Math S-21b – Summer 2024 – Homework #7

Problems due Fri, July 26:

Problem 1. Suppose **v** is an eigenvector of the $n\times n$ matrix **A** with associated eigenvalue λ, and let $q(x)$ be any polynomial. Show that **v** is also an eigenvector of the matrix $q(\mathbf{A})$. What is the associated eigenvalue? [**Note**: If $q(x)=c_m x^m + c_{m-1}x^{m-1}+\cdots+c_1 x + c_0$, then $q(\mathbf{A})=c_m\mathbf{A}^m + c_{m-1}\mathbf{A}^{m-1}+\cdots+c_1\mathbf{A}+c_0\mathbf{I}$.] For example, if **v** is an eigenvector of the $n\times n$ matrix **A** with eigenvalue 4, it will also be an eigenvector of the matrix $\mathbf{A}^2+2\mathbf{A}+3\mathbf{I}_n$. What is its eigenvalue?

Problem 2. Find a 2×2 matrix **A** such that $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$ are eigenvectors of **A**, with eigenvalues 5 and 10, respectively.

Problem 3. Two interacting populations of hares and foxes can be modeled by the recursive equations:

$$h(t+1)=4h(t)-2f(t)$$
$$f(t+1)=\ \ h(t)+\ \ f(t)$$

For each of the initial populations given in parts (a) through (c), find closed formulas for $h(t)$ and $f(t)$.

a. $h(0)=f(0)=100$ b. $h(0)=200, f(0)=100$ c. $h(0)=600, f(0)=500$

Problem 4. For the matrix $\mathbf{A}=\begin{bmatrix}4 & -2\\ 1 & 1\end{bmatrix}$, find formulas for the matrix of $\mathbf{A}^t$ and the vector $\mathbf{A}^t\begin{bmatrix}1\\2\end{bmatrix}$ where t is a positive integer.

For each of the matrices in Problems 5-8, find all (real) eigenvalues, their algebraic multiplicities, and their geometric multiplicities. Then find a basis for each eigenspace, and determine if **A** is diagonalizable. <u>If possible</u>, find an invertible **S** and a diagonal **D** such that $\mathbf{S}^{-1}\mathbf{AS}=\mathbf{D}$. ***Show your work!***

Problem 5. $\begin{bmatrix}0 & 4\\ -1 & 4\end{bmatrix}$ **Problem 6.** $\begin{bmatrix}-1 & -1 & -1\\ -1 & -1 & -1\\ -1 & -1 & -1\end{bmatrix}$ **Problem 7.** $\mathbf{A}=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$ **Problem 8.** $\begin{bmatrix}-2 & 2 & -1\\ -2 & 3 & -1\\ -4 & 4 & -1\end{bmatrix}$

Problem 9. If a 2×2 matrix **A** has two eigenvalues λ_1 and λ_2, show that $\operatorname{tr}(\mathbf{A})=\lambda_1+\lambda_2$ and $\det(\mathbf{A})=\lambda_1\lambda_2$. (The trace of a matrix is the sum of its diagonal entries.)
[**Note**: In general, if an $n\times n$ matrix **A** has n eigenvalues $\lambda_1,\ldots,\lambda_n$, listed with their algebraic multiplicities, then $\operatorname{tr}(\mathbf{A})=\lambda_1+\cdots+\lambda_n$ and $\det(\mathbf{A})=\lambda_1\lambda_2\cdots\lambda_n$.]

Problem 10. Consider an arbitrary $n\times n$ matrix **A**. What is the relationship between the characteristic polynomials **A** and $\mathbf{A}^{\mathrm{T}}$? What does your answer tell you about the eigenvalues of **A** and $\mathbf{A}^{\mathrm{T}}$?

Problem 11. Suppose that $\mathbf{B}=\mathbf{S}^{-1}\mathbf{AS}$ for some $n\times n$ matrices **A**, **B**, and **S**, i.e. **A** and **B** are *similar*.

a. Show that if **x** is in $\ker(\mathbf{B})$, then **Sx** is in $\ker(\mathbf{A})$.

b. Show that the linear transformation $T(\mathbf{x})=\mathbf{Sx}$ from $\ker(\mathbf{B})$ to $\ker(\mathbf{A})$ is an <u>isomorphism</u>.

c. Show that $\text{nullity}(\mathbf{A})=\text{nullity}(\mathbf{B})$ and $\text{rank}(\mathbf{A})=\text{rank}(\mathbf{B})$.

Problem 12. (a) Is the matrix $\mathbf{A}=\begin{bmatrix}0 & 1\\ 5 & 3\end{bmatrix}$ similar to the matrix $\mathbf{B}=\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$? Justify your answer.

(b) Is the matrix $\begin{bmatrix}-1 & 6\\ -2 & 6\end{bmatrix}$ similar to the matrix $\begin{bmatrix}1 & 2\\ -1 & 4\end{bmatrix}$? Justify your answer.

Problem 13. Some years ago, the Broadway Marketplace opened a couple of blocks from Harvard, selling fresh meat and produce and some groceries and maintaining a constant weekly customer base of 2,000 people. For argument's sake, let's say that a new store, Megamart Inc., opens nearby. Weekly surveys show that Broadway keeps 50% of its customers from one week to the next, with the rest going for the cheaper prices at the Megamart, and that Megamart retains 70% of its customers from the previous week with the rest going to Broadway. The state of grocery shopping can be represented by the vector $\mathbf{x}(t)=\begin{bmatrix} B(t) \\ M(t) \end{bmatrix}$, where $B(t)$ and $M(t)$ are the numbers of customers shopping at Broadway Marketplace and Megamart, respectively, t weeks after Megamart opened. Initially, $B(0)=2000$ and $M(0)=0$.

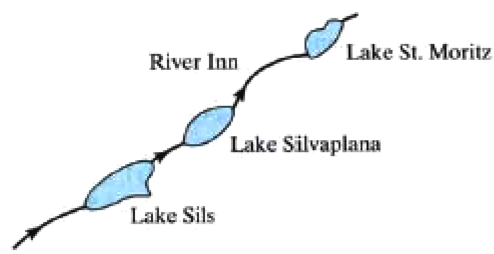
a. Find a matrix $\mathbf{A}$ such that $\mathbf{x}(t+1)=\mathbf{A}\mathbf{x}(t)$. Verify that $\mathbf{A}$ is a regular transition matrix. [See above.]

b. How many customers will shop at each store after t weeks? Give closed formulas.

c. The Broadway Marketplace fears that they must close down when they have less than 700 customers per week. Does that happen and, if so, after how many weeks will this happen?

Problem 14. In an unfortunate accident involving an Austrian truck, 100 kg of a highly toxic substance are spilled into Lake Sils, in the Swiss Engadine Valley. The river Inn carries the pollutant down to Lake Silvaplana and later to Lake St. Moritz. This sorry state, t weeks after the accident, can be described by the vector:

$$\mathbf{x}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \left.\begin{matrix} \text{pollutant in Lake Sils} \\ \text{pollutant in Lake Silvaplana} \\ \text{pollutant in Lake St. Moritz} \end{matrix}\right\} \text{(in kg).}$$



Suppose that $\mathbf{x}(t+1)=\begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}\mathbf{x}(t)$.

a. Explain the significance of the entries of the transformation matrix in practical terms.

b. Find closed formulas for the amount of pollutant in each of the three lakes t weeks after the accident. Graph the three functions against time (on the same axes). When does the pollution in Lake Silvaplana reach a maximum?

Problem 15. Find all the eigenvalues and "eigenvectors" of $[T(f)](x)=f(3x-1)$ from P_2 to P_2. Is T diagonalizable?

Problem 16. Find all the eigenvalues and "eigenvectors" of $[T(f)](x)=f(x-3)$ from P_2 to P_2. Is T diagonalizable?

For additional practice:
Section 7.1:

In Exercises 1 through 4, let **A** be an invertible $n \times n$ matrix and **v** an eigenvector of **A** with associated eigenvalue λ.

1. Is **v** an eigenvector of $\mathbf{A}^3$? If so, what is the eigenvalue?

2. Is **v** an eigenvector of $\mathbf{A}^{-1}$? If so, what is the eigenvalue?

3. Is **v** an eigenvector of $\mathbf{A} + 2\mathbf{I}_n$? If so, what is the eigenvalue?

4. Is **v** an eigenvector of 7**A**? If so, what is the eigenvalue?

5. If a vector **v** is an eigenvector of both **A** and **B**, is **v** necessarily an eigenvector of **A** + **B**?

6. If a vector **v** is an eigenvector of both **A** and **B**, is **v** necessarily an eigenvector of **AB**?

Arguing geometrically, find all eigenvectors and eigenvalues of the linear transformations in Exercises 15 through 21. Find a basis consisting of eigenvectors if possible.

15. Reflection about a line L in $\mathbf{R}^2$.

16. Rotation through an angle of 180° in $\mathbf{R}^2$.

17. Counterclockwise rotation through an angle of 45° followed by a scaling by 2 in $\mathbf{R}^2$.

18. Reflection about a plane V in $\mathbf{R}^3$.

19. Orthogonal projection onto a line L in $\mathbf{R}^3$.

20. Rotation about the $\mathbf{e}_3$-axis through an angle of 90°, counterclockwise as viewed from the positive $\mathbf{e}_3$-axis in $\mathbf{R}^3$.

21. Scaling by 5 in $\mathbf{R}^3$.

33. Find a 2×2 matrix **A** such that $\mathbf{x}(t) = \begin{bmatrix} 2^t - 6^t \\ 2^t + 6^t \end{bmatrix}$ is a trajectory of the dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$.

35. Show that similar matrices have the same eigenvalues. Hint: If **v** is an eigenvector of $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, then **Sv** is an eigenvector of **A**.

39. Find a basis of the linear space V of 2×2 matrices **A** for which $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector, and thus determine the dimension of V.

53. Three holy men (let's call them Abraham, Benjamin, and Chaim) put little stock in material things; their only earthly possession is a small purse with a bit of gold dust. Each day they get together for the following bizarre bonding ritual: Each of them takes his purse and gives his gold away to the two others, in equal parts. For example, if Abraham has 4 ounces one day, he will give 2 ounces each to Benjamin and Chaim.

    a. If Abraham starts out with 6 ounces, Benjamin with 1 ounce, and Chaim with 2 ounces, find formulas for the amounts $a(t)$, $b(t)$, and $c(t)$ each will have after t distributions.

    Hint: The vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ will be useful.

    b. Who will have the most gold after one year, that is, after 365 distributions?

Section 7.2:

For the matrices in Exercises 5 and 7, find all real eigenvalues, with their algebraic multiplicities. Show all your work. Do not use technology.

5. $\begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}$ 7. $\mathbf{I}_3$

15. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix}$, where k is an arbitrary constant. For which values of k does **A** have two distinct real eigenvalues? When is there no real eigenvalue?

21. Prove the part of Fact 7.2.8 that concerns the trace: If an $n \times n$ matrix **A** has n eigenvalues $\lambda_1, \ldots, \lambda_n$, listed with their algebraic multiplicities, then $\operatorname{tr}(\mathbf{A}) = \lambda_1 + \cdots + \lambda_n$.

23. Suppose matrix **A** is similar to **B**. What is the relationship between the characteristic polynomials **A** and **B**? What does your answer tell you about the eigenvalues of **A** and **B**?

24. Find all eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}$.

25. Consider a 2×2 matrices **A** of the form $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c, and d are positive numbers such that $a + c = b + d = 1$. (The matrix in Exercise 24 has this form.) Such a matrix is called a *regular transition matrix*. Verify that $\begin{bmatrix} b \\ c \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors of **A**. What are the associated eigenvalues? Is the absolute value of these eigenvalues more or less than 1? Sketch a phase portrait.

26. Based on your answer to Exercise 25, sketch a phase portrait of the dynamical system $\mathbf{x}(t+1) = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix} \mathbf{x}(t)$.

27. a. Based on your answer to Exercise 25, find closed formulas for the components of the dynamical system $\mathbf{x}(t+1) = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix} \mathbf{x}(t)$, with initial value $\mathbf{x}_0 = \mathbf{e}_1$. Then do the same for the initial value $\mathbf{x}_0 = \mathbf{e}_2$. Sketch the two trajectories.

    b. Consider the matrix $\mathbf{A} = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}$. Using technology, compute some powers of the matrix **A**, say, $\mathbf{A}^2$, $\mathbf{A}^5$, $\mathbf{A}^{10}$, …. What do you observe? Explain your answer carefully.

    c. If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an arbitrary regular transition matrix, what can you say about the powers $\mathbf{A}^t$ as t goes to infinity?

Section 7.3:

For the matrices in Exercises 11 and 16, find all (real) eigenvalues. Then find a basis for each eigenspace, and find an eigenbasis, if you can. Do not use technology.

11. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ 16. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$

21. Find a 2×2 matrices **A** for which $E_1 = \text{span}\begin{bmatrix}1\\2\end{bmatrix}$ and $E_2 = \text{span}\begin{bmatrix}2\\3\end{bmatrix}$. How many such matrices are there?

27. Consider a 2×2 matrices **A**. Suppose that tr(**A**) = 5 and det(**A**) = 6. Find the eigenvalues of **A**.

33. Show that if matrix **A** is similar to **B**, then $\mathbf{A}-\lambda\mathbf{I}_n$ is similar to $\mathbf{B}-\lambda\mathbf{I}_n$, for all scalars λ.
[Note: The same relationship holds between $\lambda\mathbf{I}_n-\mathbf{A}$ and $\lambda\mathbf{I}_n-\mathbf{B}$ using the same argument.]

35. Is matrix $\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}$ similar to $\begin{bmatrix}3 & 0\\1 & 2\end{bmatrix}$?

41. Consider a modification of Example 5: Suppose the transformation matrix is $\mathbf{A}=\begin{bmatrix}0 & 1.4 & 1.2\\0.8 & 0 & 0\\0 & 0.4 & 0\end{bmatrix}$. The initial populations are $j(0) = 600$, $m(0) = 100$, and $a(0) = 250$. Find closed formulas for $j(t)$, $m(t)$, and $a(t)$. Describe the long-term behavior. What can you say about the proportion $j(t) : m(t) : a(t)$ in the long term?

47. The color of snapdragons is determined by a pair of genes, which we designate by the letters *A* and *a*. The pair of genes is called the flower's *genotype*. Genotype *AA* produces red flowers, genotype *Aa* pink ones, and genotype *aa* white ones. A biologist undertakes a breeding program, starting with a large population of flowers of genotype *AA*. Each flower is fertilized with pollen from a plant of genotype *Aa* (taken from another population), and one offspring is produced. Since it is a matter of chance which of the genes a parent passes on, we expect half of the flowers in the next generation to be red (genotype *AA*) and the other half pink (genotype *Aa*). All the flowers in this generation are now fertilized with pollen from plants of genotype *Aa* (taken from another population), and so on.

a. Find closed formulas for the fractions of red, pink, and white flowers in the t-th generation. We know that $r(0) = 1$ and $p(0) = w(0) = 0$, and we found that $r(1) = p(1) = ½$ and $w(1) = 0$.

b. What is the proportion $r(t) : p(t) : w(t)$ in the long term?

Section 7.4:
In Exercises 4, 5, 6, 11, and 19, determine if **A** is diagonalizable. If possible, find an invertible **S** and a diagonal **D** such that $\mathbf{S}^{-1}\mathbf{AS}=\mathbf{D}$. Do not use technology.

4. $\mathbf{A}=\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}$ 5. $\mathbf{A}=\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ 6. $\mathbf{A}=\begin{bmatrix}2 & 0\\-1 & 2\end{bmatrix}$ 11. $\mathbf{A}=\begin{bmatrix}1 & 0 & 1\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}$ 19. $\mathbf{A}=\begin{bmatrix}1 & 1 & 1\\0 & 1 & 0\\0 & 1 & 0\end{bmatrix}$

47. Find all the eigenvalues and "eigenvectors" of $[T(f)](x) = f(-x)$ from P_2 to P_2. Is T diagonalizable?

48. Find all the eigenvalues and "eigenvectors" of $[T(f)](x) = f(2x)$ from P_2 to P_2. Is T diagonalizable?