

## Math S-21b – Summer 2011 – Homework #3

Problems due Monday, July 11:

### Section 3.1:

12. Find vectors that span the kernel of  $\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & -2 & -1 & 3 & 4 \end{bmatrix}$ . Use paper and pencil.

20. For the matrix  $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ , describe the image of the transformation  $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$  geometrically (as a line, plane, etc. in  $\mathbf{R}^2$  or  $\mathbf{R}^3$ ).

32. Give an example of a linear transformation whose image is the line spanned by  $\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$  in  $\mathbf{R}^3$ .

34. Give an example of a linear transformation whose kernel is the line spanned by  $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$  in  $\mathbf{R}^3$ .

39. Consider an  $n \times p$  matrix  $\mathbf{A}$  and a  $p \times m$  matrix  $\mathbf{B}$ .

- What is the relationship between  $\ker(\mathbf{AB})$  and  $\ker(\mathbf{B})$ ? Are they always equal? Is one of them always contained in the other?
- What is the relationship between  $\text{im}(\mathbf{A})$  and  $\text{im}(\mathbf{AB})$ ?

44. Consider a matrix  $\mathbf{A}$ , and let  $\mathbf{B} = \text{rref}(\mathbf{A})$ .

- Is  $\ker(\mathbf{A})$  necessarily equal to  $\ker(\mathbf{B})$ ? Explain.
- Is  $\text{im}(\mathbf{A})$  necessarily equal to  $\text{im}(\mathbf{B})$ ? Explain.

### Section 3.2:

24. Find a redundant column vector of the matrix  $\mathbf{A} = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$ , and write it as a linear combination of

preceding columns. Use this representation to write a nontrivial relation among the columns, and thus find a nonzero vector in the kernel of  $\mathbf{A}$ .

36. Consider a linear transformation  $T$  from  $\mathbf{R}^n$  to  $\mathbf{R}^p$  and some linearly dependent vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  in  $\mathbf{R}^n$ . Are the vectors  $T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_m)$  necessarily linearly dependent? How can you tell?

40. Consider an  $n \times p$  matrix  $\mathbf{A}$  and a  $p \times m$  matrix  $\mathbf{B}$ . We are told that the columns of  $\mathbf{A}$  and the columns of  $\mathbf{B}$  are linearly independent (respectively). Are the columns of the product  $\mathbf{AB}$  linearly independent as well? *Hint:* Exercise 3.1.51 is useful. [This exercise reads: “Consider an  $n \times p$  matrix  $\mathbf{A}$  and a  $p \times m$  matrix  $\mathbf{B}$  such that  $\ker(\mathbf{A}) = \{\mathbf{0}\}$  and  $\ker(\mathbf{B}) = \{\mathbf{0}\}$ . Find  $\ker(\mathbf{AB})$ .”]

48. Express the plane  $V$  in  $\mathbf{R}^3$  with equation  $3x_1 + 4x_2 + 5x_3 = 0$  as the kernel of a matrix  $\mathbf{A}$  and as the image of a matrix  $\mathbf{B}$ . [Note: This exercise doesn't specify the sizes of the matrices  $\mathbf{A}$  and  $\mathbf{B}$ . There are many possible solutions, including the case where both  $\mathbf{A}$  and  $\mathbf{B}$  are  $3 \times 3$  matrices. Think geometrically!]

**Section 3.3:**

24. Find the reduced row-echelon form of the matrix  $\mathbf{A} = \begin{bmatrix} 4 & 8 & 1 & 1 & 6 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}$ .

Then find a basis of the image of  $\mathbf{A}$  and a basis of the kernel of  $\mathbf{A}$ .

30. Find a basis of the subspace of  $\mathbf{R}^4$  defined by the equation  $2x_1 - x_2 + 2x_3 + 4x_4 = 0$ .

32. Find a basis of the subspace of  $\mathbf{R}^4$  that consists of all vectors perpendicular to both  $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ .

36. Can you find a  $3 \times 3$  matrix  $\mathbf{A}$  such that  $\text{im}(\mathbf{A}) = \text{ker}(\mathbf{A})$ ? Explain.

**Section 3.4:**

In Exercises 6 and 18, determine whether the vector  $\mathbf{x}$  is in the span  $V$  of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  (proceed “by inspection” if possible, and use the reduced row-echelon form if necessary). If  $\mathbf{x}$  is in  $V$ , find the coordinates of  $\mathbf{x}$  with respect to the basis  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$  of  $V$ , and write the coordinate vector  $[\mathbf{x}]_{\mathcal{B}}$ .

6.  $\mathbf{x} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$ ;  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$       18.  $\mathbf{x} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}$ ;  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

In Exercises 26 and 28, find the matrix  $\mathbf{B}$  of the linear transformation  $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$  with respect to the basis  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ .

26.  $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ ;  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$       28.  $\mathbf{A} = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$ ;  $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ ;  $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ ;  $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

42. Find a basis  $\mathcal{B}$  of  $\mathbf{R}^3$  such that the  $\mathcal{B}$ -matrix  $\mathbf{B}$  of the linear transformation given by reflection  $T$  about the plane  $x_1 - 2x_2 + 2x_3 = 0$  in  $\mathbf{R}^3$  is diagonal.

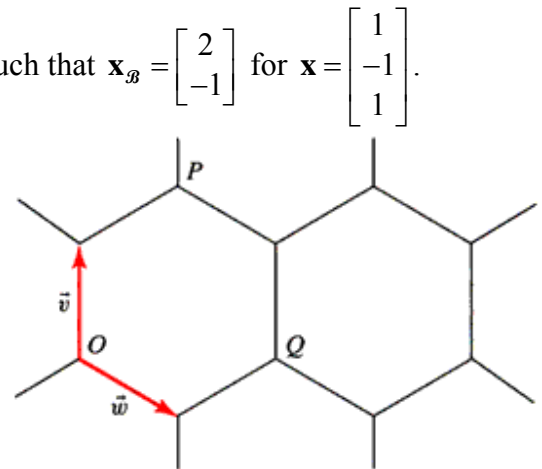
44. Consider the plane  $2x_1 - 3x_2 + 4x_3 = 0$  with basis  $\mathcal{B}$  consisting of vectors  $\begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix}$  and  $\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$ .

If  $\mathbf{x}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ , find  $\mathbf{x}$ .

46. Consider the plane  $x_1 + 2x_2 + x_3 = 0$ . Find a basis  $\mathcal{B}$  of this plane such that  $\mathbf{x}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$  for  $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ .

50. Given a hexagonal tiling of the plane, such as you might find on a kitchen floor, consider the basis  $\mathcal{B}$  of  $\mathbf{R}^2$  consisting of the vectors  $\vec{v}, \vec{w}$  in the following sketch (right):

a. Find the coordinate vectors  $[\overrightarrow{OP}]_{\mathcal{B}}$  and  $[\overrightarrow{OQ}]_{\mathcal{B}}$ . *Hint:* Sketch the coordinate grid defined by the basis  $\mathcal{B} = \{\vec{v}, \vec{w}\}$ .



b. We are told that  $\begin{bmatrix} \overline{OR} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ . Sketch the point  $R$ . Is  $R$  the vertex or the center of a tile?

c. We are told that  $\begin{bmatrix} \overline{OS} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 17 \\ 13 \end{bmatrix}$ . Is  $S$  the center or the vertex of a tile?

56. Find a basis  $\mathcal{B}$  of  $\mathbf{R}^2$  such that  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$  and  $\begin{bmatrix} 3 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ .

[Note: Read this problem very carefully. Many students get this one backwards!]

**For additional practice:**

**Section 3.1:**

For each matrix  $\mathbf{A}$  in Exercises 5, 6, and 8, find vectors that span the kernel of  $\mathbf{A}$ . Use paper and pencil.

5.  $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$

6.  $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

8.  $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

19. For the matrix  $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix}$ , describe the image of the transformation  $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$  geometrically (as a line, plane, etc. in  $\mathbf{R}^2$  or  $\mathbf{R}^3$ ).

Describe the images and kernels of the transformations in Exercises 23 through 25 geometrically.

23. Reflection about the line  $y = \frac{1}{3}x$  in  $\mathbf{R}^2$ .

24. Orthogonal projection onto the plane  $x + 2y + 3z = 0$  in  $\mathbf{R}^3$ .

25. Rotation through an angle  $\frac{\pi}{4}$  in the counterclockwise direction (in  $\mathbf{R}^2$ ).

31. Give an example of a matrix  $\mathbf{A}$  such that  $\text{im}(\mathbf{A})$  is the plane with normal vector  $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$  in  $\mathbf{R}^3$ .

**Section 3.2:**

Which of the sets  $W$  in Exercises 1 through 3 are subspaces of  $\mathbf{R}^3$ ?

1.  $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$

2.  $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \leq y \leq z \right\}$

3.  $W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x, y, z \text{ arbitrary constants} \right\}$

6. Consider two subspaces  $V$  and  $W$  of  $\mathbf{R}^n$ .

a. Is the intersection  $V \cap W$  necessarily a subspace of  $\mathbf{R}^n$ ?

b. Is the union  $V \cup W$  necessarily a subspace of  $\mathbf{R}^n$ ?

In Exercises 17 and 19, use paper and pencil to identify the redundant vectors. Thus determine whether the given vectors are linearly independent.

17.  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

19.  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$

28. Find a basis for the image of the matrix  $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$ .

37. Consider a linear transformation  $T$  from  $\mathbf{R}^n$  to  $\mathbf{R}^p$  and some linearly independent vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  in  $\mathbf{R}^n$ . Are the vectors  $T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_m)$  necessarily linearly independent? How can you tell?

41. Consider an  $m \times n$  matrix  $\mathbf{A}$  and a  $n \times m$  matrix  $\mathbf{B}$  (with  $n \neq m$ ) such that  $\mathbf{AB} = \mathbf{I}_m$ . (We say that  $\mathbf{A}$  is a *left inverse* of  $\mathbf{B}$ .) Are the columns of  $\mathbf{B}$  linearly independent? What about the columns of  $\mathbf{A}$ ?

49. Express the line  $L$  in  $\mathbf{R}^3$  spanned by the vector  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  as the image of a matrix  $\mathbf{A}$  and as the kernel of a

matrix  $\mathbf{B}$ . [Note: This exercise doesn't specify the sizes of the matrices  $\mathbf{A}$  and  $\mathbf{B}$ . There are many possible solutions, including the case where both  $\mathbf{A}$  and  $\mathbf{B}$  are  $3 \times 3$  matrices. Think geometrically!]

### Section 3.3:

23. Find the reduced row-echelon form of the matrix  $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$ .

Then find a basis of the image of  $\mathbf{A}$  and a basis of the kernel of  $\mathbf{A}$ .

27. Determine whether the vectors  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix} \right\}$  form a basis of  $\mathbf{R}^4$ .

29. Find a basis of the subspace of  $\mathbf{R}^3$  defined by the equation  $2x_1 + 3x_2 + x_3 = 0$ .

60. Consider two subspaces  $V$  and  $W$  of  $\mathbf{R}^n$ , where  $V$  is contained in  $W$ . Explain why  $\dim(V) \leq \dim(W)$ . (This statement seems intuitively rather obvious. Still, we cannot rely on our intuition when dealing with  $\mathbf{R}^n$ .)

61. Consider two subspaces  $V$  and  $W$  of  $\mathbf{R}^n$ , where  $V$  is contained in  $W$ . In Exercise 40, we learned that  $\dim(V) \leq \dim(W)$ . Show that if  $\dim(V) = \dim(W)$ , then  $V = W$ .

62. Consider a subspace  $V$  of  $\mathbf{R}^n$  with  $\dim(V) = n$ . Explain why  $V = \mathbf{R}^n$ .

81. Consider a  $4 \times 2$  matrix  $\mathbf{A}$  and a  $2 \times 5$  matrix  $\mathbf{B}$ .

a. What are the possible dimensions of the *kernel* of  $\mathbf{AB}$ ?

b. What are the possible dimensions of the *image* of  $\mathbf{AB}$ ?

### Section 3.4:

In Exercises 5, 7, and 17, determine whether the vector  $\mathbf{x}$  is in the span  $V$  of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  (proceed "by inspection" if possible, and use the reduced row-echelon form if necessary). If  $\mathbf{x}$  is in  $V$ , find the coordinates of  $\mathbf{x}$  with respect to the basis  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$  of  $V$ , and write the coordinate vector  $[\mathbf{x}]_{\mathcal{B}}$ .

5.  $\mathbf{x} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}$ ;  $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

7.  $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$ ;  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$17. \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}; \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

27. Find the matrix  $\mathbf{B}$  of the linear transformation  $T(\mathbf{x}) = \mathbf{Ax}$  with respect to the

$$\text{basis } \mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} \text{ where } \mathbf{A} = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}; \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}; \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}; \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

45. Consider the plane  $2x_1 - 3x_2 + 4x_3 = 0$ . Find a basis  $\mathcal{B}$  of this plane such that  $\mathbf{x}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  for  $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ .

55. Consider the basis  $\mathcal{B}$  of  $\mathbf{R}^2$  consisting of the vectors  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , and let  $\mathcal{R}$  be the basis consisting of  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ . Find a matrix  $\mathbf{P}$  such that  $[\mathbf{x}]_{\mathcal{R}} = \mathbf{P}[\mathbf{x}]_{\mathcal{B}}$ .

### Chapter 3 True/False

- If  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  are linearly independent vectors in  $\mathbb{R}^n$ , then they must form a basis of  $\mathbb{R}^n$ .
- There exists a  $5 \times 4$  matrix whose image consists of all of  $\mathbb{R}^5$ .
- The kernel of any invertible matrix consists of the zero vector only.
- The identity matrix  $I_n$  is similar to all invertible  $n \times n$  matrices.
- If  $2\vec{u} + 3\vec{v} + 4\vec{w} = 5\vec{u} + 6\vec{v} + 7\vec{w}$ , then vectors  $\vec{u}, \vec{v}, \vec{w}$  must be linearly dependent.
- The column vectors of a  $5 \times 4$  matrix must be linearly dependent.
- If  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  and  $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$  are any two bases of a subspace  $V$  of  $\mathbb{R}^{10}$ , then  $n$  must equal  $m$ .
- If  $A$  is a  $5 \times 6$  matrix of rank 4, then the nullity of  $A$  is 1.
- The image of a  $3 \times 4$  matrix is a subspace of  $\mathbb{R}^4$ .
- The span of vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  consists of all linear combinations of vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ .
- If vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  are linearly independent, then vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  must be linearly independent as well.
- The vectors of the form  $\begin{bmatrix} a \\ b \\ 0 \\ a \end{bmatrix}$  (where  $a$  and  $b$  are arbitrary real numbers) form a subspace of  $\mathbb{R}^4$ .
- Matrix  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  is similar to  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .
- Vectors  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$  form a basis of  $\mathbb{R}^3$ .
- If the kernel of a matrix  $A$  consists of the zero vector only, then the column vectors of  $A$  must be linearly independent.
- If the image of an  $n \times n$  matrix  $A$  is all of  $\mathbb{R}^n$ , then  $A$  must be invertible.
- If vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  span  $\mathbb{R}^4$ , then  $n$  must be equal to 4.
- If vectors  $\vec{u}, \vec{v}$ , and  $\vec{w}$  are in a subspace  $V$  of  $\mathbb{R}^n$ , then vector  $2\vec{u} - 3\vec{v} + 4\vec{w}$  must be in  $V$  as well.
- If matrix  $A$  is similar to matrix  $B$ , and  $B$  is similar to  $C$ , then  $C$  must be similar to  $A$ .
- If a subspace  $V$  of  $\mathbb{R}^n$  contains none of the standard vectors  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ , then  $V$  consists of the zero vector only.
- If  $A$  and  $B$  are  $n \times n$  matrices, and vector  $\vec{v}$  is in the kernel of both  $A$  and  $B$ , then  $\vec{v}$  must be in the kernel of matrix  $AB$  as well.
- If two nonzero vectors are linearly dependent, then each of them is a scalar multiple of the other.
- If  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are any three distinct vectors in  $\mathbb{R}^3$ , then there must be a linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  such that  $T(\vec{v}_1) = \vec{e}_1, T(\vec{v}_2) = \vec{e}_2$ , and  $T(\vec{v}_3) = \vec{e}_3$ .
- If vectors  $\vec{u}, \vec{v}, \vec{w}$  are linearly dependent, then vector  $\vec{w}$  must be a linear combination of  $\vec{u}$  and  $\vec{v}$ .
- If  $A$  and  $B$  are invertible  $n \times n$  matrices, then  $AB$  must be similar to  $BA$ .
- If  $A$  is an invertible  $n \times n$  matrix, then the kernels of  $A$  and  $A^{-1}$  must be equal.

27. Matrix  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  is similar to  $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ .
28. Vectors  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ ,  $\begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$ ,  $\begin{bmatrix} 9 \\ 8 \\ 7 \\ 6 \end{bmatrix}$ ,  $\begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$  are linearly independent.
29. If a subspace  $V$  of  $\mathbb{R}^3$  contains the standard vectors  $\vec{e}_1, \vec{e}_2, \vec{e}_3$ , then  $V$  must be  $\mathbb{R}^3$ .
30. If a  $2 \times 2$  matrix  $P$  represents the orthogonal projection onto a line in  $\mathbb{R}^2$ , then  $P$  must be similar to matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ .
31.  $\mathbb{R}^2$  is a subspace of  $\mathbb{R}^3$ .
32. If an  $n \times n$  matrix  $A$  is similar to matrix  $B$ , then  $A + 7I_n$  must be similar to  $B + 7I_n$ .
33. If  $V$  is any three-dimensional subspace of  $\mathbb{R}^5$ , then  $V$  has infinitely many bases.
34. Matrix  $I_n$  is similar to  $2I_n$ .
35. If  $AB = 0$  for two  $2 \times 2$  matrices  $A$  and  $B$ , then  $BA$  must be the zero matrix as well.
36. If  $A$  and  $B$  are  $n \times n$  matrices, and vector  $\vec{v}$  is in the image of both  $A$  and  $B$ , then  $\vec{v}$  must be in the image of matrix  $A + B$  as well.
37. If  $V$  and  $W$  are subspaces of  $\mathbb{R}^n$ , then their union  $V \cup W$  must be a subspace of  $\mathbb{R}^n$  as well.
38. If the kernel of a  $5 \times 4$  matrix  $A$  consists of the zero vector only and if  $A\vec{v} = A\vec{w}$  for two vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^4$ , then vectors  $\vec{v}$  and  $\vec{w}$  must be equal.
39. If  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  and  $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$  are two bases of  $\mathbb{R}^n$ , then there exists a linear transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  such that  $T(\vec{v}_1) = \vec{w}_1, T(\vec{v}_2) = \vec{w}_2, \dots, T(\vec{v}_n) = \vec{w}_n$ .
40. If matrix  $A$  represents a rotation through  $\pi/2$  and matrix  $B$  a rotation through  $\pi/4$ , then  $A$  is similar to  $B$ .
41. There exists a  $2 \times 2$  matrix  $A$  such that  $\text{im}(A) = \text{ker}(A)$ .
42. If two  $n \times n$  matrices  $A$  and  $B$  have the same rank, then they must be similar.
43. If  $A$  is similar to  $B$ , and  $A$  is invertible, then  $B$  must be invertible as well.
44. If  $A^2 = 0$  for a  $10 \times 10$  matrix  $A$ , then the inequality  $\text{rank}(A) \leq 5$  must hold.
45. For every subspace  $V$  of  $\mathbb{R}^3$  there exists a  $3 \times 3$  matrix  $A$  such that  $V = \text{im}(A)$ .
46. There exists a nonzero  $2 \times 2$  matrix  $A$  that is similar to  $2A$ .
47. If the  $2 \times 2$  matrix  $R$  represents the reflection about a line in  $\mathbb{R}^2$ , then  $R$  must be similar to matrix  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .
48. If  $A$  is similar to  $B$ , then there exists one and only one invertible matrix  $S$  such that  $S^{-1}AS = B$ .
49. If the kernel of a  $5 \times 4$  matrix  $A$  consists of the zero vector alone, and if  $AB = AC$  for two  $4 \times 5$  matrices  $B$  and  $C$ , then matrices  $B$  and  $C$  must be equal.
50. If  $A$  is any  $n \times n$  matrix such that  $A^2 = A$ , then the image of  $A$  and the kernel of  $A$  have only the zero vector in common.
51. There exists a  $2 \times 2$  matrix  $A$  such that  $A^2 \neq 0$  and  $A^3 = 0$ .
52. If  $A$  and  $B$  are  $n \times m$  matrices such that the image of  $A$  is a subset of the image of  $B$ , then there must exist an  $m \times m$  matrix  $C$  such that  $A = BC$ .
53. Among the  $3 \times 3$  matrices whose entries are all 0's and 1's, most are invertible.