Math S-21b – Summer 2024 – Homework #2

Problems due by Wednesday, July 3:

Problem 1. a. Consider the vector $\mathbf{v}=\begin{bmatrix}2\\3\\4\end{bmatrix}$.

Is the transformation $T(\mathbf{x})=\mathbf{v}\cdot\mathbf{x}$ (the dot product) from $\mathbf{R}^3$ to $\mathbf{R}$ linear? If so, find the matrix of T.

b. Consider an arbitrary vector $\mathbf{v}$ in $\mathbf{R}^3$. Is the transformation $T(\mathbf{x})=\mathbf{v}\cdot\mathbf{x}$ linear? If so, find the matrix of T (in terms of the components of $\mathbf{v}$).

c. Conversely, consider a linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}$. Show that there exists a vector $\mathbf{v}$ in $\mathbf{R}^3$ such that $T(\mathbf{x})=\mathbf{v}\cdot\mathbf{x}$, for all $\mathbf{x}$ in $\mathbf{R}^3$.

Problem 2. The cross product of two vectors in $\mathbf{R}^3$ is defined by $\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\times\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{bmatrix}$.

Consider an arbitrary vector $\mathbf{v}$ in $\mathbf{R}^3$. Is the transformation $T(\mathbf{x})=\mathbf{v}\times\mathbf{x}$ from $\mathbf{R}^3$ to $\mathbf{R}^3$ linear? If so, find its matrix in terms of the components of the vector $\mathbf{v}$.

Problem 3. Let L be the line in $\mathbf{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix}2\\1\\2\end{bmatrix}$. Find the <u>orthogonal projection</u> of the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ onto L.

Problem 4. Let L be the line in $\mathbf{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix}2\\1\\2\end{bmatrix}$. Find the <u>reflection</u> of the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ about the line L.

Find matrices of the linear transformations from $\mathbf{R}^3$ to $\mathbf{R}^3$ given in Problems 5-9. Some of these transformations have not been formally defined in the text. Use common sense. You may assume that all these transformations are linear.

Problem 5. The orthogonal projection onto the xy-plane.

Problem 6. The reflection about the xz-plane.

Problem 7. The rotation about the z-axis through an angle of $\pi/2$, counterclockwise as viewed from the positive z-axis.

Problem 8. The rotation about the y-axis through an angle θ, counterclockwise as viewed from the positive y-axis.

Problem 9. The reflection about the plane $y=z$.

Problem 10. One of the five given matrices represents an orthogonal projection onto a line and another represents a reflection about a line. Identify both and briefly justify your choice.
[<u>*This is not such an easy problem*</u>!]

$$\mathbf{A}=\frac{1}{3}\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}\quad \mathbf{B}=\frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\quad \mathbf{C}=\frac{1}{3}\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\quad \mathbf{D}=-\frac{1}{3}\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}\quad \mathbf{E}=\frac{1}{3}\begin{bmatrix}-1&2&2\\2&-1&2\\2&2&-1\end{bmatrix}$$

In Problems 11 and 12, find all matrices $\mathbf{X}$ that satisfy the given matrix equation.

Problem 11. $\mathbf{X}\begin{bmatrix}2&1\\4&2\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

Problem 12. $\mathbf{X}\begin{bmatrix}1&2\\3&5\end{bmatrix}=\mathbf{I}_2$

Decide whether the matrices in Problems 13-15 are invertible. If they are, find the inverse matrix. Do the computations with paper and pencil. Show all your work.

Problem 13. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **Problem 14.** $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ **Problem 15.** $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 \\ 2 & 2 & 5 & 4 \\ 0 & 3 & 0 & 1 \end{bmatrix}$

Problem 16. Consider two $n \times n$ matrices **A** and **B**, such that the product **AB** is invertible. Show that the matrices **A** and **B** are both invertible. *Hint*: $\mathbf{AB}(\mathbf{AB})^{-1} = \mathbf{I}_n$ and $(\mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I}_n$. Use Fact 2.4.8.

__Note__: "Fact 2.4.8" states that if **A** and **B** are two $n \times n$ such that $\mathbf{BA} = \mathbf{I}_n$, then (a) **A** and **B** are both invertible, (b) $\mathbf{A}^{-1} = \mathbf{B}$ and $\mathbf{B}^{-1} = \mathbf{A}$, and (c) $\mathbf{AB} = \mathbf{I}_n$.

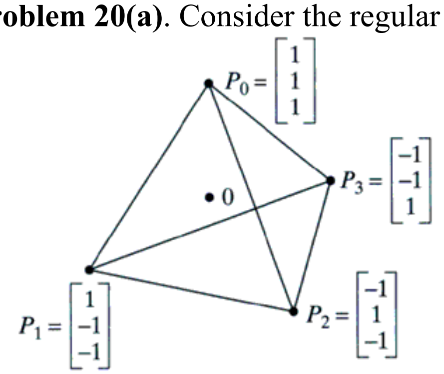
Problem 17. [2.4/67-75] For two invertible $n \times n$ matrices **A** and **B**, determine which of the formulas stated in Exercises 67 through 75 are necessarily true.

67. $(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$
68. $\mathbf{A}^2$ is invertible, and $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$
69. $\mathbf{A} + \mathbf{B}$ is invertible, and $(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{A}^{-1} + \mathbf{B}^{-1}$
70. $(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$
71. $\mathbf{ABB}^{-1}\mathbf{A}^{-1} = \mathbf{I}_n$
72. $\mathbf{ABA}^{-1} = \mathbf{B}$
73. $(\mathbf{ABA}^{-1})^3 = \mathbf{AB}^3\mathbf{A}^{-1}$
74. $(\mathbf{I}_n + \mathbf{A})(\mathbf{I}_n + \mathbf{A}^{-1}) = 2\mathbf{I}_n + \mathbf{A} + \mathbf{A}^{-1}$
75. $\mathbf{A}^{-1}\mathbf{B}$ is invertible, and $(\mathbf{A}^{-1}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}$

Problem 18. Find all linear transformations *T* from $\mathbf{R}^2$ to $\mathbf{R}^2$ such that $T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. [*Hint*: We are looking for the 2 × 2 matrices **A** such that $\mathbf{A}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{A}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. These two equations can be combined to form the matrix equation $\mathbf{A}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.]

Problem 19. Find the matrix of the linear transformation *T* from $\mathbf{R}^2$ to $\mathbf{R}^3$ with $T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix}$ and $T\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Problem 20(a). Consider the regular tetrahedron sketched below, whose center is at the origin.



Let *T* from $\mathbf{R}^3$ to $\mathbf{R}^3$ be the rotation about the axis through points 0 and P_2 that transforms P_1 into P_3. Find the images of the four corners of the tetrahedron under this transformation.

$P_0 \xrightarrow{T}$
$P_1 \to P_3$
$P_2 \to$
$P_3 \to$

Let *L* from $\mathbf{R}^3$ to $\mathbf{R}^3$ be the reflection about the plane through the points 0, P_0, and P_3. Find the images of the four corners of the tetrahedron under this transformation.

$P_0 \xrightarrow{L}$
$P_1 \to$
$P_2 \to$
$P_3 \to$

Describe the transformations in parts (a) through (c) geometrically.

a. T^{-1} b. L^{-1} c. $T^2 = T \circ T$ (the composite of *T* with itself)

d. Find the images of the four corners under the transformations $T \circ L$ and $L \circ T$. Are the two transformations the same?

e. Find the images of the four corners under the transformation $L \circ T \circ L$. Describe this transformation geometrically.

$P_0 \xrightarrow{T \circ L}$
$P_1 \to$
$P_2 \to$
$P_3 \to$

$P_0 \xrightarrow{L \circ T}$
$P_1 \to$
$P_2 \to$
$P_3 \to$

Problem 20(b). Find the matrices of the transformations *T* and *L* defined in Problem 20(a).

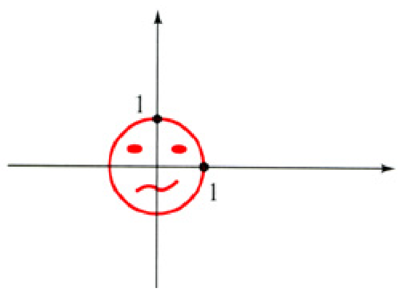
For additional practice (<u>not to be turned in</u>):
Section 2.1:

5. Consider the linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}^2$ with $T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}7\\11\end{bmatrix}$, $T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}6\\9\end{bmatrix}$, and $T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}-13\\17\end{bmatrix}$.

Find the matrix $\mathbf{A}$ of T.

6. Consider the linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^3$ given by $T\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_1\begin{bmatrix}1\\2\\3\end{bmatrix}+x_2\begin{bmatrix}4\\5\\6\end{bmatrix}$. Is this transformation linear? If so, find its matrix.

Consider the circular face in the accompanying figure. For each of the matrices $\mathbf{A}$ in Exercises 24 through 30, draw a sketch showing the effect of the linear transformation $T(\mathbf{x})=\mathbf{A}\mathbf{x}$ on this face.



24. $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ 25. $\begin{bmatrix}2&0\\0&2\end{bmatrix}$ 26. $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ 27. $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

28. $\begin{bmatrix}1&0\\0&2\end{bmatrix}$ 29. $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ 30. $\begin{bmatrix}0&0\\0&1\end{bmatrix}$

Section 2.2:

4. Interpret the following linear transformation geometrically: $T(\mathbf{x})=\begin{bmatrix}1&1\\-1&1\end{bmatrix}\mathbf{x}$.

5. The matrix $\begin{bmatrix}-0.8&-0.6\\0.6&-0.8\end{bmatrix}$ represents a rotation. Find the angle of rotation (in radians).

Section 2.3:

3. $\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}$ 4. $\begin{bmatrix}1&-1\\0&2\\2&1\end{bmatrix}\begin{bmatrix}3&2\\1&0\end{bmatrix}$ 11. $\begin{bmatrix}1&2&3\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix}$ 12. $\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1&2&3\end{bmatrix}$

In Exercises 55 and 57, find all matrices $\mathbf{X}$ that satisfy the given matrix equation.

55. $\begin{bmatrix}1&2\\2&4\end{bmatrix}\mathbf{X}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ 57. $\begin{bmatrix}1&2\\3&5\end{bmatrix}\mathbf{X}=\mathbf{I}_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

Section 2.4:

41. Which of the following linear transformations T from $\mathbf{R}^3$ to $\mathbf{R}^3$ are invertible? Find the inverse if it exists.
    a. Reflection about a plane.
    b. Orthogonal projection onto a plane.
    c. Scaling by a factor of 5 [i.e., $T(\mathbf{v})=5\mathbf{v}$, for all vectors $\mathbf{v}$].
    d. Rotation about an axis.

Chapter 2 True/False questions

1. If A is any invertible $n \times n$ matrix, then $\text{rref}(A) = I_n$.
2. The formula $(A^2)^{-1} = (A^{-1})^2$ holds for all invertible matrices A.
3. The formula $AB = BA$ holds for all $n \times n$ matrices A and B.
4. If $AB = I_n$ for two $n \times n$ matrices A and B, then A must be the inverse of B.
5. If A is a 3×4 matrix and B is a 4×5 matrix, then AB will be a 5×3 matrix.
6. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is a linear transformation.
7. The matrix $\begin{bmatrix} 5 & 6 \\ -6 & 5 \end{bmatrix}$ represents a rotation combined with a scaling.
8. If A is any invertible $n \times n$ matrix, then A commutes with A^{-1}.
9. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ y-x \end{bmatrix}$ is a linear transformation.
10. Matrix $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ represents a rotation.
11. There exists a real number k such that the matrix $\begin{bmatrix} k-2 & 3 \\ -3 & k-2 \end{bmatrix}$ fails to be invertible.
12. Matrix $\begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$ represents a rotation.
13. The formula $\det(2A) = 2\det(A)$ holds for all 2×2 matrices A.
14. There exists a matrix A such that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
15. Matrix $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is invertible.
16. Matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is invertible.
17. There exists an upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
18. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (y+1)^2-(y-1)^2 \\ (x-3)^2-(x+3)^2 \end{bmatrix}$ is a linear transformation.
19. Matrix $\begin{bmatrix} k & -2 \\ 5 & k-6 \end{bmatrix}$ is invertible for all real numbers k.
20. There exists a real number k such that the matrix $\begin{bmatrix} k-1 & -2 \\ -4 & k-3 \end{bmatrix}$ fails to be invertible.
21. The matrix product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is always a scalar multiple of I_2.
22. There exists a nonzero upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
23. There exists a positive integer n such that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^n = I_2$.
24. There exists an invertible 2×2 matrix A such that $A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
25. There exists an invertible $n \times n$ matrix with two identical rows.
26. If $A^2 = I_n$, then matrix A must be invertible.
27. There exists a matrix A such that $A\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.
28. There exists a matrix A such that $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
29. The matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ represents a reflection about a line.
30. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 3k \\ 0 & 1 \end{bmatrix}$ for all real numbers k.
31. If matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is invertible, then matrix $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ must be invertible as well.
32. If A^2 is invertible, then matrix A itself must be invertible.
33. If $A^{17} = I_2$, then matrix A must be I_2.
34. If $A^2 = I_2$, then matrix A must be either I_2 or $-I_2$.
35. If matrix A is invertible, then matrix $5A$ must be invertible as well.
36. If A and B are two 4×3 matrices such that $A\vec{v} = B\vec{v}$ for all vectors $\vec{v}$ in $\mathbb{R}^3$, then matrices A and B must be equal.
37. If matrices A and B commute, then the formula $A^2B = BA^2$ must hold.
38. If $A^2 = A$ for an invertible $n \times n$ matrix A, then A must be I_n.
39. If matrices A and B are both invertible, then matrix $A + B$ must be invertible as well.
40. The equation $A^2 = A$ holds for all 2×2 matrices A representing a projection.
41. The equation $A^{-1} = A$ holds for all 2×2 matrices A representing a reflection.
42. The formula $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$ holds for all invertible 2×2 matrices A and for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$.
43. There exist a 2×3 matrix A and a 3×2 matrix B such that $AB = I_2$.
44. There exist a 3×2 matrix A and a 2×3 matrix B such that $AB = I_3$.

45. If $A^2 + 3A + 4I_3 = 0$ for a 3×3 matrix A, then A must be invertible.

46. If A is an $n \times n$ matrix such that $A^2 = 0$, then matrix $I_n + A$ must be invertible.

47. If matrix A commutes with B, and B commutes with C, then matrix A must commute with C.

48. If T is any linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$, then $T(\vec{v} \times \vec{w}) = T(\vec{v}) \times T(\vec{w})$ for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$.

49. There exists an invertible 10×10 matrix that has 92 ones among its entries.

50. The formula $\operatorname{rref}(AB) = \operatorname{rref}(A) \operatorname{rref}(B)$ holds for all $n \times p$ matrices A and for all $p \times m$ matrices B.

51. There exists an invertible matrix S such that $S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S$ is a diagonal matrix.

52. If the linear system $A^2\vec{x} = \vec{b}$ is consistent, then the system $A\vec{x} = \vec{b}$ must be consistent as well.

53. There exists an invertible 2×2 matrix A such that $A^{-1} = -A$.

54. There exists an invertible 2×2 matrix A such that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

55. If a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents the orthogonal projection onto a line L, then the equation $a^2 + b^2 + c^2 + d^2 = 1$ must hold.

56. If A is an invertible 2×2 matrix and B is any 2×2 matrix, then the formula $\operatorname{rref}(AB) = \operatorname{rref}(B)$ must hold.