Math S-21b – Summer 2024 – Homework #6

Problems due Mon, July 22:

Section 5.5:

Problem 1. Consider the space P_2 with inner product $\langle f, g\rangle = \frac{1}{2}\int_{-1}^{1} f(t)g(t)\,dt$. Find an orthonormal basis of the space of all functions in P_2 that are orthogonal to the function $f(t) = t$.

Problem 2. Find all Fourier coefficients of the absolute value function $f(t) = |t|$ defined on the interval $[-\pi, \pi]$.

Problem 3(a). Find the Fourier coefficients of the piecewise continuous function $f(t) = \begin{Bmatrix} 0 & \text{if } t \le 0 \\ 1 & \text{if } t > 0 \end{Bmatrix}$ defined on the interval $[-\pi, \pi]$.

3(b). Apply Fact 5.5.6 to your answer in Problem 3(a). [Fact 5.5.6 states that the (infinite) sum of the squares of the Fourier coefficients of a piecewise continuous function *f* converges to $\|f\|^2$, the square of the norm of f. That is, $\frac{a_0^{\;2}}{2} + a_1^{\;2} + b_1^{\;2} + a_2^{\;2} + b_2^{\;2} + \cdots + a_n^{\;2} + b_n^{\;2} + \cdots = \|f\|^2$.]

Section 6.1:

Problem 4. Use the determinant to find out for which values of the constant *k* the matrix $\begin{bmatrix} 0 & 1 & k \\ 3 & 2k & 5 \\ 9 & 7 & 5 \end{bmatrix}$ is invertible.

Section 6.2:

In Problems 5 and 6, use Gaussian elimination, i.e. row reduction, to find the determinant of the matrix.

Problem 5. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix}$.

Problem 6. $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix}$

Problem 7. Find the determinant of the linear transformation $T(f) = 2f + 3f'$ from P_2 to P_2.

Problem 8. Find the determinant of the linear transformation $(T(f))(t) = f(3t-2)$ from P_2 to P_2.

Problem 9. Find the determinant of the linear transformation $T(\mathbf{M}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\mathbf{M} + \mathbf{M}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ from the space *V* of symmetric 2×2 matrices to *V*.

Problem 10. a. For an invertible $n\times n$ matrix **A** and an arbitrary $n\times n$ matrix **B**, show that $\text{rref}[\mathbf{A} \mid \mathbf{AB}] = [\mathbf{I}_n \mid \mathbf{B}]$.

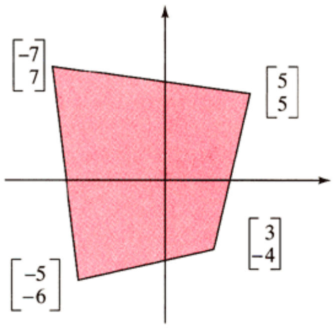
Hint: The left part of $\text{rref}[\mathbf{A} \mid \mathbf{AB}]$ is $\text{rref}(\mathbf{A}) = \mathbf{I}_n$. Write $\text{rref}[\mathbf{A} \mid \mathbf{AB}] = [\mathbf{I}_n \mid \mathbf{M}]$; we have to show that $\mathbf{M} = \mathbf{B}$. To demonstrate this, note that the columns of matrix $\begin{bmatrix} \mathbf{B} \\ -\mathbf{I}_n \end{bmatrix}$ are in the kernel of $[\mathbf{A} \mid \mathbf{AB}]$ and therefore in the kernel of $[\mathbf{I}_n \mid \mathbf{M}]$. [Note: It's best to think in terms of *block* (or *partitioned*) *matrices*.]

b. What does the formula $\text{rref}[\mathbf{A} \mid \mathbf{AB}] = [\mathbf{I}_n \mid \mathbf{M}]$ tell you if $\mathbf{B} = \mathbf{A}^{-1}$?

[**Note**: The result that $\text{rref}[\mathbf{A} \mid \mathbf{AB}] = [\mathbf{I}_n \mid \mathbf{B}]$ is an essential step in the proof that det(**AB**) = det(**A**)det(**B**).]

Section 6.3:

Problem 11. Find the area of the following region:



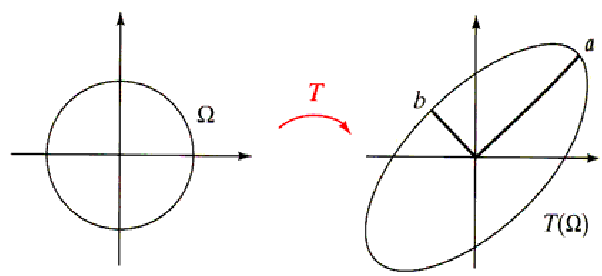
Problem 12. Find the area (or 2-volume) of the parallelogram (or 2-parallelepiped) defined by the vectors $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$.

Problem 13. Find the 3-volume of the 3-parallepipied defined by the vectors $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$.

Problem 14. If $T(\mathbf{x}) = \mathbf{Ax}$ is an invertible linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$, then the image $T(\Omega)$ of the unit circle Ω is an ellipse. (Reference: Exercise 2.2.50.) [*Note*: This observation is useful in Exercises 48 (Problem 16 below) and 49.]

a. Sketch this ellipse when $\mathbf{A} = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$, where p and q are positive. What is its area?

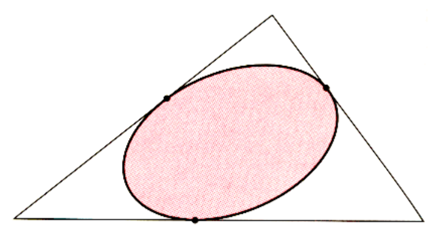
b. For an arbitrary invertible transformation $T(\mathbf{x}) = \mathbf{Ax}$, denote the lengths of the semi-major and semi-minor axes of $T(\Omega)$ by a and b, respectively. What is the relationship between a, b, and $\det(\mathbf{A})$?



c. For the transformation $T(\mathbf{x}) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\mathbf{x}$, sketch this ellipse and determine its axes.

Hint: Consider $T\begin{bmatrix}1\\1\end{bmatrix}$ and $T\begin{bmatrix}1\\-1\end{bmatrix}$.

Problem 15. What is the area of the largest ellipse you can inscribe into a triangle with side lengths 3, 4, and 5. *Hint*: The largest ellipse you can inscribe into an equilateral triangle is a circle. Then apply a linear transformation to relate the two scenarios.



For additional practice:

Section 5.5:

3. Consider a matrix $\mathbf{S}$ in $\mathbf{R}^{n\times n}$. In $\mathbf{R}^n$, define the product $\langle \mathbf{x}, \mathbf{y} \rangle = (\mathbf{Sx})^T\mathbf{Sy}$.

   a. For which choices of $\mathbf{S}$ is this an inner product?

   b. For which choices of $\mathbf{S}$ is $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$ (the dot product)?

4. In $\mathbf{R}^{n\times m}$, consider the inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \operatorname{trace}(\mathbf{A}^T\mathbf{B})$.

   a. Find a formula for this inner product in $\mathbf{R}^{n\times 1} = \mathbf{R}^n$.

   b. Find a formula for this inner product in $\mathbf{R}^{1\times m}$, i.e. the space of row vectors with m components.

9. Recall that a function $f(t)$ from **R** to **R** is called *even* if $f(-t) = f(t)$ for all *t*, and *odd* if $f(-t) = -f(t)$ for all *t*. Show that if $f(x)$ is an odd continuous function and $g(x)$ is an even continuous function, then functions $f(x)$ and $g(x)$ are orthogonal in the space *C*[–1, 1] with the inner product defined in Example 1, i.e. $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$

19. For which 2×2 matrices **A** is $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^{\mathrm{T}} \mathbf{A} \mathbf{w}$ an inner product on $\mathbf{R}^2$?
[*Hint*: Be prepared to complete a square.]

20. Consider the inner product $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^{\mathrm{T}} \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \mathbf{w}$ in $\mathbf{R}^2$. (See Exercise 19.)

a. Find all vectors in $\mathbf{R}^2$ that are perpendicular to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with respect to this inner product.

b. Find an orthonormal basis of $\mathbf{R}^2$ with respect to this inner product.

26. Find the Fourier coefficients of the piecewise continuous function $f(t) = \begin{Bmatrix} -1 & \text{if } -\pi < t < 0 \\ 1 & \text{if } 0 \le t \le \pi \end{Bmatrix}$ extended periodically. Sketch the graphs of the first few Fourier polynomials. [A graphing calculator may be useful.]

28. Apply Fact 5.5.6 to your answer in Exercise 26. [Fact 5.5.6 states that the (infinite) sum of the squares of the Fourier coefficients of a piecewise continuous function *f* converges to $\|f\|^2$, the square of the norm of *f*.]

Section 6.1:
In Exercises 16 and 17, use the determinant to find out for which values of the constant *k* the matrix is invertible.

16. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & k & 5 \\ 6 & 7 & 8 \end{bmatrix}$ 17. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & k & -1 \\ 1 & k^2 & 1 \end{bmatrix}$

In the following problems, use the determinant to find out for which values of the constant λ the matrix $\mathbf{A} - \lambda \mathbf{I}_n$ fails to be invertible.

26. $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$ 30. $\mathbf{A} = \begin{bmatrix} 4 & 2 & 0 \\ 4 & 6 & 0 \\ 5 & 2 & 3 \end{bmatrix}$

34. Find the determinant of the matrix $\begin{bmatrix} 4 & 5 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 2 & 7 & 1 & 4 \\ 1 & 8 & 2 & 3 \end{bmatrix}$. [**Do it <u>without</u> the calculator.**]

43. If **A** is an $n \times n$ matrix and *k* is an arbitrary constant, what is the relationship between det(**A**) and det(–**A**)?

44. If **A** is an $n \times n$ matrix and *k* is an arbitrary constant, what is the relationship between det(**A**) and det(*k***A**)?

Section 6.2:

5. Use Gaussian elimination, i.e. row reduction, to find the determinant of the matrix $\begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix}$.

25. Find the determinant of the linear transformation $T(\mathbf{M}) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$ from the space V of upper triangular 2×2 matrices to V.

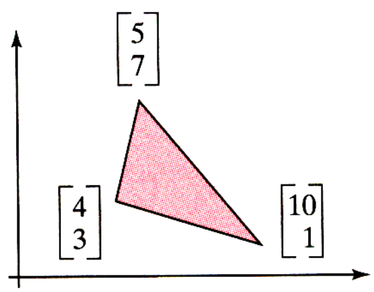
40. If **A** is an orthogonal matrix, what are the possible values of det(**A**)?

41. Consider a skew-symmetric $n \times n$ matrix **A**, where n is odd. Show that **A** is noninvertible, by showing that det(**A**) = 0.

43. Consider two vectors **v** and **w** in $\mathbf{R}^n$. Form the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$.Express det($\mathbf{A}^T\mathbf{A}$) in terms of $\|\mathbf{v}\|$, $\|\mathbf{w}\|$, and $\mathbf{v} \cdot \mathbf{w}$. What can you say about the sign of the result?

Section 6.3:

3. Find the area of the following triangle:



19. A basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of $\mathbf{R}^3$ is called positively oriented if $\mathbf{v}_1$ encloses an acute angle with $\mathbf{v}_2 \times \mathbf{v}_3$. Illustrate this definition with a sketch. Show that the basis is positively oriented if (and only if) $\det\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ is positive.

20. We say that a linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}^3$ *preserves orientation* if it transforms any positively oriented basis into another positively oriented basis. (See Exercise 19.) Explain why a linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ preserves orientation if (and only if) det(**A**) is positive.

23. Use Cramer's rule to solve the system $\begin{Bmatrix} 5x_1 - 3x_2 = 1 \\ -6x_1 + 7x_2 = 0 \end{Bmatrix}$.

24. Use Cramer's rule to solve the system $\begin{Bmatrix} 2x + 3y & & = 8 \\ & 4y + 5z & = 3 \\ 6x & + 7z & = -1 \end{Bmatrix}$.

49. What are the lengths of the semiaxes of the largest ellipse you can inscribe into a triangle with sides 3, 4, and 5? See Exercise 48.

Chapter 6 True/False Exercises

1. If $A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ is any 3×3 matrix, then $\det A = \vec{u} \cdot (\vec{v} \times \vec{w})$.

2. $\det(4A) = 4 \det A$ for all 4×4 matrices A.

3. $\det(A + B) = \det A + \det B$ for all 5×5 matrices A and B.

4. The equation $\det(-A) = \det A$ holds for all 6×6 matrices.

5. If all the entries of a 7×7 matrix A are 7, then $\det A$ must be 7^7.

6. An 8×8 matrix fails to be invertible if (and only if) its determinant is nonzero.

7. If B is obtained be multiplying a column of A by 9, then the equation $\det B = 9 \det A$ must hold.

8. $\det(A^{10}) = (\det A)^{10}$ for all 10×10 matrices A.

9. The determinant of any diagonal $n \times n$ matrix is the product of its diagonal entries.

10. If matrix B is obtained by swapping two rows of an $n \times n$ matrix A, then the equation $\det B = -\det A$ must hold.

11. Matrix $\begin{bmatrix} 9 & 100 & 3 & 7 \\ 5 & 4 & 100 & 8 \\ 100 & 9 & 8 & 7 \\ 6 & 5 & 4 & 100 \end{bmatrix}$ is invertible.

12. If A is an invertible $n \times n$ matrix, then $\det(A^T)$ must equal $\det(A^{-1})$.

13. If the determinant of a 4×4 matrix A is 4, then its rank must be 4.

14. There exists a nonzero 4×4 matrix A such that $\det A = \det(4A)$.

15. If two $n \times n$ matrices A and B are similar, then the equation $\det A = \det B$ must hold.

16. The determinant of all orthogonal matrices is 1.

17. If A is any $n \times n$ matrix, then $\det(AA^T) = \det(A^T A)$.

18. There exists an invertible matrix of the form $\begin{bmatrix} a & e & f & j \\ b & 0 & g & 0 \\ c & 0 & h & 0 \\ d & 0 & i & 0 \end{bmatrix}$.

19. The matrix $\begin{bmatrix} k^2 & 1 & 4 \\ k & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$ is invertible for all positive constants k.

20. $\det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = 1$.

21. There exists a 4×4 matrix A whose entries are all 1 or -1, and such that $\det A = 16$.

22. If the determinant of a 2×2 matrix A is 4, then the inequality $\|A\vec{v}\| \leq 4\|\vec{v}\|$ must hold for all vectors $\vec{v}$ in $\mathbb{R}^2$.

23. If $A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ is a 3×3 matrix, then the formula $\det(A) = \vec{v} \cdot (\vec{u} \times \vec{w})$ must hold.

24. There exist invertible 2×2 matrices A and B such that $\det(A + B) = \det A + \det B$.

25. If all the entries of a square matrix are 1 or 0, then $\det A$ must be 1, 0, or -1.

26. If all the entries of a square matrix A are integers and $\det A = 1$, then the entries of matrix A^{-1} must be integers as well.

27. If all the columns of a square matrix A are unit vectors, then the determinant of A must be less than or equal to 1.

28. If A is any noninvertible square matrix, then $\det A = \det(\text{rref } A)$.

29. If the determinant of a square matrix is -1, then A must be an orthogonal matrix.

30. If all the entries of an invertible matrix A are integers, then the entries of A^{-1} must be integers as well.

31. There exist invertible 3×3 matrices A and S such that $S^{-1}AS = 2A$.

32. There exist invertible 3×3 matrices A and S such that $S^T AS = -A$.

33. If A is any symmetric matrix, then $\det A = 1$ or $\det A = -1$.

34. If A is any skew-symmetric 4×4 matrix, then $\det A = 0$.

35. If $\det A = \det B$ for two $n \times n$ matrices A and B, then A must be similar to B.

36. Suppose A is an $n \times n$ matrix and B is obtained from A by swapping two rows of A. If $\det B < \det A$, then A must be invertible.

37. If an $n \times n$ matrix A is invertible, then there must be an $(n-1) \times (n-1)$ submatrix of A (obtained by deleting a row and a column of A) that is invertible as well.

38. If all the entries of matrices A and A^{-1} are integers, then the equation $\det A = \det(A^{-1})$ must hold.

39. If a square matrix A is invertible, then its classical adjoint $\text{adj}(A)$ is invertible as well.

40. There exists a 3×3 matrix A such that $A^2 = -I_3$.

41. If all the diagonal entries of an $n \times n$ matrix A are odd integers and all the other entries are even integers, then A must be an invertible matrix.

42. If all the diagonal entries of an $n \times n$ matrix A are even integers and all the other entries are odd integers, then A must be an invertible matrix.

43. For every nonzero 2×2 matrix A there exists a 2×2 matrix B such that $\det(A + B) \neq \det A + \det B$.

44. If A is a 4×4 matrix whose entries are all 1 or -1, then $\det A$ must be divisible by 8 [i.e., $\det A = 8k$ for some integer k].

45. If A is an invertible $n \times n$ matrix, then A must commute with its adjoint, $\text{adj}(A)$.

46. There exists a real number k such that the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & k & 7 \\ 8 & 9 & 8 & 7 \\ 0 & 0 & 6 & 5 \end{bmatrix}$$

is invertible.

47. If A and B are orthogonal $n \times n$ matrices such that $\det A = \det B = 1$, then matrices A and B must commute.