

Math S-21b – Summer 2024 – Homework #3

Problems due Monday, July 8:

Problem 1. Find vectors that span the kernel of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix}$. Use paper and pencil.

Problem 2. Give an example of a linear transformation whose image is the line spanned by $\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$ in \mathbf{R}^3 .

Problem 3. Give an example of a linear transformation whose kernel is the line spanned by $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ in \mathbf{R}^3 .

Problem 4. Consider an $n \times p$ matrix \mathbf{A} and a $p \times m$ matrix \mathbf{B} .

- What is the relationship between $\ker(\mathbf{AB})$ and $\ker(\mathbf{B})$? Are they always equal? Is one of them always contained in the other?
- What is the relationship between $\text{im}(\mathbf{A})$ and $\text{im}(\mathbf{AB})$?

Problem 5. Consider a matrix \mathbf{A} , and let $\mathbf{B} = \text{rref}(\mathbf{A})$.

- Is $\ker(\mathbf{A})$ necessarily equal to $\ker(\mathbf{B})$? Explain.
- Is $\text{im}(\mathbf{A})$ necessarily equal to $\text{im}(\mathbf{B})$? Explain.

Problem 6. Consider a linear transformation T from \mathbf{R}^n to \mathbf{R}^p and some linearly dependent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ in \mathbf{R}^n . Are the vectors $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_m)\}$ necessarily linearly dependent? How can you tell?

Problem 7(a). Consider an $n \times p$ matrix \mathbf{A} and a $p \times m$ matrix \mathbf{B} such that $\ker(\mathbf{A}) = \{\mathbf{0}\}$ and $\ker(\mathbf{B}) = \{\mathbf{0}\}$. Find $\ker(\mathbf{AB})$.

(b) Consider an $n \times p$ matrix \mathbf{A} and a $p \times m$ matrix \mathbf{B} . We are told that the columns of \mathbf{A} and the columns of \mathbf{B} are linearly independent (respectively). Are the columns of the product \mathbf{AB} linearly independent as well? [*Hint:* Problem 7a above is useful.]

Problem 8. Express the plane V in \mathbf{R}^3 with equation $3x_1 + 4x_2 + 5x_3 = 0$ as the kernel of a matrix \mathbf{A} and as the image of a matrix \mathbf{B} . [*Note:* This exercise doesn't specify the sizes of the matrices \mathbf{A} and \mathbf{B} . There are many possible solutions, including the case where both \mathbf{A} and \mathbf{B} are 3×3 matrices. Think geometrically!]

Problem 9. Find the reduced row-echelon form of the matrix $\mathbf{A} = \begin{bmatrix} 4 & 8 & 1 & 1 & 6 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}$.

Then find a basis of the image of \mathbf{A} and a basis of the kernel of \mathbf{A} .

Problem 10. Find a basis of the subspace of \mathbf{R}^4 defined by the equation $2x_1 - x_2 + 2x_3 + 4x_4 = 0$.

Problem 11. Find a basis of the subspace of \mathbf{R}^4 that consists of all vectors perpendicular to both $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$.

In Problems 12 and 13, determine whether the vector \mathbf{x} is in the span V of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ (proceed “by inspection” if possible, and use the reduced row-echelon form if necessary). If \mathbf{x} is in V , find the coordinates of \mathbf{x} with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ of V , and write the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.

Problem 12. $\mathbf{x} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

Problem 13. $\mathbf{x} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Problem 14. Find the matrix of the linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$

where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 15. Find the matrix of the linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

where $\mathbf{A} = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

Problem 16. Consider the plane with equation $2x_1 - 3x_2 + 4x_3 = 0$ with the basis \mathcal{B} consisting of

vectors $\begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find \mathbf{x} .

Problem 17. Find a basis \mathcal{B} of the plane $x_1 + 2x_2 + x_3 = 0$. such that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ for $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Problem 18. Given a hexagonal tiling of the plane, such as you might find on a kitchen floor, consider the basis \mathcal{B} of \mathbf{R}^2 consisting of the vectors \vec{v}, \vec{w} in the following sketch:

a. Find the coordinate vectors $[\overline{OP}]_{\mathcal{B}}$ and $[\overline{OQ}]_{\mathcal{B}}$.

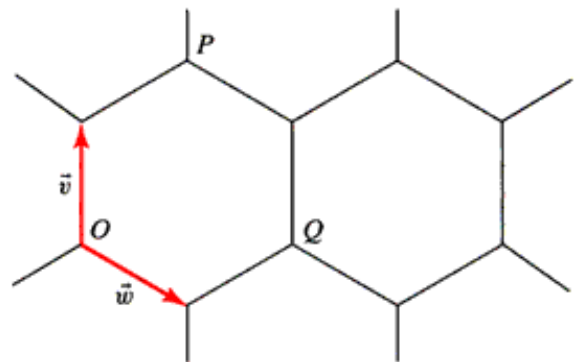
Hint: Sketch the coordinate grid defined by the basis $\mathcal{B} = \{\vec{v}, \vec{w}\}$.

b. We are told that $[\overline{OR}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Sketch the point R .

Is R the vertex or the center of a tile?

c. We are told that $[\overline{OS}]_{\mathcal{B}} = \begin{bmatrix} 17 \\ 13 \end{bmatrix}$.

Is S the center or the vertex of a tile?



Problem 19. Find a basis \mathcal{B} of \mathbf{R}^2 such that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

[Note: Read this problem very carefully. Many students get this one backwards!]

For additional practice:

Section 3.1:

For each matrix \mathbf{A} in Exercises 5, 6, and 8, find vectors that span the kernel of \mathbf{A} . Use paper and pencil.

5. $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$

6. $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

8. $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

19. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix}$, describe the image of the transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ geometrically (as a line, plane, etc. in \mathbf{R}^2 or \mathbf{R}^3).

20. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, describe the image of the transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ geometrically (as a line, plane, etc. in \mathbf{R}^2 or \mathbf{R}^3).

Describe the images and kernels of the transformations in Exercises 23 through 25 geometrically.

23. Reflection about the line $y = \frac{1}{3}x$ in \mathbf{R}^2 .

24. Orthogonal projection onto the plane $x + 2y + 3z = 0$ in \mathbf{R}^3 .

25. Rotation through an angle $\frac{\pi}{4}$ in the counterclockwise direction (in \mathbf{R}^2).

31. Give an example of a matrix \mathbf{A} such that $\text{im}(\mathbf{A})$ is the plane with normal vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ in \mathbf{R}^3 .

Section 3.2:

Which of the sets W in Exercises 1 through 3 are subspaces of \mathbf{R}^3 ?

1. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$ 2. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \leq y \leq z \right\}$ 3. $W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x, y, z \text{ arbitrary constants} \right\}$

6. Consider two subspaces V and W of \mathbf{R}^n .

- a. Is the intersection $V \cap W$ necessarily a subspace of \mathbf{R}^n ?
- b. Is the union $V \cup W$ necessarily a subspace of \mathbf{R}^n ?

In Exercises 17 and 19, use paper and pencil to identify the redundant vectors. Thus determine whether the given vectors are linearly independent.

17. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

19. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$

24. Find a redundant column vector of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$, and write it as a linear combination of

preceding columns. Use this representation to write a nontrivial relation among the columns, and thus find a nonzero vector in the kernel of \mathbf{A} .

28. Find a basis for the image of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$.

37. Consider a linear transformation T from \mathbf{R}^n to \mathbf{R}^p and some linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ in \mathbf{R}^n . Are the vectors $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_m)\}$ necessarily linearly independent? How can you tell?

41. Consider an $m \times n$ matrix \mathbf{A} and an $n \times m$ matrix \mathbf{B} (with $n \neq m$) such that $\mathbf{AB} = \mathbf{I}_m$. (We say that \mathbf{A} is a *left inverse* of \mathbf{B} .) Are the columns of \mathbf{B} linearly independent? What about the columns of \mathbf{A} ?

49. Express the line L in \mathbf{R}^3 spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the image of a matrix \mathbf{A} and as the kernel of a

matrix \mathbf{B} . [Note: This exercise doesn't specify the sizes of the matrices \mathbf{A} and \mathbf{B} . There are many possible solutions, including the case where both \mathbf{A} and \mathbf{B} are 3×3 matrices. Think geometrically!]

Section 3.3:

23. Find the reduced row-echelon form of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$.

Then find a basis of the image of \mathbf{A} and a basis of the kernel of \mathbf{A} .

27. Determine whether the vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix} \right\}$ form a basis of \mathbf{R}^4 .

29. Find a basis of the subspace of \mathbf{R}^3 defined by the equation $2x_1 + 3x_2 + x_3 = 0$.

36. Can you find a 3×3 matrix \mathbf{A} such that $\text{im}(\mathbf{A}) = \text{ker}(\mathbf{A})$? Explain.

60. Consider two subspaces V and W of \mathbf{R}^n , where V is contained in W . Explain why $\dim(V) \leq \dim(W)$. (This statement seems intuitively rather obvious. Still, we cannot rely on our intuition when dealing with \mathbf{R}^n .)

61. Consider two subspaces V and W of \mathbf{R}^n , where V is contained in W . In Exercise 40, we learned that $\dim(V) \leq \dim(W)$. Show that if $\dim(V) = \dim(W)$, then $V = W$.

62. Consider a subspace V of \mathbf{R}^n with $\dim(V) = n$. Explain why $V = \mathbf{R}^n$.

81. Consider a 4×2 matrix \mathbf{A} and a 2×5 matrix \mathbf{B} .

a. What are the possible dimensions of the *kernel* of \mathbf{AB} ?

b. What are the possible dimensions of the *image* of \mathbf{AB} ?

Section 3.4:

In Exercises 5, 7, and 17, determine whether the vector \mathbf{x} is in the span V of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ (proceed "by inspection" if possible, and use the reduced row-echelon form if necessary). If \mathbf{x} is in V , find the coordinates of \mathbf{x} with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ of V , and write the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.

5. $\mathbf{x} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

7. $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$17. \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}; \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

27. Find the matrix \mathbf{B} of the linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with respect to the

$$\text{basis } \mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} \text{ where } \mathbf{A} = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}; \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}; \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}; \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

42. Find a basis \mathcal{B} of \mathbf{R}^3 such that the \mathcal{B} -matrix \mathbf{B} of the linear transformation given by reflection T about the plane $x_1 - 2x_2 + 2x_3 = 0$ in \mathbf{R}^3 is diagonal.

45. Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$. Find a basis \mathcal{B} of this plane such that $\mathbf{x}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

55. Consider the basis \mathcal{B} of \mathbf{R}^2 consisting of the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and let \mathcal{R} be the basis consisting of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find a matrix \mathbf{P} such that $[\mathbf{x}]_{\mathcal{R}} = \mathbf{P}[\mathbf{x}]_{\mathcal{B}}$.

Chapter 3 True/False

1. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent vectors in \mathbb{R}^n , then they must form a basis of \mathbb{R}^n .
2. There exists a 5×4 matrix whose image consists of all of \mathbb{R}^5 .
3. The kernel of any invertible matrix consists of the zero vector only.
4. The identity matrix I_n is similar to all invertible $n \times n$ matrices.
5. If $2\vec{u} + 3\vec{v} + 4\vec{w} = 5\vec{u} + 6\vec{v} + 7\vec{w}$, then vectors $\vec{u}, \vec{v}, \vec{w}$ must be linearly dependent.
6. The column vectors of a 5×4 matrix must be linearly dependent.
7. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ and $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$ are any two bases of a subspace V of \mathbb{R}^{10} , then n must equal m .
8. If A is a 5×6 matrix of rank 4, then the nullity of A is 1.
9. The image of a 3×4 matrix is a subspace of \mathbb{R}^4 .
10. The span of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ consists of all linear combinations of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

11. If vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent, then vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ must be linearly independent as well.

12. The vectors of the form $\begin{bmatrix} a \\ b \\ 0 \\ a \end{bmatrix}$ (where a and b are arbitrary real numbers) form a subspace of \mathbb{R}^4 .

13. Matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

14. Vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ form a basis of \mathbb{R}^3 .

15. If the kernel of a matrix A consists of the zero vector only, then the column vectors of A must be linearly independent.

16. If the image of an $n \times n$ matrix A is all of \mathbb{R}^n , then A must be invertible.

17. If vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span \mathbb{R}^4 , then n must be equal to 4.

18. If vectors \vec{u}, \vec{v} , and \vec{w} are in a subspace V of \mathbb{R}^n , then vector $2\vec{u} - 3\vec{v} + 4\vec{w}$ must be in V as well.

19. If matrix A is similar to matrix B , and B is similar to C , then C must be similar to A .

20. If a subspace V of \mathbb{R}^n contains none of the standard vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$, then V consists of the zero vector only.
21. If A and B are $n \times n$ matrices, and vector \vec{v} is in the kernel of both A and B , then \vec{w} must be in the kernel of matrix AB as well.
22. If two nonzero vectors are linearly dependent, then each of them is a scalar multiple of the other.
23. If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are any three distinct vectors in \mathbb{R}^3 , then there must be a linear transformation T from \mathbb{R}^3 to \mathbb{R}^3 such that $T(\vec{v}_1) = \vec{e}_1$, $T(\vec{v}_2) = \vec{e}_2$, and $T(\vec{v}_3) = \vec{e}_3$.
24. If vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent, then vector \vec{w} must be a linear combination of \vec{u} and \vec{v} .
25. If A and B are invertible $n \times n$ matrices, then AB must be similar to BA .
26. If A is an invertible $n \times n$ matrix, then the kernels of A and A^{-1} must be equal.
27. Matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
28. Vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$ are linearly independent.
29. If a subspace V of \mathbb{R}^3 contains the standard vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, then V must be \mathbb{R}^3 .
30. If a 2×2 matrix P represents the orthogonal projection onto a line in \mathbb{R}^2 , then P must be similar to matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
31. \mathbb{R}^2 is a subspace of \mathbb{R}^3 .
32. If an $n \times n$ matrix A is similar to matrix B , then $A + 7I_n$ must be similar to $B + 7I_n$.
33. If V is any three-dimensional subspace of \mathbb{R}^5 , then V has infinitely many bases.
34. Matrix I_n is similar to $2I_n$.
35. If $AB = 0$ for two 2×2 matrices A and B , then BA must be the zero matrix as well.
36. If A and B are $n \times n$ matrices, and vector \vec{v} is in the image of both A and B , then \vec{v} must be in the image of matrix $A + B$ as well.
37. If V and W are subspaces of \mathbb{R}^n , then their union $V \cup W$ must be a subspace of \mathbb{R}^n as well.
38. If the kernel of a 5×4 matrix A consists of the zero vector only and if $A\vec{v} = A\vec{w}$ for two vectors \vec{v} and \vec{w} in \mathbb{R}^4 , then vectors \vec{v} and \vec{w} must be equal.
39. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ and $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$ are two bases of \mathbb{R}^n , then there exists a linear transformation T from \mathbb{R}^n to \mathbb{R}^n such that $T(\vec{v}_1) = \vec{w}_1$, $T(\vec{v}_2) = \vec{w}_2$, \dots , $T(\vec{v}_n) = \vec{w}_n$.
40. If matrix A represents a rotation through $\pi/2$ and matrix B a rotation through $\pi/4$, then A is similar to B .
41. There exists a 2×2 matrix A such that $\text{im}(A) = \text{ker}(A)$.
42. If two $n \times n$ matrices A and B have the same rank, then they must be similar.
43. If A is similar to B , and A is invertible, then B must be invertible as well.
44. If $A^2 = 0$ for a 10×10 matrix A , then the inequality $\text{rank}(A) \leq 5$ must hold.
45. For every subspace V of \mathbb{R}^3 there exists a 3×3 matrix A such that $V = \text{im}(A)$.
46. There exists a nonzero 2×2 matrix A that is similar to $2A$.
47. If the 2×2 matrix R represents the reflection about a line in \mathbb{R}^2 , then R must be similar to matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
48. If A is similar to B , then there exists one and only one invertible matrix S such that $S^{-1}AS = B$.
49. If the kernel of a 5×4 matrix A consists of the zero vector alone, and if $AB = AC$ for two 4×5 matrices B and C , then matrices B and C must be equal.
50. If A is any $n \times n$ matrix such that $A^2 = A$, then the image of A and the kernel of A have only the zero vector in common.
51. There exists a 2×2 matrix A such that $A^2 \neq 0$ and $A^3 = 0$.
52. If A and B are $n \times m$ matrices such that the image of A is a subset of the image of B , then there must exist an $m \times m$ matrix C such that $A = BC$.
53. Among the 3×3 matrices whose entries are all 0's and 1's, most are invertible.