Math S-21b – Summer 2024 – Homework #4

Problems due Friday, July 12:

Find a **basis** for each of the spaces in Problems 1 to 3 and determine its **dimension**.

Problem 1. The space of all matrices $\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathbf{R}^{2\times 2}$ such that $a+d=0$.

Problem 2. The space of all polynomials $f(t)$ in P_3 such that $f(1)=0$ and $\int_{-1}^{1} f(t)\,dt=0$.

Problem 3. The space of all 2×2 matrices **A** such that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\mathbf{A}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 4. (a) Is the transformation $T(\mathbf{M})=\mathbf{M}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ from $\mathbf{R}^{2\times 2}$ to $\mathbf{R}^{2\times 2}$ linear?

If it is, determine whether it is an isomorphism.

(b) Find the kernel and nullity of the transformation in 4(a).

Problem 5. (a) Is the transformation $[T(f)](t)=f''(t)+4f'(t)$ from P_2 to P_2 linear?

(b) Find the image, rank, kernel and nullity of the transformation in (a).

(c) Find the matrix of the linear transformation $T(f)=f''+4f'$ from P_2 to P_2 relative to the basis $\mathcal{U}=\{1,t,t^2\}$. Is T an isomorphism? Why or why not?

Problem 6. Find the kernel and nullity of the transformation $T(f)=f-f'$ from C^∞ to C^∞.

[C^∞ denotes the linear space consisting of all infinitely differentiable functions of one variable.]

Problem 7. (a) Find the matrix $\mathbf{A}=[T]_{\mathcal{U}}$ of the linear transformation $T(\mathbf{M})=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\mathbf{M}$ from $\mathbf{R}^{2\times 2}$ to $\mathbf{R}^{2\times 2}$

with respect to the (standard) basis $\mathcal{U}=\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

Is T an isomorphism? If not find bases for the kernel and image of T, and thus determine the rank of T.

(b) Find the matrix $\mathbf{B}=[T]_{\mathcal{B}}$ of the linear transformation $T(\mathbf{M})=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\mathbf{M}$ from $\mathbf{R}^{2\times 2}$ to $\mathbf{R}^{2\times 2}$ with respect to the basis $\mathcal{B}=\left\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right\}$.

(c) Find the change of basis matrix **S** from coordinates relative to the basis $\mathcal{B}$ in 7(b) to coordinates relative to the standard basis $\mathcal{U}$ considered in 7(a). [**Note**: $[\mathbf{M}]_{\mathcal{U}}=\mathbf{S}[\mathbf{M}]_{\mathcal{B}}$]

(d) Verify the formula $\mathbf{SB}=\mathbf{AS}$ (that is, $\mathbf{B}=\mathbf{S}^{-1}\mathbf{AS}$) for the matrices **B** and **A** you found in 7(a) and 7(b), respectively.

Problem 8. (a) Find the matrix $\mathbf{A}=[T]_{\mathcal{U}}$ of the linear transformation $[T(f)](t)=f(2t-1)$ from P_2 to P_2 relative to the basis $\mathcal{U}=\{1,t,t^2\}$. Is T an isomorphism? If not, find bases for the kernel and image of T, and thus determine the rank of T.

(b) Find the matrix $\mathbf{B}=[T]_{\mathcal{B}}$ of the linear transformation $[T(f)](t)=f(2t-1)$ from P_2 to P_2 relative to the basis $\mathcal{B}=\{1,t-1,(t-1)^2\}$.

(c) Find the change of basis matrix **S** from coordinates relative to the basis $\mathcal{B}$ in 8(b) to coordinates relative to the standard basis $\mathcal{U}$ considered in 8(a). Then find the change of basis matrix from $\mathcal{U}$ to $\mathcal{B}$.

(d) Verify the formula $\mathbf{SB}=\mathbf{AS}$ (that is, $\mathbf{B}=\mathbf{S}^{-1}\mathbf{AS}$) for the matrices **B** and **A** you found in 8(a) and 8(b), respectively.

For additional practice:

Section 4.1:

Which of the subsets of P_2 given in Exercises 1, 2, and 3 are subspaces of P_2? Find a basis for those that are subspaces. [P_2 is the linear space consisting of polynomials of degree less than or equal to 2.]

1. $\{p(t): p(0)=2\}$ 2. $\{p(t): p(0)=0\}$ 3. $\{p(t): p'(1)=p(2)\}$ (p' denotes the derivative.)

Which of the subsets of $\mathbf{R}^{3\times 3}$ such given in Exercises 9, 10, and 11 are subspaces of $\mathbf{R}^{3\times 3}$?

9. The 3×3 matrices whose entries are all greater than or equal to zero.

10. The 3×3 matrices $\mathbf{A}$ such that the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is in the kernel of $\mathbf{A}$.

11. The 3×3 matrices in reduced row-echelon form.

25. Find a basis for the space of all polynomials $f(t)$ in P_2 such that $f(1)=0$ and determine its dimension.

29. Find a basis for the space of all 2×2 matrices $\mathbf{A}$ such that $\mathbf{A}\begin{bmatrix}1&1\\1&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ and determine its dimension.

Section 4.2:

2. Is the transformation $T(\mathbf{M})=7\mathbf{M}$ from $\mathbf{R}^{2\times 2}$ to $\mathbf{R}^{2\times 2}$ linear? If so, determine whether it is an isomorphism.

4. Is the transformation $T(\mathbf{M})=\det(\mathbf{M})$ from $\mathbf{R}^{2\times 2}$ to $\mathbf{R}$ linear? If so, determine whether it is an isomorphism.

67. For which constants k is the linear transformation $T(\mathbf{M})=\begin{bmatrix}2&3\\0&4\end{bmatrix}\mathbf{M}-\mathbf{M}\begin{bmatrix}3&0\\0&k\end{bmatrix}$ an isomorphism from $\mathbf{R}^{2\times 2}$ to $\mathbf{R}^{2\times 2}$.

81. In this exercise, we will outline a proof of the Rank-Nullity Theorem: If T is a linear transformation from V to W, where V is finite-dimensional, then $\dim(V)=\dim(\text{im}(T))+\dim(\ker(T))=\text{rank}(T)+\text{nullity}(T)$.

    a. Explain why $\ker(T)$ and $\text{im}(T)$ are finite dimensional. *Hint*: Use Exercises 4.1.54 and 4.1.57.

    Now, consider a basis $\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ of $\ker(T)$, where $n=\text{nullity}(T)$, and a basis $\{\mathbf{w}_1,\cdots,\mathbf{w}_r\}$ of $\text{im}(T)$, where $r=\text{rank}(T)$. Consider vectors $\{\mathbf{u}_1,\cdots,\mathbf{u}_r\}$ in V such that $T(\mathbf{u}_i)=\mathbf{w}_i$ for $i=1,\cdots,r$. Our goal is to show that the $r+n$ vectors $\mathbf{u}_1,\cdots,\mathbf{u}_r,\mathbf{v}_1,\cdots,\mathbf{v}_n$ form a basis of V. This will prove our claim.

    b. Show that the vectors $\mathbf{u}_1,\cdots,\mathbf{u}_r,\mathbf{v}_1,\cdots,\mathbf{v}_n$ are linearly independent. *Hint*: Consider a relation $c_1\mathbf{u}_1+\cdots c_r\mathbf{u}_r+d_1\mathbf{v}_1+\cdots d_n\mathbf{v}_n=\mathbf{0}$, apply linear transformation T to both sides, and take it from there.

    c. Show that the vectors $\mathbf{u}_1,\cdots,\mathbf{u}_r,\mathbf{v}_1,\cdots,\mathbf{v}_n$ span V. *Hint*: Consider an arbitrary vector $\mathbf{v}$ in V, and write $T(\mathbf{v})=d_1\mathbf{w}_1+\cdots d_r\mathbf{w}_r$. Now show that the vector $\mathbf{v}-d_1\mathbf{u}_1+\cdots d_r\mathbf{u}_r$ is in the kernel of T, so that $\mathbf{v}-d_1\mathbf{u}_1+\cdots d_r\mathbf{u}_r$ can be written as a linear combination of $\mathbf{v}_1,\cdots,\mathbf{v}_n$.

Section 4.3:

1. Are the polynomials $f(t)=7+3t+t^2$, $g(t)=9+9t+4t^2$, and $h(t)=3+2t+t^2$ linearly independent?

TRUE or FALSE?

1. The space $\mathbb{R}^{2\times 3}$ is 5-dimensional.

2. If $f_1, \ldots, f_n$ is a basis of a linear space V, then any element of V can be written as a linear combination of $f_1, \ldots, f_n$.

3. The space P_1 is isomorphic to $\mathbb{C}$.

4. If the kernel of a linear transformation T from P_4 to P_4 is $\{0\}$, then T must be an isomorphism.

5. If W_1 and W_2 are subspaces of a linear space V, then the intersection $W_1 \cap W_2$ must be a subspace of V as well.

6. If T is a linear transformation from P_6 to $\mathbb{R}^{2\times 2}$, then the kernel of T must be 3-dimensional.

7. The polynomials of degree less than 7 form a 7-dimensional subspace of the linear space of all polynomials.

8. The function $T(f) = 3f - 4f'$ from C^∞ to C^∞ is a linear transformation.

9. The lower triangular 2×2 matrices form a subspace of the space of all 2×2 matrices.

10. The kernel of a linear transformation is a subspace of the domain.

11. The linear transformation $T(f) = f + f''$ from C^∞ to C^∞ is an isomorphism.

12. All linear transformations from P_3 to $\mathbb{R}^{2\times 2}$ are isomorphisms.

13. If T is a linear transformation from V to V, then the intersection of $\operatorname{im}(T)$ and $\ker(T)$ must be $\{0\}$.

14. The space of all upper triangular 4×4 matrices is isomorphic to the space of all lower triangular 4×4 matrices.

15. Every polynomial of degree 3 can be expressed as a linear combination of the polynomial $(t-3)$, $(t-3)^2$, and $(t-3)^3$.

16. If a linear space V can be spanned by 10 elements, then the dimension of V must be ≤ 10.

17. The function $T(M) = \det(M)$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}$ is a linear transformation.

18. There exists a 2×2 matrix A such that the space of all matrices commuting with A is 1-dimensional.

19. All bases of P_3 contain at least one polynomial of degree ≤ 2.

20. If T is an isomorphism, then T^{-1} must be an isomorphism as well.

21. If the image of a linear transformation T from P to P is all of P, then T must be an isomorphism.

22. If f_1, f_2, f_3 is a basis of a linear space V, then f_1, $f_1 + f_2$, $f_1 + f_2 + f_3$ must be a basis of V as well.

23. If a, b, and c are distinct real numbers, then the polynomials $(x-b)(x-c)$, $(x-a)(x-c)$, and $(x-a)(x-b)$ must be linearly independent.

24. The linear transformation $T\big(f(t)\big) = f(4t-3)$ from P to P is an isomorphism.

25. The linear transformation $T(M) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} M$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ has rank 1.

26. If the matrix of a linear transformation T (with respect to some basis) is $\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix}$, then there must exist a nonzero element f in the domain of T such that $T(f) = 3f$.

27. The kernel of the linear transformation $T\big(f(t)\big) = f(t^2)$ from P to P is $\{0\}$.

28. If S is any invertible 2×2 matrix, then the linear transformation $T(M) = SMS$ is an isomorphism from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

29. There exists a 2×2 matrix A such that the space of all matrices commuting with A is 2-dimensional.

30. There exists a basis of $\mathbb{R}^{2\times 2}$ that consists of four invertible matrices.

31. If W is a subspace of V, and if W is finite dimensional, then V must be finite dimensional as well.

32. There exists a linear transformation from $\mathbb{R}^{3\times 3}$ to $\mathbb{R}^{2\times 2}$ whose kernel consists of all lower triangular 3×3 matrices, while the image consists of all upper triangular 2×2 matrices.

33. Every two-dimensional subspace of $\mathbb{R}^{2\times 2}$ contains at least one invertible matrix.

34. If $\mathfrak{A} = (f, g)$ and $\mathfrak{B} = (f, f+g)$ are two bases of a linear space V, then the change of basis matrix from $\mathfrak{A}$ to $\mathfrak{B}$ is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

35. If the matrix of a linear transformation T with respect to a basis (f, g) is $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then the matrix of T with respect to the basis (g, f) is $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.

36. The linear transformation $T(f) = f'$ from P_n to P_n has rank n, for all positive integers n.

37. If the matrix of a linear transformation T (with respect to some basis) is $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$, then T must be an isomorphism.

38. There exists a subspace of $\mathbb{R}^{3\times 4}$ that is isomorphic to P_9.

39. There exist two distinct subspaces W_1 and W_2 of $\mathbb{R}^{2\times 2}$ whose union $W_1 \cup W_2$ is a subspace of $\mathbb{R}^{2\times 2}$ as well.

40. There exists a linear transformation from P to P_5 whose image is all of P_5.

41. If $f_1, \ldots, f_n$ are polynomials such that the degree of f_k is k (for $k = 1, \ldots, n$), then $f_1, \ldots, f_n$ must be linearly independent.

42. The transformation $D(f) = f'$ from C^∞ to C^∞ is an isomorphism.

43. If T is a linear transformation from P_4 to W with $\operatorname{im}(T) = W$, then the inequality $\dim(W) \leq 5$ must hold.

44. The kernel of the linear transformation

$$T(f(t)) = \int_0^1 f(t)\,dt$$

from P to $\mathbb{R}$ is finite dimensional.

45. If T is a linear transformation from V to V, then $\{f \text{ in } V : T(f) = f\}$ must be a subspace of V.

46. If T is a linear transformation from P_6 to P_6 that transforms t^k into a polynomial of degree k (for $k = 1, \ldots, 6$), then T must be an isomorphism.

47. There exist invertible 2×2 matrices P and Q such that the linear transformation $T(M) = PM - MQ$ is an isomorphism.

48. There exists a linear transformation from P_6 to $\mathbb{C}$ whose kernel is isomorphic to $\mathbb{R}^{2\times 2}$.

49. If f_1, f_2, f_3 is a basis of a linear space V, and if f is any element of V, then the elements $f_1 + f$, $f_2 + f$, $f_3 + f$ must form a basis of V as well.

50. There exists a two-dimensional subspace of $\mathbb{R}^{2\times 2}$ whose nonzero elements are all invertible.

51. The space P_{11} is isomorphic to $\mathbb{R}^{3\times 4}$.

52. If T is a linear transformation from V to W, and if both $\text{im}(T)$ and $\ker(T)$ are finite dimensional, then W must be finite dimensional.

53. If T is a linear transformation from V to $\mathbb{R}^{2\times 2}$ with $\ker(T) = \{0\}$, then the inequality $\dim(V) \le 4$ must hold.

54. The function

$$T(f(t)) = \frac{d}{dt}\int_2^{3t+4} f(x)\,dx$$

from P_5 to P_5 is an isomorphism.

55. Any 4-dimensional linear space has infinitely many 3-dimensional subspaces.

56. If the matrix of a linear transformation T (with respect to some basis) is $\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix}$, then there must exist a nonzero element f in the domain of T such that $T(f) = 4f$.

57. If the image of a linear transformation T is infinite dimensional, then the domain of T must be infinite dimensional.

58. There exists a 2×2 matrix A such that the space of all matrices commuting with A is 3-dimensional.

59. If A, B, C, and D are noninvertible 2×2 matrices, then the matrices AB, AC, and AD must be linearly dependent.

60. There exist two distinct 3-dimensional subspaces W_1 and W_2 of P_4 such that the union $W_1 \cup W_2$ is a subspace of P_4 as well.

61. If the elements $f_1, \ldots, f_n$ (where $f_1 \neq 0$) are linearly dependent, then one element f_k can be expressed *uniquely* as a linear combination of the preceding elements $f_1, \ldots, f_{k-1}$.

62. There exists a 3×3 matrix P such that the linear transformation $T(M) = MP - PM$ from $\mathbb{R}^{3\times 3}$ to $\mathbb{R}^{3\times 3}$ is an isomorphism.

63. If f_1, f_2, f_3, f_4, f_5 are elements of a linear space V, and if there are exactly two redundant elements in the list f_1, f_2, f_3, f_4, f_5, then there must be exactly two redundant elements in the list f_2, f_4, f_5, f_1, f_3 as well.

64. There exists a linear transformation T from P_6 to P_6 such that the kernel of T is isomorphic to the image of T.

65. If T is a linear transformation from V to W, and if both $\text{im}(T)$ and $\ker(T)$ are finite dimensional, then V must be finite dimensional.

66. If the matrix of a linear transformation T (with respect to some basis) is $\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix}$, then there must exist a nonzero element f in the domain of T such that $T(f) = 5f$.

67. Every three-dimensional subspace of $\mathbb{R}^{2\times 2}$ contains at least one invertible matrix.