Math S-21b – Summer 2023 – Homework #3

Problems due Wednesday, July 5:

Problem 1. (3.1/11) Find vectors that span the kernel of $\mathbf{A}=\begin{bmatrix}1 & 0 & 2 & 4\\ 0 & 1 & -3 & -1\\ 3 & 4 & -6 & 8\\ 0 & -1 & 3 & 4\end{bmatrix}$. Use paper and pencil.

Problem 2. (3.1/32) Give an example of a linear transformation whose <u>image</u> is the line spanned by $\begin{bmatrix}7\\6\\5\end{bmatrix}$ in $\mathbf{R}^3$.

Problem 3. (3.1/34)

Give an example of a linear transformation whose <u>kernel</u> is the line spanned by $\begin{bmatrix}-1\\1\\2\end{bmatrix}$ in $\mathbf{R}^3$.

Problem 4. (3.1/39) Consider an $n\times p$ matrix **A** and a $p\times m$ matrix **B**.

a. What is the relationship between ker(**AB**) and ker(**B**)? Are they always equal? Is one of them always contained in the other?

b. What is the relationship between im(**A**) and im(**AB**)?

Problem 5. (3.1/44) Consider a matrix **A**, and let **B** = rref(**A**).

a. Is ker(**A**) necessarily equal to ker(**B**)? Explain.

b. Is im(**A**) necessarily equal to im(**B**)? Explain.

Problem 6. (3.2/36) Consider a linear transformation T from $\mathbf{R}^n$ to $\mathbf{R}^p$ and some linearly dependent vectors $\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_m\}$ in $\mathbf{R}^n$. Are the vectors $\{T(\mathbf{v}_1),T(\mathbf{v}_2),\cdots,T(\mathbf{v}_m)\}$ necessarily linearly dependent? How can you tell?

Problem 7(a). (3.1/51) Consider an $n\times p$ matrix **A** and a $p\times m$ matrix **B** such that $\ker(\mathbf{A})=\{\mathbf{0}\}$ and $\ker(\mathbf{B})=\{\mathbf{0}\}$. Find ker(**AB**).

(b) (3.2/40) Consider an $n\times p$ matrix **A** and a $p\times m$ matrix **B**. We are told that the columns of **A** and the columns of **B** are linearly independent (respectively). Are the columns of the product **AB** linearly independent as well? [*Hint*: Exercise 3.1.51 (7a above) is useful.]

Problem 8. (3.2/48) Express the plane V in $\mathbf{R}^3$ with equation $3x_1+4x_2+5x_3=0$ as the kernel of a matrix **A** and as the image of a matrix **B**. [*Note*: This exercise doesn't specify the sizes of the matrices **A** and **B**. There are many possible solutions, including the case where both **A** and **B** are 3×3 matrices. Think geometrically!]

Problem 9. (3.3/24) Find the reduced row-echelon form of the matrix $\mathbf{A}=\begin{bmatrix}4 & 8 & 1 & 1 & 6\\ 3 & 6 & 1 & 2 & 5\\ 2 & 4 & 1 & 9 & 10\\ 1 & 2 & 3 & 2 & 0\end{bmatrix}$.

Then find a basis of the image of **A** and a basis of the kernel of **A**.

Problem 10. (3.3/30) Find a basis of the subspace of $\mathbf{R}^4$ defined by the equation $2x_1-x_2+2x_3+4x_4=0$.

Problem 11. (3.3/32)

Find a basis of the subspace of $\mathbf{R}^4$ that consists of all vectors perpendicular to both $\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\2\\3\end{bmatrix}$.

In Problems 12 and 13, determine whether the vector $\mathbf{x}$ is in the span V of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ (proceed "by inspection" if possible, and use the reduced row-echelon form if necessary). If $\mathbf{x}$ is in V, find the coordinates of $\mathbf{x}$ with respect to the basis $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ of V, and write the coordinate vector $[\mathbf{x}]_{\mathscr{B}}$.

Problem 12. (3.4/6) $\mathbf{x} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

Problem 13. (3.4/18) $\mathbf{x} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Problem 14. (3.4/26) Find the matrix of the linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with respect to the basis $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 15. (3.4/28) Find the matrix of the linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with respect to the basis $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{A} = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

Problem 16. (3.4/44) Consider the plane with equation $2x_1 - 3x_2 + 4x_3 = 0$ with the basis $\mathscr{B}$ consisting of vectors $\begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$. If $[\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $\mathbf{x}$.

Problem 17. (3.4/46) Find a basis $\mathscr{B}$ of the plane $x_1 + 2x_2 + x_3 = 0$. such that $[\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ for $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

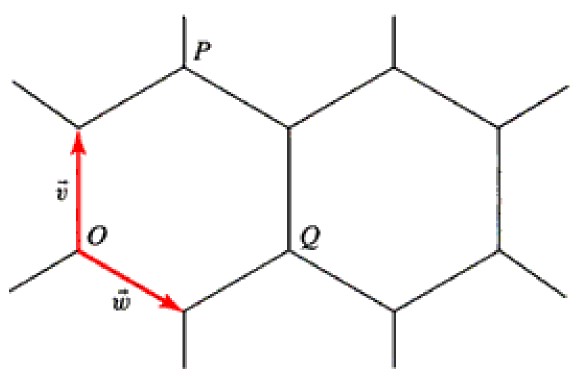
Problem 18. (3.4/50) Given a hexagonal tiling of the plane, such as you might find on a kitchen floor, consider the basis $\mathscr{B}$ of $\mathbf{R}^2$ consisting of the vectors $\vec{v}$, $\vec{w}$ in the following sketch:

a. Find the coordinate vectors $\left[\overrightarrow{OP}\right]_{\mathscr{B}}$ and $\left[\overrightarrow{OQ}\right]_{\mathscr{B}}$.

Hint: Sketch the coordinate grid defined by the basis $\mathscr{B} = \{\vec{v}, \vec{w}\}$.

b. We are told that $\left[\overrightarrow{OR}\right]_{\mathscr{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Sketch the point R. Is R the vertex or the center of a tile?

c. We are told that $\left[\overrightarrow{OS}\right]_{\mathscr{B}} = \begin{bmatrix} 17 \\ 13 \end{bmatrix}$. Is S the center or the vertex of a tile?



Problem 19. (3.4/56) Find a basis $\mathscr{B}$ of $\mathbf{R}^2$ such that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathscr{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}_{\mathscr{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

[Note: Read this problem <u>very</u> carefully. Many students get this one backwards!]

For additional practice:
Section 3.1:
For each matrix **A** in Exercises 5, 6, and 8, find vectors that span the kernel of **A**. Use paper and pencil.

5. $\mathbf{A}=\begin{bmatrix}1&1&1\\1&2&3\\1&3&5\end{bmatrix}$ 6. $\mathbf{A}=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$ 8. $\mathbf{A}=\begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}$

19. For the matrix $\mathbf{A}=\begin{bmatrix}1&2&3&4\\-2&-4&-6&-8\end{bmatrix}$, describe the image of the transformation $T(\mathbf{x})=\mathbf{A}\mathbf{x}$ geometrically (as a line, plane, etc. in $\mathbf{R}^2$ or $\mathbf{R}^3$).

20. For the matrix $\mathbf{A}=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$, describe the image of the transformation $T(\mathbf{x})=\mathbf{A}\mathbf{x}$ geometrically (as a line, plane, etc. in $\mathbf{R}^2$ or $\mathbf{R}^3$).

Describe the images and kernels of the transformations in Exercises 23 through 25 geometrically.

23. Reflection about the line $y=\frac{1}{3}x$ in $\mathbf{R}^2$.

24. Orthogonal projection onto the plane $x+2y+3z=0$ in $\mathbf{R}^3$.

25. Rotation through an angle $\pi/4$ in the counterclockwise direction (in $\mathbf{R}^2$).

31. Give an example of a matrix **A** such that im(**A**) is the plane with normal vector $\begin{bmatrix}1\\3\\2\end{bmatrix}$ in $\mathbf{R}^3$**.**

Section 3.2:
Which of the sets W in Exercises 1 through 3 are subspaces of $\mathbf{R}^3$?

1. $W=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}: x+y+z=1\right\}$ 2. $W=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}: x\le y\le z\right\}$ 3. $W=\left\{\begin{bmatrix}x+2y+3z\\4x+5y+6z\\7x+8y+9z\end{bmatrix}: x, y, z \text{ arbitrary constants}\right\}$

6. Consider two subspaces V and W of $\mathbf{R}^n$.
   a. Is the intersection $V\cap W$ necessarily a subspace of $\mathbf{R}^n$?
   b. Is the union $V\cup W$ necessarily a subspace of $\mathbf{R}^n$?

In Exercises 17 and 19, use paper and pencil to identify the redundant vectors. Thus determine whether the given vectors are linearly independent.

17. $\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}1\\3\\6\end{bmatrix}$ 19. $\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}2\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}3\\4\\5\\0\end{bmatrix}$

24. Find a redundant column vector of the matrix $\mathbf{A}=\begin{bmatrix}1&3&6\\1&2&5\\1&1&4\end{bmatrix}$, and write it as a linear combination of preceding columns. Use this representation to write a nontrivial relation among the columns, and thus find a nonzero vector in the kernel of **A**.

28. Find a basis for the image of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$.

37. Consider a linear transformation T from $\mathbf{R}^n$ to $\mathbf{R}^p$ and some linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$ in $\mathbf{R}^n$. Are the vectors $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \cdots, T(\mathbf{v}_m)\}$ necessarily linearly independent? How can you tell?

41. Consider an $m \times n$ matrix **A** and an $n \times m$ matrix **B** (with $n \neq m$) such that $\mathbf{AB} = \mathbf{I}_m$. (We say that **A** is a *left inverse* of **B**.) Are the columns of **B** linearly independent? What about the columns of **A**?

49. Express the line L in $\mathbf{R}^3$ spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the image of a matrix **A** and as the kernel of a matrix **B**. [*Note*: This exercise doesn't specify the sizes of the matrices **A** and **B**. There are many possible solutions, including the case where both **A** and **B** are 3×3 matrices. Think geometrically!]

Section 3.3:

23. Find the reduced row-echelon form of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$.

Then find a basis of the image of **A** and a basis of the kernel of **A**.

27. Determine whether the vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix} \right\}$ form a basis of $\mathbf{R}^4$.

29. Find a basis of the subspace of $\mathbf{R}^3$ defined by the equation $2x_1 + 3x_2 + x_3 = 0$.

36. Can you find a 3×3 matrix **A** such that im(**A**) = ker(**A**)? Explain.

60. Consider two subspaces V and W of $\mathbf{R}^n$, where V is contained in W. Explain why $\dim(V) \leq \dim(W)$. (This statement seems intuitively rather obvious. Still, we cannot rely on our intuition when dealing with $\mathbf{R}^n$.)

61. Consider two subspaces V and W of $\mathbf{R}^n$, where V is contained in W. In Exercise 40, we learned that $\dim(V) \leq \dim(W)$. Show that if $\dim(V) = \dim(W)$, then $V = W$.

62. Consider a subspace V of $\mathbf{R}^n$ with $\dim(V) = n$. Explain why $V = \mathbf{R}^n$.

81. Consider a 4×2 matrix **A** and a 2×5 matrix **B**.

    a. What are the possible dimensions of the *kernel* of **AB**?

    b. What are the possible dimensions of the *image* of **AB**?

Section 3.4:

In Exercises 5, 7, and 17, determine whether the vector **x** is in the span V of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ (proceed "by inspection" if possible, and use the reduced row-echelon form if necessary). If **x** is in V, find the coordinates of **x** with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ of V, and write the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.

5. $\mathbf{x} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

7. $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

17. $\mathbf{x}=\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$; $\mathbf{v}_1=\begin{bmatrix}1\\0\\2\\0\end{bmatrix}$, $\mathbf{v}_2=\begin{bmatrix}0\\1\\3\\0\end{bmatrix}$, $\mathbf{v}_3=\begin{bmatrix}0\\0\\4\\1\end{bmatrix}$

27. Find the matrix **B** of the linear transformation T(**x**) = **Ax** with respect to the basis $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_m\}$ where $\mathbf{A}=\begin{bmatrix}4&2&-4\\2&1&-2\\-4&-2&4\end{bmatrix}$; $\mathbf{v}_1=\begin{bmatrix}2\\1\\-2\end{bmatrix}$; $\mathbf{v}_2=\begin{bmatrix}0\\2\\1\end{bmatrix}$; $\mathbf{v}_3=\begin{bmatrix}1\\0\\1\end{bmatrix}$.

42. Find a basis $\mathcal{B}$ of $\mathbf{R}^3$ such that the $\mathcal{B}$-matrix **B** of the linear transformation given by reflection T about the plane $x_1-2x_2+2x_3=0$ in $\mathbf{R}^3$ is diagonal.

45. Consider the plane $2x_1-3x_2+4x_3=0$ Find a basis $\mathcal{B}$ of this plane such that $\mathbf{x}_{\mathcal{B}}=\begin{bmatrix}2\\3\end{bmatrix}$ for $\mathbf{x}=\begin{bmatrix}2\\0\\-1\end{bmatrix}$.

55. Consider the basis $\mathcal{B}$ of $\mathbf{R}^2$ consisting of the vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$, and let $\mathcal{R}$ be the basis consisting of $\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\4\end{bmatrix}$. Find a matrix **P** such that $[\mathbf{x}]_{\mathcal{R}}=\mathbf{P}[\mathbf{x}]_{\mathcal{B}}$.

Chapter 3 True/False

1. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent vectors in $\mathbb{R}^n$, then they must form a basis of $\mathbb{R}^n$.
2. There exists a 5×4 matrix whose image consists of all of $\mathbb{R}^5$.
3. The kernel of any invertible matrix consists of the zero vector only.
4. The identity matrix I_n is similar to all invertible $n\times n$ matrices.
5. If $2\vec{u}+3\vec{v}+4\vec{w}=5\vec{u}+6\vec{v}+7\vec{w}$, then vectors $\vec{u}, \vec{v}, \vec{w}$ must be linearly dependent.
6. The column vectors of a 5×4 matrix must be linearly dependent.
7. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ are any two bases of a subspace V of $\mathbb{R}^{10}$, then n must equal m.
8. If A is a 5×6 matrix of rank 4, then the nullity of A is 1.
9. The image of a 3×4 matrix is a subspace of $\mathbb{R}^4$.
10. The span of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ consists of all linear combinations of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.
11. If vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent, then vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ must be linearly independent as well.
12. The vectors of the form $\begin{bmatrix}a\\b\\0\\a\end{bmatrix}$ (where a and b are arbitrary real numbers) form a subspace of $\mathbb{R}^4$.
13. Matrix $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ is similar to $\begin{bmatrix}0&1\\1&0\end{bmatrix}$.
14. Vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}$ form a basis of $\mathbb{R}^3$.
15. If the kernel of a matrix A consists of the zero vector only, then the column vectors of A must be linearly independent.
16. If the image of an $n\times n$ matrix A is all of $\mathbb{R}^n$, then A must be invertible.
17. If vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span $\mathbb{R}^4$, then n must be equal to 4.
18. If vectors $\vec{u}, \vec{v}$, and $\vec{w}$ are in a subspace V of $\mathbb{R}^n$, then vector $2\vec{u}-3\vec{v}+4\vec{w}$ must be in V as well.
19. If matrix A is similar to matrix B, and B is similar to C, then C must be similar to A.

20. If a subspace V of $\mathbb{R}^n$ contains none of the standard vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, then V consists of the zero vector only.

21. If A and B are $n \times n$ matrices, and vector $\vec{v}$ is in the kernel of both A and B, then $\vec{v}$ must be in the kernel of matrix AB as well.

22. If two nonzero vectors are linearly dependent, then each of them is a scalar multiple of the other.

23. If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are any three distinct vectors in $\mathbb{R}^3$, then there must be a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that $T(\vec{v}_1) = \vec{e}_1$, $T(\vec{v}_2) = \vec{e}_2$, and $T(\vec{v}_3) = \vec{e}_3$.

24. If vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent, then vector $\vec{w}$ must be a linear combination of $\vec{u}$ and $\vec{v}$.

25. If A and B are invertible $n \times n$ matrices, then AB must be similar to BA.

26. If A is an invertible $n \times n$ matrix, then the kernels of A and A^{-1} must be equal.

27. Matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

28. Vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$ are linearly independent.

29. If a subspace V of $\mathbb{R}^3$ contains the standard vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, then V must be $\mathbb{R}^3$.

30. If a 2×2 matrix P represents the orthogonal projection onto a line in $\mathbb{R}^2$, then P must be similar to matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

31. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

32. If an $n \times n$ matrix A is similar to matrix B, then $A + 7I_n$ must be similar to $B + 7I_n$.

33. If V is any three-dimensional subspace of $\mathbb{R}^5$, then V has infinitely many bases.

34. Matrix I_n is similar to $2I_n$.

35. If $AB = 0$ for two 2×2 matrices A and B, then BA must be the zero matrix as well.

36. If A and B are $n \times n$ matrices, and vector $\vec{v}$ is in the image of both A and B, then $\vec{v}$ must be in the image of matrix $A + B$ as well.

37. If V and W are subspaces of $\mathbb{R}^n$, then their union $V \cup W$ must be a subspace of $\mathbb{R}^n$ as well.

38. If the kernel of a 5×4 matrix A consists of the zero vector only and if $A\vec{v} = A\vec{w}$ for two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^4$, then vectors $\vec{v}$ and $\vec{w}$ must be equal.

39. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are two bases of $\mathbb{R}^n$, then there exists a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(\vec{v}_1) = \vec{w}_1$, $T(\vec{v}_2) = \vec{w}_2, \ldots, T(\vec{v}_n) = \vec{w}_n$.

40. If matrix A represents a rotation through $\pi/2$ and matrix B a rotation through $\pi/4$, then A is similar to B.

41. There exists a 2×2 matrix A such that $\text{im}(A) = \ker(A)$.

42. If two $n \times n$ matrices A and B have the same rank, then they must be similar.

43. If A is similar to B, and A is invertible, then B must be invertible as well.

44. If $A^2 = 0$ for a 10×10 matrix A, then the inequality $\text{rank}(A) \leq 5$ must hold.

45. For every subspace V of $\mathbb{R}^3$ there exists a 3×3 matrix A such that $V = \text{im}(A)$.

46. There exists a nonzero 2×2 matrix A that is similar to $2A$.

47. If the 2×2 matrix R represents the reflection about a line in $\mathbb{R}^2$, then R must be similar to matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

48. If A is similar to B, then there exists one *and only one* invertible matrix S such that $S^{-1}AS = B$.

49. If the kernel of a 5×4 matrix A consists of the zero vector alone, and if $AB = AC$ for two 4×5 matrices B and C, then matrices B and C must be equal.

50. If A is any $n \times n$ matrix such that $A^2 = A$, then the image of A and the kernel of A have only the zero vector in common.

51. There exists a 2×2 matrix A such that $A^2 \neq 0$ and $A^3 = 0$.

52. If A and B are $n \times m$ matrices such that the image of A is a subset of the image of B, then there must exist an $m \times m$ matrix C such that $A = BC$.

53. Among the 3×3 matrices whose entries are all 0's and 1's, most are invertible.