvectors of an $n \times n$ matrix A, then A must be diagonal.

4. If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ must be an eigenvector of A^3 as well.

5. There exists a diagonalizable 5×5 matrix with only two distinct eigenvalues (over $\mathbb{C}$).

6. There exists a real 5×5 matrix without any real eigenvalues.

7. If 0 is an eigenvalue of a matrix A, then $\det A = 0$.

8. The eigenvalues of a 2×2 matrix A are the solutions of the equation $\lambda^2 - (\mathrm{tr}A)\lambda + (\det A) = 0$.

9. The eigenvalues of any triangular matrix are its diagonal entries.

10. The trace of any square matrix is the sum of its diagonal entries.

11. Any rotation-scaling matrix in $\mathbb{R}^{2\times 2}$ is diagonalizable over $\mathbb{C}$.

12. If A is a noninvertible $n \times n$ matrix, then the geometric multiplicity of eigenvalue 0 is $n - \mathrm{rank}(A)$.

13. If matrix A is diagonalizable, then its transpose A^T must be diagonalizable as well.

14. If A and B are two 3×3 matrices such that $\mathrm{tr}\, A = \mathrm{tr}\, B$ and $\det A = \det B$, then A and B must have the same eigenvalues.

15. If 1 is the only eigenvalue of an $n \times n$ matrix A, then A must be I_n.

16. If A and B are $n \times n$ matrices, if α is an eigenvalue of A, and if β is an eigenvalue of B, then $\alpha\beta$ must be an eigenvalue of AB.

17. If 3 is an eigenvalue of an $n \times n$ matrix A, then 9 must be an eigenvalue of A^2.

18. The matrix of any orthogonal projection onto a subspace V of $\mathbb{R}^n$ is diagonalizable.

19. If matrices A and B have the same eigenvalues (over $\mathbb{C}$), with the same algebraic multiplicities, then matrices A and B must have the same trace.

20. If a real matrix A has only the eigenvalues 1 and -1, then A must be orthogonal.

21. If an invertible matrix A is diagonalizable, then A^{-1} must be diagonalizable as well.

22. If $\det(A) = \det(A^T)$, then matrix A must be symmetric.

23. If matrix $A = \begin{bmatrix} 7 & a & b \\ 0 & 7 & c \\ 0 & 0 & 7 \end{bmatrix}$ is diagonalizable, then a, b, and c must all be zero.

24. If two $n \times n$ matrices A and B are diagonalizable, then $A + B$ must be diagonalizable as well.

25. All diagonalizable matrices are invertible.

26. If vector $\vec{v}$ is an eigenvector of both A and B, then $\vec{v}$ must be an eigenvector of $A + B$.

27. If matrix A^2 is diagonalizable, then matrix A must be diagonalizable as well.

28. The determinant of a matrix is the product of its eigenvalues (over $\mathbb{C}$), counted with their algebraic multiplicities.

29. All lower triangular matrices are diagonalizable (over $\mathbb{C}$).

30. If two $n \times n$ matrices A and B are diagonalizable, then AB must be diagonalizable as well.

31. If $\vec{u}$, $\vec{v}$, $\vec{w}$ are eigenvectors of a 4×4 matrix A, with associated eigenvalues 3, 7, and 11, respectively, then vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ must be linearly independent.

32. If a 4×4 matrix A is diagonalizable, then the matrix $A + 4I_4$ must be diagonalizable as well.

33. If an $n \times n$ matrix A is diagonalizable, then A must have n distinct eigenvalues.

34. If two 3×3 matrices A and B both have the eigenvalues 1, 2, and 3, then A must be similar to B.

35. If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ must be an eigenvector of A^T as well.

36. All invertible matrices are diagonalizable.

37. If $\vec{v}$ and $\vec{w}$ are linearly independent eigenvectors of matrix A, then $\vec{v} + \vec{w}$ must be an eigenvector of A as well.

38. If a 2×2 matrix R represents a reflection about a line L, then R must be diagonalizable.

39. If A is a 2×2 matrix such that $\mathrm{tr}\, A = 1$ and $\det A = -6$, then A must be diagonalizable.

40. If a matrix is diagonalizable, then the algebraic multiplicity of each of its eigenvalues λ must equal the geometric multiplicity of λ.

41. All orthogonal matrices are diagonalizable (over $\mathbb{R}$).

42. If A is an $n \times n$ matrix and λ is an eigenvalue of the block matrix $M = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$, then λ must be an eigenvalue of matrix A.

43. If two matrices A and B have the same characteristic polynomials, then they must be similar.

44. If A is a diagonalizable 4×4 matrix with $A^4 = 0$, then A must be the zero matrix.

45. If an $n \times n$ matrix A is diagonalizable (over $\mathbb{R}$), then every vector $\vec{v}$ in $\mathbb{R}^n$ can be expressed as a sum of eigenvectors of A.

46. If vector $\vec{v}$ is an eigenvector of both A and B, then $\vec{v}$ is an eigenvector of AB.

47. Similar matrices have the same characteristic polynomials.

48. If a matrix A has k distinct eigenvalues, then $\operatorname{rank}(A) \geq k$.

49. If the rank of a square matrix A is 1, then all the nonzero vectors in the image of A are eigenvectors of A.

50. If the rank of an $n \times n$ matrix A is 1, then A must be diagonalizable.

51. If A is a 4×4 matrix with $A^4 = 0$, then 0 is the only eigenvalue of A.

52. If two $n \times n$ matrices A and B are both diagonalizable, then they must commute.

53. If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ must be in the kernel of A or in the image of A.

54. All symmetric 2×2 matrices are diagonalizable (over $\mathbb{R}$).

55. If A is a 2×2 matrix with eigenvalues 3 and 4 and if $\vec{u}$ is a unit eigenvector of A, then the length of vector $A\vec{u}$ cannot exceed 4.

56. If $\vec{u}$ is a nonzero vector in $\mathbb{R}^n$, then $\vec{u}$ must be an eigenvector of matrix $\vec{u}\vec{u}^T$.

57. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is an eigenbasis for both A and B, then matrices A and B must commute.

58. If $\vec{v}$ is an eigenvector of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\vec{v}$ must be an eigenvector of its classical adjoint $\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ as well.

Chapter 8 True/False

1. If A is an orthogonal matrix, then there must exist a symmetric invertible matrix S such that $S^{-1}AS$ is diagonal.

2. The singular value of the 2×1 matrix $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is 5.

3. The function $q(x_1, x_2) = 3x_1^2 + 4x_1x_2 + 5x_2$ is a quadratic form.

4. The singular values of any matrix A are the eigenvalues of matrix A^TA.

5. If matrix A is positive definite, then all the eigenvalues of A must be positive.

6. The function $q(\vec{x}) = \vec{x}^T \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \vec{x}$ is a quadratic form.

7. The singular values of any diagonal matrix D are the absolute values of the diagonal entries of D.

8. The equation $2x^2 + 5xy + 3y^2 = 1$ defines an ellipse.

9. All symmetric matrices are diagonalizable.

10. If the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite, then a must be positive.

11. If the singular values of a 2×2 matrix A are 3 and 4, then there must exist a unit vector $\vec{u}$ in $\mathbb{R}^2$ such that $\|A\vec{u}\| = 4$.

12. The determinant of a negative definite 4×4 matrix must be positive.

13. If A is a symmetric matrix such that $A\vec{v} = 3\vec{v}$ and $A\vec{w} = 4\vec{w}$, then the equation $\vec{v} \cdot \vec{w} = 0$ must hold.

14. Matrix $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ is negative definite.

15. All skew-symmetric matrices are diagonalizable (over $\mathbb{R}$).

16. If A is any matrix, then matrix AA^T is diagonalizable.

17. All positive definite matrices are invertible.

18. Matrix $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ is diagonalizable.

19. The singular values of any triangular matrix are the absolute values of its diagonal entries.

20. If A is any matrix, then matrix A^TA is the transpose of AA^T.

21. If $\vec{v}$ and $\vec{w}$ are linearly independent eigenvectors of a symmetric matrix A, then $\vec{w}$ must be orthogonal to $\vec{v}$.

22. For any $n \times m$ matrix A there exists an orthogonal $m \times m$ matrix S such that the columns of matrix AS are orthogonal.

23. If A is a symmetric $n \times n$ matrix such that $A^n = 0$, then A must be the zero matrix.

24. If $q(\vec{x})$ is a positive definite quadratic form, then so is $kq(\vec{x})$, for any scalar k.

25. If A is an invertible symmetric matrix, then A^2 must be positive definite.

26. If the two columns $\vec{v}$ and $\vec{w}$ of a 2×2 matrix A are orthogonal, then the singular values of A must be $\|\vec{v}\|$ and $\|\vec{w}\|$.

27. If A and S are invertible $n \times n$ matrices, then matrices A and $S^T AS$ must be similar.

28. If A is negative definite, then all the diagonal entries of A must be negative.

29. If the positive definite matrix A is similar to the symmetric matrix B, then B must be positive definite as well.

30. If A is a symmetric matrix, then there must exist an orthogonal matrix S such that SAS^T is diagonal.

31. If A and B are 2×2 matrices, then the singular values of matrices AB and BA must be the same.

32. If A is any orthogonal matrix, then matrix $A + A^{-1}$ is diagonalizable (over $\mathbb{R}$).

33. The product of two quadratic forms in 3 variables must be a quadratic form as well.

34. The function $q(\vec{x}) = \vec{x}^T \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x}$ is a quadratic form.

35. If the determinants of all the principal submatrices of a symmetric 3×3 matrix A are negative, then A must be negative definite.

36. If A and B are positive definite $n \times n$ matrices, then matrix $A + B$ must be positive definite as well.

37. If A is a positive definite $n \times n$ matrix and $\vec{x}$ is a nonzero vector in $\mathbb{R}^n$, then the angle between $\vec{x}$ and $A\vec{x}$ must be acute.

38. If the 2×2 matrix A has the singular values 2 and 3 and the 2×2 matrix B has the singular values 4 and 5, then both singular values of matrix AB must be ≤ 15.

39. The equation $A^T A = AA^T$ holds for all square matrices A.

40. For every symmetric $n \times n$ matrix A there exists a constant k such that $A + kI_n$ is positive definite.

41. If matrix $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ is positive definite, then af must exceed c^2.

42. If A is positive definite, then all the entries of A must be positive or zero.

43. If A is indefinite, then 0 must be an eigenvalue of A.

44. If A is a 2×2 matrix with singular values 3 and 5, then there must exist a unit vector $\vec{u}$ in $\mathbb{R}^2$ such that $\|A\vec{u}\| = 4$.

45. If A is skew symmetric, then A^2 must be negative semidefinite.

46. The product of the n singular values of an $n \times n$ matrix A must be $|\det A|$.

47. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, then there exist exactly 4 orthogonal 2×2 matrices S such that $S^{-1}AS$ is diagonal.

48. The sum of two quadratic forms in 3 variables must be a quadratic form as well.

49. The eigenvalues of a symmetric matrix A must be equal to the singular values of A.

50. Similar matrices must have the same singular values.

51. If A is a symmetric 2×2 matrix with eigenvalues 1 and 2, then the angle between $\vec{x}$ and $A\vec{x}$ must be less than $\pi/6$, for all nonzero vectors $\vec{x}$ in $\mathbb{R}^2$.

52. If both singular values of a 2×2 matrix A are less than 5, then all the entries of A must be less than 5.

53. If A is a positive definite matrix, then the largest entry of A must be on the diagonal.

54. If A and B are real symmetric matrices such that $A^3 = B^3$, then A must be equal to B.