Math S-21b – Summer 2023 – Homework #2

Problems due by Friday, June 30:

Problem 1. (2.1/43) a. Consider the vector $\mathbf{v}=\begin{bmatrix}2\\3\\4\end{bmatrix}$.

Is the transformation $T(\mathbf{x})=\mathbf{v}\cdot\mathbf{x}$ (the dot product) from $\mathbf{R}^3$ to $\mathbf{R}$ linear? If so, find the matrix of T.

b. Consider an arbitrary vector $\mathbf{v}$ in $\mathbf{R}^3$. Is the transformation $T(\mathbf{x})=\mathbf{v}\cdot\mathbf{x}$ linear? If so, find the matrix of T (in terms of the components of $\mathbf{v}$).

c. Conversely, consider a linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}$. Show that there exists a vector $\mathbf{v}$ in $\mathbf{R}^3$ such that $T(\mathbf{x})=\mathbf{v}\cdot\mathbf{x}$, for all $\mathbf{x}$ in $\mathbf{R}^3$.

Problem 2. (2.1/44) The cross product of two vectors in $\mathbf{R}^3$ is defined by $\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\times\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{bmatrix}$.

Consider an arbitrary vector $\mathbf{v}$ in $\mathbf{R}^3$. Is the transformation $T(\mathbf{x})=\mathbf{v}\times\mathbf{x}$ from $\mathbf{R}^3$ to $\mathbf{R}^3$ linear? If so, find its matrix in terms of the components of the vector $\mathbf{v}$.

Problem 3. (2.2/6) Let L be the line in $\mathbf{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix}2\\1\\2\end{bmatrix}$.

Find the <u>orthogonal projection</u> of the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ onto L.

Problem 4. (2.2/7) Let L be the line in $\mathbf{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix}2\\1\\2\end{bmatrix}$.

Find the <u>reflection</u> of the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ about the line L.

Find matrices of the linear transformations from $\mathbf{R}^3$ to $\mathbf{R}^3$ given in Problems 5-9. Some of these transformations have not been formally defined in the text. Use common sense. You may assume that all these transformations are linear.

Problem 5. (2.2/19) The orthogonal projection onto the xy-plane.

Problem 6. (2.2/20) The reflection about the xz-plane.

Problem 7. (2.2/21) The rotation about the z-axis through an angle of $\pi/2$, counterclockwise as viewed from the positive z-axis.

Problem 8. (2.2/22) The rotation about the y-axis through an angle θ, counterclockwise as viewed from the positive y-axis.

Problem 9. (2.2/23) The reflection about the plane $y=z$.

Problem 10. (2.2/34) One of the five given matrices represents an orthogonal projection onto a line and another represents a reflection about a line. Identify both and briefly justify your choice. [*<u>This is not such an easy problem</u>*!]

$$\mathbf{A}=\frac{1}{3}\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}\qquad \mathbf{B}=\frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\qquad \mathbf{C}=\frac{1}{3}\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\qquad \mathbf{D}=-\frac{1}{3}\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}\qquad \mathbf{E}=\frac{1}{3}\begin{bmatrix}-1&2&2\\2&-1&2\\2&2&-1\end{bmatrix}$$

In Problems 11 and 12, find all matrices $\mathbf{X}$ that satisfy the given matrix equation.

Problem 11. (2.3/56) $\mathbf{X}\begin{bmatrix}2&1\\4&2\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

Problem 12. (2.3/58) $\mathbf{X}\begin{bmatrix}1&2\\3&5\end{bmatrix}=\mathbf{I}_2$

Decide whether the matrices in Problems 13-15 are invertible. If they are, find the inverse matrix. Do the computations with paper and pencil. Show all your work.

Problem 13. (2.4/2) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **Problem 14**. (2.4/4) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ **Problem 15**. (2.4/12) $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 \\ 2 & 2 & 5 & 4 \\ 0 & 3 & 0 & 1 \end{bmatrix}$

Problem 16. (2.4/66) Consider two $n \times n$ matrices **A** and **B**, such that the product **AB** is invertible. Show that the matrices **A** and **B** are both invertible. *Hint*: $\mathbf{AB}(\mathbf{AB})^{-1} = \mathbf{I}_n$ and $(\mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I}_n$. Use Fact 2.4.8.

Problem 17. (2.4/67-75) For two invertible $n \times n$ matrices **A** and **B**, determine which of the formulas stated in Exercises 67 through 75 are necessarily true.

67. $(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$
68. $\mathbf{A}^2$ is invertible, and $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$
69. **A** + **B** is invertible, and $(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{A}^{-1} + \mathbf{B}^{-1}$
70. $(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$
71. $\mathbf{ABB}^{-1}\mathbf{A}^{-1} = \mathbf{I}_n$
72. $\mathbf{ABA}^{-1} = \mathbf{B}$
73. $(\mathbf{ABA}^{-1})^3 = \mathbf{AB}^3\mathbf{A}^{-1}$
74. $(\mathbf{I}_n + \mathbf{A})(\mathbf{I}_n + \mathbf{A}^{-1}) = 2\mathbf{I}_n + \mathbf{A} + \mathbf{A}^{-1}$
75. $\mathbf{A}^{-1}\mathbf{B}$ is invertible, and $(\mathbf{A}^{-1}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}$

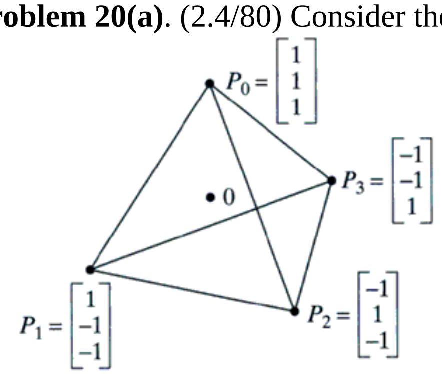
Problem 18. (2.4/76) Find all linear transformations T from $\mathbf{R}^2$ to $\mathbf{R}^2$ such that $T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

[*Hint*: We are looking for the 2 × 2 matrices **A** such that $\mathbf{A}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{A}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. These two equations can be combined to form the matrix equation $\mathbf{A}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.]

Problem 19. (2.4/78) Find the matrix of the linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^3$ with

$T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix}$ and $T\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Problem 20(a). (2.4/80) Consider the regular tetrahedron sketched below, whose center is at the origin.



Let T from $\mathbf{R}^3$ to $\mathbf{R}^3$ be the rotation about the axis through points 0 and P_2 that transforms P_1 into P_3. Find the images of the four corners of the tetrahedron under this transformation.

$P_0 \xrightarrow{T}$
$P_1 \rightarrow P_3$
$P_2 \rightarrow$
$P_3 \rightarrow$

Let L from $\mathbf{R}^3$ to $\mathbf{R}^3$ be the reflection about the plane through the points 0, P_0, and P_3. Find the images of the four corners of the tetrahedron under this transformation.

$P_0 \xrightarrow{L}$
$P_1 \rightarrow$
$P_2 \rightarrow$
$P_3 \rightarrow$

Describe the transformations in parts (a) through (c) geometrically.

a. T^{-1} b. L^{-1} c. $T^2 = T \circ T$ (the composite of T with itself)

d. Find the images of the four corners under the transformations $T \circ L$ and $L \circ T$. Are the two transformations the same?

e. Find the images of the four corners under the transformation $L \circ T \circ L$. Describe this transformation geometrically.

$P_0 \xrightarrow{T \circ L}$
$P_1 \rightarrow$
$P_2 \rightarrow$
$P_3 \rightarrow$

$P_0 \xrightarrow{L \circ T}$
$P_1 \rightarrow$
$P_2 \rightarrow$
$P_3 \rightarrow$

Problem 20(b). (2.4/81) Find the matrices of the transformations T and L defined in Exercise 80.

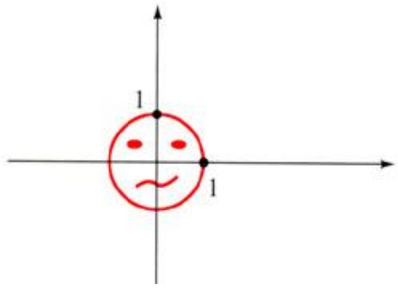
For additional practice (<u>not to be turned in</u>):
Section 2.1:

5. Consider the linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}^2$ with $T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}7\\11\end{bmatrix}$, $T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}6\\9\end{bmatrix}$, and $T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}-13\\17\end{bmatrix}$.

   Find the matrix $\mathbf{A}$ of T.

6. Consider the linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^3$ given by $T\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_1\begin{bmatrix}1\\2\\3\end{bmatrix}+x_2\begin{bmatrix}4\\5\\6\end{bmatrix}$. Is this transformation linear? If so, find its matrix.

Consider the circular face in the accompanying figure. For each of the matrices $\mathbf{A}$ in Exercises 24 through 30, draw a sketch showing the effect of the linear transformation $T(\mathbf{x})=\mathbf{A}\mathbf{x}$ on this face.



24. $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ 25. $\begin{bmatrix}2&0\\0&2\end{bmatrix}$ 26. $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ 27. $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

28. $\begin{bmatrix}1&0\\0&2\end{bmatrix}$ 29. $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ 30. $\begin{bmatrix}0&0\\0&1\end{bmatrix}$

Section 2.2:

4. Interpret the following linear transformation geometrically: $T(\mathbf{x})=\begin{bmatrix}1&1\\-1&1\end{bmatrix}\mathbf{x}$.

5. The matrix $\begin{bmatrix}-0.8&-0.6\\0.6&-0.8\end{bmatrix}$ represents a rotation. Find the angle of rotation (in radians).

Section 2.3:

3. $\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}$ 4. $\begin{bmatrix}1&-1\\0&2\\2&1\end{bmatrix}\begin{bmatrix}3&2\\1&0\end{bmatrix}$ 11. $\begin{bmatrix}1&2&3\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix}$ 12. $\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1&2&3\end{bmatrix}$

In Exercises 55 and 57, find all matrices $\mathbf{X}$ that satisfy the given matrix equation.

55. $\begin{bmatrix}1&2\\2&4\end{bmatrix}\mathbf{X}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ 57. $\begin{bmatrix}1&2\\3&5\end{bmatrix}\mathbf{X}=\mathbf{I}_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

Section 2.4:

41. Which of the following linear transformations T from $\mathbf{R}^3$ to $\mathbf{R}^3$ are invertible? Find the inverse if it exists.
    a. Reflection about a plane.
    b. Orthogonal projection onto a plane.
    c. Scaling by a factor of 5 [i.e., $T(\mathbf{v})=5\mathbf{v}$, for all vectors $\mathbf{v}$].
    d. Rotation about an axis.

Chapter 2 True/False questions

1. If A is any invertible $n \times n$ matrix, then $\text{rref}(A) = I_n$.
2. The formula $(A^2)^{-1} = (A^{-1})^2$ holds for all invertible matrices A.
3. The formula $AB = BA$ holds for all $n \times n$ matrices A and B.
4. If $AB = I_n$ for two $n \times n$ matrices A and B, then A must be the inverse of B.
5. If A is a 3×4 matrix and B is a 4×5 matrix, then AB will be a 5×3 matrix.
6. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is a linear transformation.
7. The matrix $\begin{bmatrix} 5 & 6 \\ -6 & 5 \end{bmatrix}$ represents a rotation combined with a scaling.
8. If A is any invertible $n \times n$ matrix, then A commutes with A^{-1}.
9. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ y-x \end{bmatrix}$ is a linear transformation.
10. Matrix $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ represents a rotation.
11. There exists a real number k such that the matrix $\begin{bmatrix} k-2 & 3 \\ -3 & k-2 \end{bmatrix}$ fails to be invertible.
12. Matrix $\begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$ represents a rotation.
13. The formula $\det(2A) = 2\det(A)$ holds for all 2×2 matrices A.
14. There exists a matrix A such that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
15. Matrix $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is invertible.
16. Matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is invertible.
17. There exists an upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
18. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (y+1)^2 - (y-1)^2 \\ (x-3)^2 - (x+3)^2 \end{bmatrix}$ is a linear transformation.
19. Matrix $\begin{bmatrix} k & -2 \\ 5 & k-6 \end{bmatrix}$ is invertible for all real numbers k.
20. There exists a real number k such that the matrix $\begin{bmatrix} k-1 & -2 \\ -4 & k-3 \end{bmatrix}$ fails to be invertible.
21. The matrix product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is always a scalar multiple of I_2.
22. There exists a nonzero upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
23. There exists a positive integer n such that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^n = I_2$.
24. There exists an invertible 2×2 matrix A such that $A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
25. There exists an invertible $n \times n$ matrix with two identical rows.
26. If $A^2 = I_n$, then matrix A must be invertible.
27. There exists a matrix A such that $A\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.
28. There exists a matrix A such that $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
29. The matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ represents a reflection about a line.
30. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 3k \\ 0 & 1 \end{bmatrix}$ for all real numbers k.
31. If matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is invertible, then matrix $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ must be invertible as well.
32. If A^2 is invertible, then matrix A itself must be invertible.
33. If $A^{17} = I_2$, then matrix A must be I_2.
34. If $A^2 = I_2$, then matrix A must be either I_2 or $-I_2$.
35. If matrix A is invertible, then matrix $5A$ must be invertible as well.
36. If A and B are two 4×3 matrices such that $A\vec{v} = B\vec{v}$ for all vectors $\vec{v}$ in $\mathbb{R}^3$, then matrices A and B must be equal.
37. If matrices A and B commute, then the formula $A^2B = BA^2$ must hold.
38. If $A^2 = A$ for an invertible $n \times n$ matrix A, then A must be I_n.
39. If matrices A and B are both invertible, then matrix $A + B$ must be invertible as well.
40. The equation $A^2 = A$ holds for all 2×2 matrices A representing a projection.
41. The equation $A^{-1} = A$ holds for all 2×2 matrices A representing a reflection.
42. The formula $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$ holds for all invertible 2×2 matrices A and for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$.
43. There exist a 2×3 matrix A and a 3×2 matrix B such that $AB = I_2$.
44. There exist a 3×2 matrix A and a 2×3 matrix B such that $AB = I_3$.

45. If $A^2 + 3A + 4I_3 = 0$ for a 3×3 matrix A, then A must be invertible.

46. If A is an $n \times n$ matrix such that $A^2 = 0$, then matrix $I_n + A$ must be invertible.

47. If matrix A commutes with B, and B commutes with C, then matrix A must commute with C.

48. If T is any linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$, then $T(\vec{v} \times \vec{w}) = T(\vec{v}) \times T(\vec{w})$ for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$.

49. There exists an invertible 10×10 matrix that has 92 ones among its entries.

50. The formula $\text{rref}(AB) = \text{rref}(A)\,\text{rref}(B)$ holds for all $n \times p$ matrices A and for all $p \times m$ matrices B.

51. There exists an invertible matrix S such that $S^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}S$ is a diagonal matrix.

52. If the linear system $A^2\vec{x} = \vec{b}$ is consistent, then the system $A\vec{x} = \vec{b}$ must be consistent as well.

53. There exists an invertible 2×2 matrix A such that $A^{-1} = -A$.

54. There exists an invertible 2×2 matrix A such that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

55. If a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents the orthogonal projection onto a line L, then the equation $a^2 + b^2 + c^2 + d^2 = 1$ must hold.

56. If A is an invertible 2×2 matrix and B is any 2×2 matrix, then the formula $\text{rref}(AB) = \text{rref}(B)$ must hold.