Math S-21b – Summer 2023 – Homework #5

Problems due Thurs, July 13:

Section 5.1:

Problem 1. (5.1/12) Give an algebraic proof for the *triangle inequality* $\|\mathbf{v}+\mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$. Draw a sketch.

[Hint: Expand $\|\mathbf{v}+\mathbf{w}\|^2 \le (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w})$. Then use the Cauchy-Schwarz inequality ($|\mathbf{v}\cdot\mathbf{w}| \le \|\mathbf{v}\|\|\mathbf{w}\|$).]

Problem 2. (5.1/16) Consider the vectors $\mathbf{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$ in $\mathbf{R}^4$. Can you find a vector $\mathbf{u}_4$ in $\mathbf{R}^4$ such that the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are orthonormal? If so, how many such vectors are there?

Problem 3. (5.1/17) Find a basis for $W^\perp$, where $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right\}$.

Problem 4. (5.1/26) Find the orthogonal projection of $\begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$ onto the subspace of $\mathbf{R}^3$ spanned by $\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$.

Section 5.2:

Problem 5. (5.2/14, 28) Using paper and pencil, perform the Gram-Schmidt process on the sequence of vectors $\left\{ \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} \right\}$ and then use your calculations find the QR-factorization of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 7 & 7 & 8 \\ 1 & 2 & 1 \\ 7 & 7 & 6 \end{bmatrix}$.

Problem 6. (5.2/34) Find an orthonormal basis of the kernel of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

Section 5.3:

Problem 7. (5.3/31) Are the *rows* of an orthogonal matrix **A** necessarily orthonormal?

Problem 8. (5.3/40) Consider the subspace W of $\mathbf{R}^4$ spanned by the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -5 \\ 3 \end{bmatrix}$.

Find the matrix of the orthogonal projection onto W.

Problem 9. (5.3/44) Consider an $n \times m$ matrix **A**. Find dim(im(**A**)) + dim(ker($\mathbf{A}^T$)), in terms of m and n.

Problem 10. (5.3/46) Consider a QR-factorization **M** = **QR**. Show that $\mathbf{R} = \mathbf{Q}^T\mathbf{M}$.

Section 5.4:

Problem 11. (5.4/4) Let **A** be an $n \times m$ matrix. Is the formula $(\ker \mathbf{A})^\perp = \operatorname{im}(\mathbf{A}^T)$ necessarily true? Explain.

Problem 12. (5.4/5) Let V be the solution space of the linear system $\left\{ \begin{array}{l} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + 2x_2 + 5x_3 + 4x_4 = 0 \end{array} \right\}$.

Find a basis for $V^{\perp}$.

Problem 13. (5.4/6) If **A** is an $n \times m$ matrix, is the formula im(**A**) = im(**AA**^{T}) necessarily true? Explain.

Problem 14. (5.4/7) Consider a symmetric $n \times n$ matrix **A**. What is the relationship between im(**A**) and ker(**A**)?

Problem 15. (5.4/10) Consider a consistent system $\mathbf{Ax} = \mathbf{b}$.

a. Show that this system has a solution $\mathbf{x}_0$ in $(\ker \mathbf{A})^{\perp}$. *Hint*: An arbitrary solution **x** of the system can be written as $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_0$, where $\mathbf{x}_h$ is in ker(**A**) and $\mathbf{x}_0$ is in $(\ker \mathbf{A})^{\perp}$.

b. Show that the system $\mathbf{Ax} = \mathbf{b}$ has only one solution in $(\ker \mathbf{A})^{\perp}$.
Hint: If $\mathbf{x}_0$ and $\mathbf{x}_1$ are two solutions in $(\ker \mathbf{A})^{\perp}$, think about $\mathbf{x}_1 - \mathbf{x}_0$.

c. If $\mathbf{x}_0$ is the solution in $(\ker \mathbf{A})^{\perp}$ and $\mathbf{x}_1$ is another solution of the system $\mathbf{Ax} = \mathbf{b}$, show that $\|\mathbf{x}_0\| < \|\mathbf{x}_1\|$.
The vector $\mathbf{x}_0$ is called the *minimal solution* of the linear system $\mathbf{Ax} = \mathbf{b}$.

Problem 16. (5.4/16) Use the formula $(\operatorname{im} \mathbf{A})^{\perp} = \ker(\mathbf{A}^T)$ to prove the equation: rank(**A**) = rank(**A**^{T})

Problem 17. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix}$, find a basis for <u>each</u> of the ***Four Fundamental Subspaces***:

(a) im(**A**) (b) ker(**A**) (c) im(**A**^{T}) (d) ker(**A**^{T})

Problem 18. (5.4/22) Find the least-squares solution $\mathbf{x}^*$ of the system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$.

Determine the error $\|\mathbf{b} - \mathbf{Ax}^*\|$.

Problem 19. (5.4/32) Fit a quadratic polynomial to the data points (0, 27), (1, 0), (2, 0), (3, 0), using least squares. Sketch the solution.

Problem 20: (5.4/40) Consider the data in the following table: [we'll seek a relation of the form $D = ka^n$]

Planet	***a*** Mean Distance from the Sun (in Astronomical Units)	***D*** Period of Revolution (in Earth Years)
Mercury	0.387	0.241
Earth	1.000	1.000
Jupiter	5.203	11.86
Uranus	19.19	84.04
Pluto	39.53	248.6

(a) Using logarithms, fit a function of the form $\ln(D) = c + n\ln(a)$ to the data points $(\ln(a_i), \ln(D_i))$, using least squares.

(b) Use your answer in part (a) to fit a power function $D = ka^n$ to the data points (a_i, D_i).

(c) Explain in terms of Kepler's laws of planetary motion. Explain why the constant k is close to 1.

For additional practice:
Section 5.1:

15. Consider the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ in $\mathbf{R}^4$. Find a basis of the subspace of $\mathbf{R}^4$ consisting of all vectors perpendicular (orthogonal) to $\mathbf{v}$.

18. Here is an infinite dimension version of Euclidean space: In the space of all infinite sequences, consider the subspace l_2 of square-summable sequences [i.e., those sequences $(x_1, x_2, \ldots)$ for which the infinite series $x_1^2 + x_2^2 + \cdots$ converges]. For $\mathbf{x}$ and $\mathbf{y}$ in l_2, we define $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots}$ and $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots$. (Why does the series $x_1 y_1 + x_2 y_2 + \cdots$ converge?)

   a. Check that $\mathbf{x} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots)$ is in l_2, and find $\|\mathbf{x}\|$. Recall the formula for the geometric series:

   $$1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a}, \text{ if } -1 < a < 1.$$

   b. Find the angle between $(1, 0, 0, \ldots)$ and $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$.

   c. Give an example of a sequence $(x_1, x_2, \ldots)$ that converges to 0 (i.e., $\lim_{n\to\infty} x_n = 0$) but does not belong to l_2.

   d. Let L be the subspace of l_2 spanned by $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$. Find the orthogonal projection of $(1, 0, 0, \ldots)$ onto L.

   Note: The Hilbert space l_2 was initially used mostly in physics: Werner Heisenberg's formulation of quantum mechanics is in terms of l_2. Today, this space is used in many other applications, including economics. (See, for example, the work of the economist Andreu Mas-Colell of the University of Barcelona.)

28. Find the orthogonal projection of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ onto the subspace of $\mathbf{R}^4$ spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

29. Consider the orthonormal vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ in $\mathbf{R}^{10}$.
   Find the length of the vector $\mathbf{x} = 7\mathbf{u}_1 - 3\mathbf{u}_2 + 2\mathbf{u}_3 + \mathbf{u}_4 - \mathbf{u}_5$.

Section 5.2:

6, 20. Using paper and pencil, perform the Gram-Schmidt process on the sequence of vectors $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \right\}$ and then use your calculations find the QR-factorization of the matrix $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 7 \end{bmatrix}$..

8, 22. Using paper and pencil, perform the Gram-Schmidt process on the sequence of vectors $\left\{ \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 7 \\ -2 \end{bmatrix} \right\}$ and then use your calculations find the QR-factorization of the matrix $\begin{bmatrix} 5 & 3 \\ 4 & 6 \\ 2 & 7 \\ 2 & -2 \end{bmatrix}$.

33. Find an orthonormal basis of the kernel of the matrix $\mathbf{A}=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & -1 & 1\end{bmatrix}$.

38. Find the QR-factorization of the matrix $\mathbf{A}=\begin{bmatrix}0 & -3 & 0\\ 0 & 0 & 0\\ 2 & 0 & 0\\ 0 & 0 & 4\end{bmatrix}$.

40. Consider an invertible $n \times n$ matrix **A** whose columns are orthogonal, but not necessarily orthonormal. What does the QR-factorization of **A** look like?

41. Consider an upper triangular $n \times n$ matrix **A**. What does the QR-factorization of **A** look like?

Section 5.3:
If the $n \times n$ matrices **A** and **B** are orthogonal matrices, which of the matrices in Exercises 5 through 11 must be orthogonal as well?

5. 3**A** 6. –**B** 7. **AB** 8. **A** + **B** 9. $\mathbf{B}^{-1}$ 10. $\mathbf{B}^{-1}\mathbf{AB}$ 11. $\mathbf{A}^{\mathrm{T}}$

37. Is there an orthogonal transformation T from $\mathbf{R}^3$ to $\mathbf{R}^3$ such that $T\begin{bmatrix}2\\3\\0\end{bmatrix}=\begin{bmatrix}3\\0\\2\end{bmatrix}$ and $T\begin{bmatrix}-3\\2\\0\end{bmatrix}=\begin{bmatrix}2\\-3\\0\end{bmatrix}$?

42. Let **A** be the matrix of an orthogonal projection. Find $\mathbf{A}^2$ in two ways:
   a. Geometrically. (Consider what happens when you apply an orthogonal projection twice.)
   b. By computation, using the formula given in Fact 5.3.10 (matrix of an orthogonal projection in terms of an orthonormal basis for a given subspace).

45. For which $n \times m$ matrices **A** does the equation $\dim(\ker(\mathbf{A})) = \dim(\ker(\mathbf{A}^{\mathrm{T}}))$ hold? Explain.

47. If $\mathbf{A}=\mathbf{QR}$ is a QR-factorization, what is the relationship between $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\mathbf{R}^{\mathrm{T}}\mathbf{R}$?

Section 5.4:

1. Consider the subspace im(**A**) of $\mathbf{R}^2$, where $\mathbf{A}=\begin{bmatrix}2 & 4\\ 3 & 6\end{bmatrix}$. Find a basis of $\ker(\mathbf{A}^{\mathrm{T}})$, and draw a sketch illustrating the formula $(\operatorname{im}\mathbf{A})^{\perp}=\ker(\mathbf{A}^{\mathrm{T}})$ in this case.

2. Consider the subspace im(**A**) of $\mathbf{R}^2$, where $\mathbf{A}=\begin{bmatrix}1 & 1\\ 1 & 2\\ 1 & 3\end{bmatrix}$. Find a basis of $\ker(\mathbf{A}^{\mathrm{T}})$, and draw a sketch illustrating the formula $(\operatorname{im}\mathbf{A})^{\perp}=\ker(\mathbf{A}^{\mathrm{T}})$ in this case.

15. Consider an $m \times n$ matrix **A** with $\ker(\mathbf{A}) = \{\mathbf{0}\}$. Show that there exists an $n \times m$ matrix **B** such that $\mathbf{BA}=\mathbf{I}_n$.
Hint: $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is invertible.

17. Does the equation $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}^{\mathrm{T}}\mathbf{A})$ hold for all $n \times m$ matrices **A**? Explain.

18. Does the equation $\operatorname{rank}(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = \operatorname{rank}(\mathbf{A}\mathbf{A}^{\mathrm{T}})$ hold for all $n \times m$ matrices **A**? Explain.
Hint: Exercise 17 is useful.

20. By using paper and pencil, find the least-squares solution $\mathbf{x}^*$ of the system $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{A}=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}3\\3\\3\end{bmatrix}$. Verify that the vector $\mathbf{b}-\mathbf{Ax}^*$ is perpendicular to the image of **A**.

31. Fit a linear function of the form $f(t)=c_0+c_1t$ to the data points (0, 3), (1, 3), (1, 6), using least squares. Sketch the solution. [Note: Strictly speaking, a function of this form is not a linear function in the sense that we use in this course. More properly, this might be called an "affine" function.]

38. In the accompanying table, we list the height *h*, the gender *g*, and the weight *w* of some young adults.

Height *h* (in inches above 5 ft.)	Gender *g* (1 = "female", 0 = "male")	Weight *w* (in pounds)
2	1	110
12	0	180
5	1	120
11	1	160
6	0	160

Fit a function of the form $w=c_0+c_1h+c_2g$ to these data, using least squares. Before you do the computations, think about the signs of c_1 and c_2. What signs would you expect if these data were representative of the general population? Why? What is the sign of c_0? What is the practical significance of c_0?

41. In the accompanying table, we list the public debt *D* of the United States (in billions of dollars), in the year *t* (as of September 30).

t	1970	1975	1980	1985	1990	1995
D	370	533	908	1823	3233	4974

a. Fit a linear function of the form $\log(D)=c_0+c_1t$ to the data points $(t_i,\log(D_i))$, using least squares. Use the result to fit an exponential function to the data points (t_i,D_i).

b. What debt does your formula in part (a) predict for the year 2000? What about the year 2010?

c. On Sept 30, 2000, the debt was 5,674 billion dollars. What happened?

42. If **A** is any matrix, show that the linear transformation $L(\mathbf{x})=\mathbf{Ax}$ from $\text{im}(\mathbf{A}^T)$ to $\text{im}(\mathbf{A})$ is an isomorphism. This provides yet another proof of the formula $\text{rank}(\mathbf{A})=\text{rank}(\mathbf{A}^T)$.

Chapter 5 True/False Exercises

1. If A and B are symmetric $n\times n$ matrices, then $A+B$ must be symmetric as well.

2. If matrices A and S are orthogonal, then $S^{-1}AS$ is orthogonal as well.

3. All nonzero symmetric matrices are invertible.

4. If A is an $n\times n$ matrix such that $AA^T=I_n$, then A must be an orthogonal matrix.

5. If $\vec{u}$ is a unit vector in $\mathbb{R}^n$, and $L=\text{span}(\vec{u})$, then $\text{proj}_L(\vec{x})=(\vec{x}\cdot\vec{u})\vec{x}$ for all vectors $\vec{x}$ in $\mathbb{R}^n$.

6. If A is a symmetric matrix, then $7A$ must be symmetric as well.

7. If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(\vec{e}_1), T(\vec{e}_2),\ldots,T(\vec{e}_n)$ are all unit vectors, then T must be an orthogonal transformation.

8. If A is an invertible matrix, then the equation $(A^T)^{-1}=(A^{-1})^T$ must hold.

9. If matrix A is orthogonal, then matrix A^2 must be orthogonal as well.

10. The equation $(AB)^T=A^TB^T$ holds for all $n\times n$ matrices A and B.

11. If matrix A is orthogonal, then A^T must be orthogonal as well.

12. If A and B are symmetric $n\times n$ matrices, then AB must be symmetric as well.

13. If matrices A and B commute, then A must commute with B^T as well.

14. If A is any matrix with $\ker(A)=\{\vec{0}\}$, then the matrix AA^T represents the orthogonal projection onto the image of A.

15. If A and B are symmetric $n\times n$ matrices, then $ABBA$ must be symmetric as well.

16. If matrices A and B commute, then matrices A^T and B^T must commute as well.

17. There exists a subspace V of $\mathbb{R}^5$ such that $\dim(V)=\dim(V^\perp)$, where $V^\perp$ denotes the orthogonal complement of V.

18. Every invertible matrix A can be expressed as the product of an orthogonal matrix and an upper triangular matrix.

19. If $\vec{x}$ and $\vec{y}$ are two vectors in $\mathbb{R}^n$, then the equation $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ must hold.

20. The equation $\det(A^T) = \det(A)$ holds for all 2×2 matrices A.

21. If A and B are orthogonal 2×2 matrices, then $AB = BA$.

22. If A is a symmetric matrix, vector $\vec{v}$ is in the image of A, and $\vec{w}$ is in the kernel of A, then the equation $\vec{v} \cdot \vec{w} = 0$ must hold.

23. The formula $\ker(A) = \ker(A^T A)$ holds for all matrices A.

24. If $A^T A = AA^T$ for an $n \times n$ matrix A, then A must be orthogonal.

25. The determinant of all orthogonal 2×2 matrices is 1.

26. If A is any square matrix, then matrix $\frac{1}{2}(A - A^T)$ is skew-symmetric.

27. The entries of an orthogonal matrix are all less than or equal to 1.

28. Every nonzero subspace of $\mathbb{R}^n$ has an orthonormal basis.

29. $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ is an orthogonal matrix.

30. If V is a subspace of $\mathbb{R}^n$ and $\vec{x}$ is a vector in $\mathbb{R}^n$, then vector $\text{proj}_V \vec{x}$ must be orthogonal to vector $\vec{x} - \text{proj}_V \vec{x}$.

31. There exist orthogonal 2×2 matrices A and B such that $A + B$ is orthogonal as well.

32. If $\|A\vec{x}\| \leq \|\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$, then A must represent the orthogonal projection onto a subspace V of $\mathbb{R}^n$.

33. If A is an invertible matrix such that $A^{-1} = A$, then A must be orthogonal.

34. If the entries of two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are all positive, then $\vec{v}$ and $\vec{w}$ must enclose an acute angle.

35. The formula $(\ker B)^{\perp} = \text{im}(B^T)$ holds for all matrices B.

36. The matrix $A^T A$ is symmetric for all matrices A.

37. If matrix A is similar to B and A is orthogonal, then B must be orthogonal as well.

38. The formula $\text{im}(B) = \text{im}(B^T B)$ holds for all square matrices B.

39. If matrix A is symmetric and matrix S is orthogonal, then matrix $S^{-1}AS$ must be symmetric.

40. If A is a square matrix such that $A^T A = AA^T$, then $\ker(A) = \ker(A^T)$.

41. Any square matrix can be written as the sum of a symmetric and a skew-symmetric matrix.

42. If $x_1, x_2, \ldots, x_n$ are any real numbers, then the inequality

$$\left(\sum_{k=1}^{n} x_k\right)^2 \leq n \sum_{k=1}^{n} (x_k^2)$$

must hold.

43. If $AA^T = A^2$ for a 2×2 matrix A, then A must be symmetric.

44. If V is a subspace of $\mathbb{R}^n$ and $\vec{x}$ is a vector in $\mathbb{R}^n$, then the inequality $\vec{x} \cdot (\text{proj}_V \vec{x}) \geq 0$ must hold.

45. If A is an $n \times n$ matrix such that $\|A\vec{u}\| = 1$ for all unit vectors $\vec{u}$, then A must be an orthogonal matrix.

46. If A is any symmetric 2×2 matrix, then there must exist a real number x such that matrix $A - xI_2$ fails to be invertible.

47. There exists a basis of $\mathbb{R}^{2\times 2}$ that consists of orthogonal matrices.

48. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then the matrix Q in the QR factorization of A is a rotation matrix.

49. There exists a linear transformation L from $\mathbb{R}^{3\times 3}$ to $\mathbb{R}^{2\times 2}$ whose kernel is the space of all skew-symmetric 3×3 matrices.

50. If a 3×3 matrix A represents the orthogonal projection onto a plane V in $\mathbb{R}^3$, then there must exist an orthogonal 3×3 matrix S such that $S^T AS$ is diagonal.