

Math S-21b – Summer 2009 – Homework #3

Problems due Monday, July 6:

Section 3.1:

12. Find vectors that span the kernel of $\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & -2 & -1 & 3 & 4 \end{bmatrix}$. Use paper and pencil.

20. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, describe the image of the transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ geometrically (as a line, plane, etc. in \mathbf{R}^2 or \mathbf{R}^3).

32. Give an example of a linear transformation whose image is the line spanned by $\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$ in \mathbf{R}^3 .

34. Give an example of a linear transformation whose kernel is the line spanned by $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ in \mathbf{R}^3 .

39. Consider an $n \times p$ matrix \mathbf{A} and a $p \times m$ matrix \mathbf{B} .

- What is the relationship between $\ker(\mathbf{AB})$ and $\ker(\mathbf{B})$? Are they always equal? Is one of them always contained in the other?
- What is the relationship between $\text{im}(\mathbf{A})$ and $\text{im}(\mathbf{AB})$?

44. Consider a matrix \mathbf{A} , and let $\mathbf{B} = \text{rref}(\mathbf{A})$.

- Is $\ker(\mathbf{A})$ necessarily equal to $\ker(\mathbf{B})$? Explain.
- Is $\text{im}(\mathbf{A})$ necessarily equal to $\text{im}(\mathbf{B})$? Explain.

Section 3.2:

24. Find a redundant column vector of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$, and write it as a linear combination of

preceding columns. Use this representation to write a nontrivial relation among the columns, and thus find a nonzero vector in the kernel of \mathbf{A} .

36. Consider a linear transformation T from \mathbf{R}^n to \mathbf{R}^p and some linearly dependent vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ in \mathbf{R}^n . Are the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_m)$ necessarily linearly dependent? How can you tell?

40. Consider an $n \times p$ matrix \mathbf{A} and a $p \times m$ matrix \mathbf{B} . We are told that the columns of \mathbf{A} and the columns of \mathbf{B} are linearly independent (respectively). Are the columns of the product \mathbf{AB} linearly independent as well? *Hint:* Exercise 3.1.51 is useful. [This exercise reads: “Consider an $n \times p$ matrix \mathbf{A} and a $p \times m$ matrix \mathbf{B} such that $\ker(\mathbf{A}) = \{\mathbf{0}\}$ and $\ker(\mathbf{B}) = \{\mathbf{0}\}$. Find $\ker(\mathbf{AB})$.”]

48. Express the plane V in \mathbf{R}^3 with equation $3x_1 + 4x_2 + 5x_3 = 0$ as the kernel of a matrix \mathbf{A} and as the image of a matrix \mathbf{B} . [Note: This exercise doesn't specify the sizes of the matrices \mathbf{A} and \mathbf{B} . There are many possible solutions, including the case where both \mathbf{A} and \mathbf{B} are 3×3 matrices. Think geometrically!]

Section 3.3:

24. Find the reduced row-echelon form of the matrix $\mathbf{A} = \begin{bmatrix} 4 & 8 & 1 & 1 & 6 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}$.

Then find a basis of the image of \mathbf{A} and a basis of the kernel of \mathbf{A} .

30. Find a basis of the subspace of \mathbf{R}^4 defined by the equation $2x_1 - x_2 + 2x_3 + 4x_4 = 0$.

32. Find a basis of the subspace of \mathbf{R}^4 that consists of all vectors perpendicular to both $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$.

36. Can you find a 3×3 matrix \mathbf{A} such that $\text{im}(\mathbf{A}) = \text{ker}(\mathbf{A})$? Explain.

Section 3.4:

In Exercises 6 and 18, determine whether the vector \mathbf{x} is in the span V of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ (proceed “by inspection” if possible, and use the reduced row-echelon form if necessary). If \mathbf{x} is in V , find the coordinates of \mathbf{x} with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ of V , and write the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.

6. $\mathbf{x} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ 18. $\mathbf{x} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

In Exercises 26 and 28, find the matrix \mathbf{B} of the linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$.

26. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ 28. $\mathbf{A} = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

42. Find a basis \mathcal{B} of \mathbf{R}^3 such that the \mathcal{B} -matrix \mathbf{B} of the linear transformation given by reflection T about the plane $x_1 - 2x_2 + 2x_3 = 0$ in \mathbf{R}^3 is diagonal.

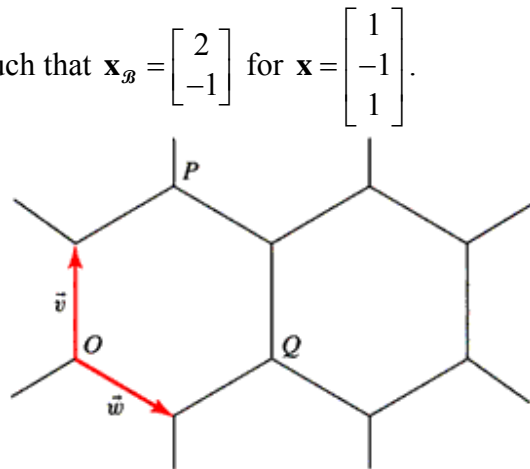
44. Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$ with basis \mathcal{B} consisting of vectors $\begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$.

If $\mathbf{x}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find \mathbf{x} .

46. Consider the plane $x_1 + 2x_2 + x_3 = 0$. Find a basis \mathcal{B} of this plane such that $\mathbf{x}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ for $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

50. Given a hexagonal tiling of the plane, such as you might find on a kitchen floor, consider the basis \mathcal{B} of \mathbf{R}^2 consisting of the vectors \vec{v}, \vec{w} in the following sketch (right):

a. Find the coordinate vectors $[\overrightarrow{OP}]_{\mathcal{B}}$ and $[\overrightarrow{OQ}]_{\mathcal{B}}$. *Hint:* Sketch the coordinate grid defined by the basis $\mathcal{B} = \{\vec{v}, \vec{w}\}$.



b. We are told that $\begin{bmatrix} \overline{OR} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Sketch the point R . Is R the vertex or the center of a tile?

c. We are told that $\begin{bmatrix} \overline{OS} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 17 \\ 13 \end{bmatrix}$. Is S the center or the vertex of a tile?

56. Find a basis \mathcal{B} of \mathbf{R}^2 such that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

[Note: Read this problem very carefully. Many students get this one backwards!]

For additional practice:

Section 3.1:

For each matrix \mathbf{A} in Exercises 5, 6, and 8, find vectors that span the kernel of \mathbf{A} . Use paper and pencil.

5. $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$

6. $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

8. $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

19. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix}$, describe the image of the transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ geometrically (as a line, plane, etc. in \mathbf{R}^2 or \mathbf{R}^3).

Describe the images and kernels of the transformations in Exercises 23 through 25 geometrically.

23. Reflection about the line $y = \frac{1}{3}x$ in \mathbf{R}^2 .

24. Orthogonal projection onto the plane $x + 2y + 3z = 0$ in \mathbf{R}^3 .

25. Rotation through an angle $\frac{\pi}{4}$ in the counterclockwise direction (in \mathbf{R}^2).

31. Give an example of a matrix \mathbf{A} such that $\text{im}(\mathbf{A})$ is the plane with normal vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ in \mathbf{R}^3 .

Section 3.2:

Which of the sets W in Exercises 1 through 3 are subspaces of \mathbf{R}^3 ?

1. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$ 2. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \leq y \leq z \right\}$ 3. $W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x, y, z \text{ arbitrary constants} \right\}$

6. Consider two subspaces V and W of \mathbf{R}^n .

a. Is the intersection $V \cap W$ necessarily a subspace of \mathbf{R}^n ?

b. Is the union $V \cup W$ necessarily a subspace of \mathbf{R}^n ?

In Exercises 17 and 19, use paper and pencil to identify the redundant vectors. Thus determine whether the given vectors are linearly independent.

$$17. \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \qquad 19. \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$$

$$28. \text{ Find a basis for the image of the matrix } \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}.$$

37. Consider a linear transformation T from \mathbf{R}^n to \mathbf{R}^p and some linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ in \mathbf{R}^n . Are the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_m)$ necessarily linearly independent? How can you tell?

41. Consider an $m \times n$ matrix \mathbf{A} and a $n \times m$ matrix \mathbf{B} (with $n \neq m$) such that $\mathbf{AB} = \mathbf{I}_m$. (We say that \mathbf{A} is a *left inverse* of \mathbf{B} .) Are the columns of \mathbf{B} linearly independent? What about the columns of \mathbf{A} ?

$$49. \text{ Express the line } L \text{ in } \mathbf{R}^3 \text{ spanned by the vector } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ as the image of a matrix } \mathbf{A} \text{ and as the kernel of a}$$

matrix \mathbf{B} . [Note: This exercise doesn't specify the sizes of the matrices \mathbf{A} and \mathbf{B} . There are many possible solutions, including the case where both \mathbf{A} and \mathbf{B} are 3×3 matrices. Think geometrically!]

Section 3.3:

$$23. \text{ Find the reduced row-echelon form of the matrix } \mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}.$$

Then find a basis of the image of \mathbf{A} and a basis of the kernel of \mathbf{A} .

$$27. \text{ Determine whether the vectors } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix} \right\} \text{ form a basis of } \mathbf{R}^4.$$

29. Find a basis of the subspace of \mathbf{R}^3 defined by the equation $2x_1 + 3x_2 + x_3 = 0$.

40. Consider two subspaces V and W of \mathbf{R}^n , where V is contained in W . Explain why $\dim(V) \leq \dim(W)$. (This statement seems intuitively rather obvious. Still, we cannot rely on our intuition when dealing with \mathbf{R}^n .)

41. Consider two subspaces V and W of \mathbf{R}^n , where V is contained in W . In Exercise 40, we learned that $\dim(V) \leq \dim(W)$. Show that if $\dim(V) = \dim(W)$, then $V = W$.

42. Consider a subspace V of \mathbf{R}^n with $\dim(V) = n$. Explain why $V = \mathbf{R}^n$.

61. Consider a 4×2 matrix \mathbf{A} and a 2×5 matrix \mathbf{B} .

a. What are the possible dimensions of the *kernel* of \mathbf{AB} ?

b. What are the possible dimensions of the *image* of \mathbf{AB} ?

Section 3.4:

In Exercises 5, 7, and 17, determine whether the vector \mathbf{x} is in the span V of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ (proceed “by inspection” if possible, and use the reduced row-echelon form if necessary). If \mathbf{x} is in V , find the coordinates of \mathbf{x} with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ of V , and write the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.

$$5. \mathbf{x} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}; \mathbf{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

$$7. \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}; \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$17. \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}; \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

27. Find the matrix \mathbf{B} of the linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with respect to the

$$\text{basis } \mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} \text{ where } \mathbf{A} = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}; \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}; \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}; \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

45. Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$. Find a basis \mathcal{B} of this plane such that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

55. Consider the basis \mathcal{B} of \mathbf{R}^2 consisting of the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and let \mathcal{R} be the basis consisting of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find a matrix \mathbf{P} such that $[\mathbf{x}]_{\mathcal{R}} = \mathbf{P}[\mathbf{x}]_{\mathcal{B}}$.