Math S-21b – Summer 2024 – Homework #1

Problems due (in Canvas) no later than Friday, June 28:

Section 1.1:

In Problems 1, 2, and 3, find all solutions of the linear systems using elimination. Then check your solutions.

Problem 1 (1.1/1) $\begin{Bmatrix} x+2y=1 \\ 2x+3y=1 \end{Bmatrix}$ **Problem 2** (1.1/3) $\begin{Bmatrix} 2x+4y=3 \\ 3x+6y=2 \end{Bmatrix}$ **Problem 3** (1.1/7) $\begin{Bmatrix} x+2y+3z=1 \\ x+3y+4z=3 \\ x+4y+5z=4 \end{Bmatrix}$

Problem 4 (1.1/17) Find all solutions of the linear system $\begin{Bmatrix} x+2y=a \\ 3x+5y=b \end{Bmatrix}$, where a and b are arbitrary constants.

Problem 5 (1.1/25) Consider the linear system $\begin{Bmatrix} x + y - z=-2 \\ 3x-5y+13z=18 \\ x-2y+5z=k \end{Bmatrix}$, where k is an arbitrary number.

a. For which value(s) of k does this system have one or infinitely many solutions?
b. For each value of k you found in part a, how many solutions does the system have?
c. Find all solutions for each value of k.

Problem 6 (1.1/29) Find the polynomial of degree 2 [a polynomial of the form $f(t)=a+bt+ct^2$] whose graph goes through the points $(1,-1)$, $(2,3)$, and $(3,13)$. Sketch the graph of this polynomial.

Section 1.2:

Problem 7 (1.2/10) Find all solutions of the equations $\begin{Bmatrix} 4x_1+3x_2+2x_3-x_4=4 \\ 5x_1+4x_2+3x_3-x_4=4 \\ -2x_1-2x_2-x_3+2x_4=-3 \\ 11x_1+6x_2+4x_3+x_4=11 \end{Bmatrix}$ with paper and pencil using Gauss-Jordan elimination. Show all your work.

Problem 8 (1.2/16) Solve the linear system $\begin{Bmatrix} 3x_1+6x_2+9x_3+5x_4+25x_5=53 \\ 7x_1+14x_2+21x_3+9x_4+53x_5=105 \\ -4x_1-8x_2-12x_3+5x_4-10x_5=11 \end{Bmatrix}$. You may use technology.

Problem 9 (1.2/30) Find the polynomial of degree 3 [a polynomial of the form $f(t)=a+bt+ct^2+dt^3$] whose graph goes through the points $(0,1)$, $(1,0)$, $(-1,0)$, and $(2,-15)$. Sketch the graph of this cubic.

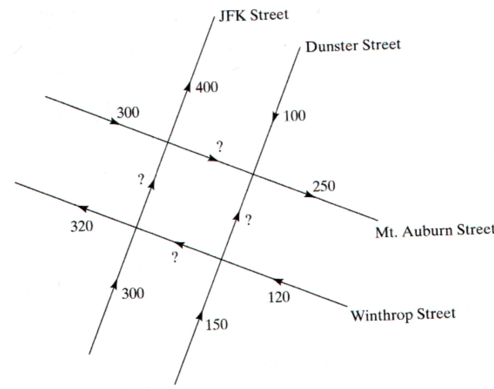
Problem 10 (1.2/34) The dot product of two vectors $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y}=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ in $\mathbf{R}^n$ is defined by $\mathbf{x}\cdot\mathbf{y}=x_1y_1+x_2y_2+\cdots+x_ny_n$. Note that the dot product of two vectors is a scalar. We say that the vectors $\mathbf{x}$ and $\mathbf{y}$ are *perpendicular* (or *orthogonal*) if $\mathbf{x}\cdot\mathbf{y}=0$. Find all vectors in $\mathbf{R}^3$ perpendicular to $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$.

Problem 11 (1.2/36) Find all solutions x_1, x_2, x_3 of the equation $\mathbf{b}=x_1\mathbf{v}_1+x_2\mathbf{v}_2+x_3\mathbf{v}_3$, where

$$\mathbf{b}=\begin{bmatrix} -8 \\ -1 \\ 2 \\ 15 \end{bmatrix},\ \mathbf{v}_1=\begin{bmatrix} 1 \\ 4 \\ 7 \\ 5 \end{bmatrix},\ \mathbf{v}_2=\begin{bmatrix} 2 \\ 5 \\ 8 \\ 3 \end{bmatrix},\ \mathbf{v}_3=\begin{bmatrix} 4 \\ 6 \\ 9 \\ 1 \end{bmatrix}.$$

Problem 12 (1.2/42) The accompanying sketch represents a maze of one-way streets in a city in the United States. The traffic volume through certain blocks during an hour has been measured. Suppose that the vehicles leaving the area during this hour were exactly the same as those entering it.

What can you say about the traffic volume at the four locations indicated by a question mark? Can you figure out exactly how much traffic there was on each block? If not, describe one possible scenario. For each of the four locations, find the highest and lowest possible traffic volume.



Problem 13 (1.2/70) "A rooster is worth five coins, a hen three coins, and 3 chicks one coin. With 100 coins we buy 100 of them. How many roosters, hens, and chicks can we buy?"
(From the *Mathematical Manual* by Zhang Qiujian, Chapter 3, Problem 38; 5th century A.D.)
Commentary: This famous *Hundred Fowl Problem* has reappeared in countless variations in Indian, Arabic, and European texts; it has remained popular to this day.

Section 1.3:

Find the rank of the matrices in Exercises 14 through 16.

Problem 14 (1.3/2) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ **Problem 15** (1.3/3) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **Problem 16** (1.3/4) $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

Problem 17 (1.3/26) Let **A** be a 4×3 matrix, and let **b** and **c** be two vectors in $\mathbf{R}^4$. We are told that the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution. What can you say about the number of solutions of the system $\mathbf{Ax} = \mathbf{c}$?

Problem 18 (1.3/47) A linear system of the form $\mathbf{Ax} = \mathbf{0}$ is called *homogeneous*. [Matrices and vectors are indicated in **bold**.] Justify the following facts:

a. All homogeneous systems are consistent.

b. A homogeneous system with fewer equations than unknowns has infinitely many solutions.

c. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the homogeneous system $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{x}_1 + \mathbf{x}_2$ is a solution as well.

d. If $\mathbf{x}$ is a solution of the homogeneous system $\mathbf{Ax} = \mathbf{0}$ and if k is an arbitrary constant, then $k\mathbf{x}$ is a solution as well.

Problem 19 (1.3/48) Consider a solution $\mathbf{x}_1$ of the linear system $\mathbf{Ax} = \mathbf{b}$.

Justify the facts stated in parts (a) and (b):

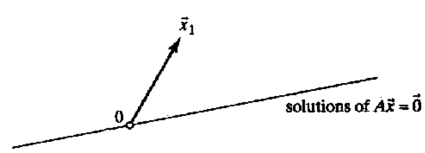
a. If $\mathbf{x}_h$ is a solution of the system $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{x}_1 + \mathbf{x}_h$ is a solution of the system $\mathbf{Ax} = \mathbf{b}$.

b. If $\mathbf{x}_2$ is another solution of the system $\mathbf{Ax} = \mathbf{b}$, then $\mathbf{x}_2 - \mathbf{x}_1$ is a solution of the system $\mathbf{Ax} = \mathbf{0}$.

c. Now suppose **A** is a 2×2 matrix. A solution vector $\mathbf{x}_1$ of the system $\mathbf{Ax} = \mathbf{b}$ is shown in the accompanying figure. We are told that the solutions of the system $\mathbf{Ax} = \mathbf{0}$ form the line shown in the sketch. Draw the line consisting of all solutions of the system $\mathbf{Ax} = \mathbf{b}$.

If you are puzzled by the generality of this problem, think about an example first:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$



Problem 20 (1.3/56) Is the vector $\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix} 1 \\ 7 \\ 1 \\ 9 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 9 \\ 2 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 4 \\ 7 \\ 9 \end{bmatrix}$? Explain.

Additional practice problems (optional, **<u>don't turn in</u>**)
Section 1.1:

11. Find all solutions of the linear system $\begin{Bmatrix} x-2y=2 \\ 3x+5y=17 \end{Bmatrix}$. Represent your solutions graphically, as intersections of lines in the xy-plane.

15. Find all solutions of the linear system $\begin{Bmatrix} x+y-\ z=0 \\ 4x-y+5z=0 \\ 6x+y+4z=0 \end{Bmatrix}$. Describe your solutions in terms of intersecting planes. You need not sketch these planes.

Section 1.2:

In exercises 5, 9, and 11, find all solutions of the equations with paper and pencil using Gauss-Jordan elimination. Show all your work.

5. $\begin{Bmatrix} x_3+x_4=0 \\ x_2+x_3=0 \\ x_1+x_2=0 \\ x_1+x_4=0 \end{Bmatrix}$ 9. $\begin{Bmatrix} x_4+2x_5-x_6=2 \\ x_1+2x_2+x_5-x_6=0 \\ x_1+2x_2+2x_3-x_5+x_6=2 \end{Bmatrix}$ 11. $\begin{Bmatrix} x_1+2x_3+4x_4=-8 \\ x_2-3x_3-x_4=6 \\ 3x_1+4x_2-6x_3+8x_4=0 \\ -x_2+3x_3+4x_4=-12 \end{Bmatrix}$

20. We say that two $n\times m$ matrices in reduced row-echelon form are of the same type if they contain the same number of leading 1's in the same positions. For example, $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are of the same type. How many types of 2×2 matrices in reduced row-echelon form are there?

21. How many types of 3×2 matrices in reduced row-echelon form are there? (See Exercise 20.)

22. How many types of 2×3 matrices in reduced row-echelon form are there? (See Exercise 20.)

Chapter 1 True/False Questions (optional, <u>don't turn in</u>)

1. There exists a 3×4 matrix with rank 4.
2. If A is a 3×4 matrix and vector $\vec{v}$ is in $\mathbb{R}^4$, then vector $A\vec{v}$ is in $\mathbb{R}^3$.
3. If the 4×4 matrix A has rank 4, then any linear system with coefficient matrix A will have a unique solution.
4. There exists a system of three linear equations with three unknowns that has exactly three solutions.
5. There exists a 5×5 matrix A of rank 4 such that the system $A\vec{x}=\vec{0}$ has only the solution $\vec{x}=\vec{0}$.
6. If matrix A is in rref, then at least one of the entries in each column must be 1.
7. If A is an $n\times n$ matrix and $\vec{x}$ is a vector in $\mathbb{R}^n$, then the product $A\vec{x}$ is a linear combination of the columns of matrix A.
8. If vector $\vec{u}$ is a linear combination of vectors $\vec{v}$ and $\vec{w}$, then we can write $\vec{u}=a\vec{v}+b\vec{w}$ for some scalars a and b.
9. Matrix $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is in rref.
10. A system of four linear equations in three unknowns is always inconsistent.
11. If A is a nonzero matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then the rank of A must be 2.
12. $\operatorname{rank}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}=3$
13. The system $A\vec{x}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is inconsistent for all 4×3 matrices A.

14. There exists a 2×2 matrix A such that

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}.$$

15. $\text{rank}\begin{bmatrix}2&2&2\\2&2&2\\2&2&2\end{bmatrix} = 2$

16. $\begin{bmatrix}11&13&15\\17&19&21\end{bmatrix}\begin{bmatrix}-1\\3\\-1\end{bmatrix} = \begin{bmatrix}13\\19\\21\end{bmatrix}$

17. There exists a matrix A such that $A\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}3\\5\\7\end{bmatrix}$.

18. Vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is a linear combination of vectors

$$\begin{bmatrix}4\\5\\6\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}7\\8\\9\end{bmatrix}.$$

19. The system $\begin{bmatrix}1&2&3\\4&5&6\\0&0&0\end{bmatrix}\vec{x} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ is inconsistent.

20. There exists a 2×2 matrix A such that $A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$.

21. If A and B are any two 3×3 matrices of rank 2, then A can be transformed into B by means of elementary row operations.

22. If vector $\vec{u}$ is a linear combination of vectors $\vec{v}$ and $\vec{w}$, and $\vec{v}$ is a linear combination of vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$, then $\vec{u}$ must be a linear combination of $\vec{p}$, $\vec{q}$, $\vec{r}$, and $\vec{w}$.

23. A linear system with fewer unknowns than equations must have infinitely many solutions or none.

24. The rank of any upper triangular matrix is the number of nonzero entries on its diagonal.

25. If the system $A\vec{x} = \vec{b}$ has a unique solution, then A must be a square matrix.

26. If A is any 4×3 matrix, then there exists a vector $\vec{b}$ in $\mathbb{R}^4$ such that the system $A\vec{x} = \vec{b}$ is inconsistent.

27. There exist scalars a and b such that matrix

$$\begin{bmatrix}0&1&a\\-1&0&b\\-a&-b&0\end{bmatrix}$$

has rank 3.

28. If $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^4$, then $\vec{v}$ must be a linear combination of $\vec{v}$ and $\vec{w}$.

29. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are nonzero vectors in $\mathbb{R}^2$, then $\vec{w}$ must be a linear combination of $\vec{u}$ and $\vec{v}$.

30. If $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^4$, then the zero vector in $\mathbb{R}^4$ must be a linear combination of $\vec{v}$ and $\vec{w}$.

31. There exists a 4×3 matrix A of rank 3 such that $A\begin{bmatrix}1\\2\\3\end{bmatrix} = \vec{0}$.

32. The system $A\vec{x} = \vec{b}$ is inconsistent if (and only if) rref(A) contains a row of zeros.

33. If A is a 4×3 matrix of rank 3 and $A\vec{v} = A\vec{w}$ for two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$, then vectors $\vec{v}$ and $\vec{w}$ must be equal.

34. If A is a 4×4 matrix and the system $A\vec{x} = \begin{bmatrix}2\\3\\4\\5\end{bmatrix}$ has a unique solution, then the system $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$.

35. If vector $\vec{u}$ is a linear combination of vectors $\vec{v}$ and $\vec{w}$, then $\vec{w}$ must be a linear combination of $\vec{u}$ and $\vec{v}$.

36. If $A = \begin{bmatrix}\vec{u} & \vec{v} & \vec{w}\end{bmatrix}$ and $\text{rref}(A) = \begin{bmatrix}1&0&2\\0&1&3\\0&0&0\end{bmatrix}$, then the equation $\vec{w} = 2\vec{u} + 3\vec{v}$ must hold.

37. If A and B are matrices of the same size, then the formula $\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)$ must hold.

38. If A and B are any two $n \times n$ matrices of rank n, then A can be transformed into B by means of elementary row operations.

39. If a vector $\vec{v}$ in $\mathbb{R}^4$ is a linear combination of $\vec{u}$ and $\vec{w}$, and if A is a 5×4 matrix, then $A\vec{v}$ must be a linear combination of $A\vec{u}$ and $A\vec{w}$.

40. If matrix E is in reduced row-echelon form, and if we omit a row of E, then the remaining matrix must be in reduced row-echelon form as well.

41. The linear system $A\vec{x} = \vec{b}$ is consistent if (and only if) $\text{rank}(A) = \text{rank}\left[A \,\vdots\, \vec{b}\right]$.

42. If A is a 3×4 matrix of rank 3, then the system $A\vec{x} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ must have infinitely many solutions.

43. If two matrices A and B have the same reduced row-echelon form, then the equations $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ must have the same solutions.

44. If matrix E is in reduced row-echelon form, and if we omit a column of E, then the remaining matrix must be in reduced row-echelon form as well.

45. If A and B are two 2×2 matrices such that the equations $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solutions, then rref(A) must be equal to rref(B).