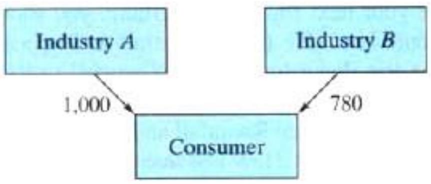
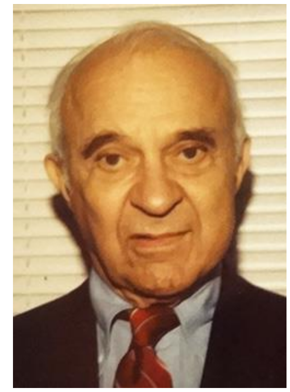
Math S-21b – Summer 2023 – Graduate Credit #1
Additional Problems for Graduate Credit students

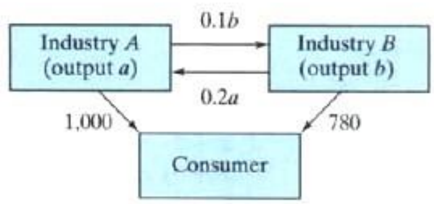
Section 1.1:

20. The Russian-born U.S. economist and Nobel laureate Wassily Leontief (1906-1999) was interested in the following question: What output should each of the industries in an economy produce to satisfy the total demand for all products? Here, we consider a very simple example of input-output analysis, an economy with only two industries, A and B. Assume that the consumer demand for their products is, respectively, 1000 and 780, in millions of dollars per year. What outputs a and b (in millions of dollars per year) should the two industries generate to satisfy the demand?

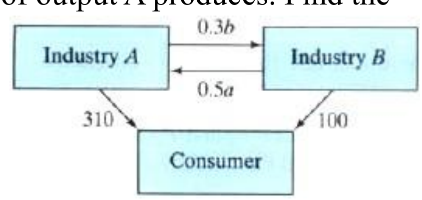




You may be tempted to say 1000 and 780, respectively, but things are not quite as simple as that. We have to take into account the interindustry demand as well. Let us say that industry A produces electricity. Of course, producing almost any product will require electric power. Suppose that industry B needs 10¢ worth of electricity for each \$1 of output B produces and that industry A needs 20¢ worth of B's products for each \$1 of output A produces. Find the outputs a and b needed to satisfy both consumer and interindustry demand.



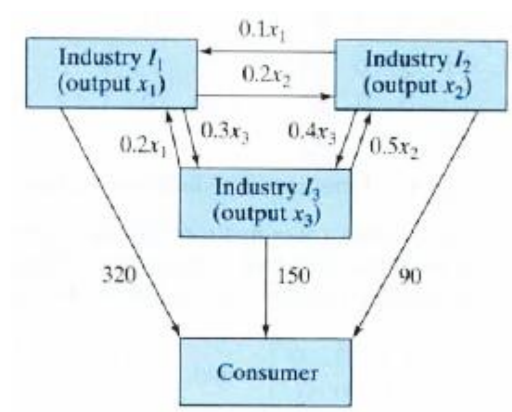
21. Find the outputs a and b needed to satisfy the consumer and interindustry demands given in the following figure (see Exercise 20.):



Section 1.2:

37. For some background on this exercise, see Exercise 1.1.20.

Consider an economy with three industries, I_1, I_2, I_3. What outputs x_1, x_2, and x_3 should they produce to satisfy both consumer demand and interindustry demand? The demands put on the three industries are shown in the accompanying figure.



38. If we consider more than three industries in an input-output model, it is cumbersome to represent all the demands in a diagram as in Exercise 37. Suppose we have industries $I_1, I_2, \cdots, I_n$ with outputs $x_1, x_2, \cdots, x_n$. The *output vector* is

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. The *consumer demand* vector is $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, where b_i is the consumer demand on industry I_i.

The demand vector for industry I_j is $\mathbf{v}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$ where a_{ij} is the demand industry I_j puts on industry I_i, for each \$1 of output industry I_j produces. For example, $a_{32} = 0.5$ means that industry I_2 needs 50¢ worth of products from industry I_3 for each \$1 of goods I_2 produces. The coefficient a_{ii} need not be 0: Producing a product may require goods or services from the same industry.

a. Find the four demand vectors for the economy in Exercise 37.

b. What is the meaning in economic terms of $x_j\mathbf{v}_j$?

c. What is the meaning in economic terms of $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n + \mathbf{b}$?

d. What is the meaning in economic terms of the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n + \mathbf{b} = \mathbf{x}$?

39. Consider the economy of Israel in 1958. [Ref.: W. Leontief: Input-Output Economics, Oxford University Press, 1966.] The three industries considered here are:

I_1: agriculture;
I_2: manufacturing;
I_3: energy.

Outputs and demands are measured in millions of Israeli pounds, the currency of Israel at that time.

We are told that $\mathbf{b} = \begin{bmatrix} 13.2 \\ 17.6 \\ 1.8 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 0.293 \\ 0.014 \\ 0.044 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0.207 \\ 0.01 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0.017 \\ 0.216 \end{bmatrix}$.

a. Why do the first components of $\mathbf{v}_2$ and $\mathbf{v}_3$ equal 0?

b. Find the outputs x_1, x_2, and x_3 required to satisfy demand.

Section 2.4:

49. ***Input-Output Analysis***. (This exercise builds on Exercises 1.1.20, 1.2.37, 1.2.38, and 1.2.39.) Consider the industries $J_1, J_2, ..., J_n$ in an economy. Suppose the consumer demand vector is $\mathbf{b}$, the output vector is $\mathbf{x}$ and the demand of the jth industry is $\mathbf{v}_j$. (The ith component a_{ij} of $\mathbf{v}_j$ is the demand industry J_j puts on industry J_i, per unit of output of J_j.) As we have seen in Exercise 1.2.38, the output $\mathbf{x}$ just meets the aggregate demand if

$$\underbrace{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \cdots x_n\mathbf{v}_n + \mathbf{b}}_{\text{aggregate demand}} = \underbrace{\mathbf{x}}_{\text{output}} .$$

This equation can be written more succinctly as

$$\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \mathbf{b} = \mathbf{x}$$

or $\mathbf{Ax}+\mathbf{b}=\mathbf{x}$. The matrix $\mathbf{A}$ is called the *technology matrix* of this economy; its coefficients a_{ij} describe the interindustry demand, which depend on the technology used in the production process. The equation

$$\mathbf{Ax}+\mathbf{b}=\mathbf{x}$$

describes a linear system, which we can write in the customary form:

$$\begin{aligned} \mathbf{x}-\mathbf{Ax}&=\mathbf{b} \\ \mathbf{I}_n\mathbf{x}-\mathbf{Ax}&=\mathbf{b} \\ (\mathbf{I}_n-\mathbf{A})\mathbf{x}&=\mathbf{b} \end{aligned}$$

If we want to know the output $\mathbf{x}$ required to satisfy a given consumer demand $\mathbf{b}$ (this was our objective in the previous exercises), we can solve this linear system, preferably via the augmented matrix.

In economics, however, we often ask the other questions: If $\mathbf{b}$ changes, how will $\mathbf{x}$ change in response. If the consumer demand on one industry increases by 1 unit and the consumer demand on the other industries remains unchanged, how will $\mathbf{x}$ change? If we ask questions like these, we think of the output $\mathbf{x}$ as a *function* of the consumer demand $\mathbf{b}$.

If the matrix $(\mathbf{I}_n-\mathbf{A})$ is invertible, we can express $\mathbf{x}$ as a function $\mathbf{b}$ (in fact, as a linear transformation):

$$\mathbf{x}=(\mathbf{I}_n-\mathbf{A})^{-1}\mathbf{b}$$

a. Consider the economy of Israel in 1958 (discussed in Exercise 1.2.39). Find the technology matrix $\mathbf{A}$, the matrix $(\mathbf{I}_n-\mathbf{A})$, and its inverse $(\mathbf{I}_n-\mathbf{A})^{-1}$.

b. In the example discussed in part (a), suppose the consumer demand on agriculture (Industry 1) is 1 unit (1 million pounds), and the demands on the other two industries are zero. What output $\mathbf{x}$ is required in this case? How does your answer relate to the matrix $(\mathbf{I}_n-\mathbf{A})^{-1}$?

c. Explain, in terms of economics, why the diagonal elements of the matrix $(\mathbf{I}_n-\mathbf{A})^{-1}$ you found in part (a) must be at least 1.

d. If the consumer demand on manufacturing increases by 1 (from whatever it was), and the consumer demand on the other two industries remains the same, how will the output have to change? How does your answer relate to the matrix $(\mathbf{I}_n-\mathbf{A})^{-1}$?

e. Using your answers in parts (a) through (d) as a guide, explain in general (not just for this example) what the columns and the entries of the matrix $(\mathbf{I}_n-\mathbf{A})^{-1}$ tell you, in terms of economics. Those who have studied multivariable calculus may wish to consider the partial derivatives $\dfrac{\partial x_i}{\partial b_j}$.

50. This exercise refers to Exercise 49a. Consider the entry $k=a_{11}=0.293$ of the technology matrix $\mathbf{A}$. Verify that the entry in the first row and the first column of $(\mathbf{I}_n-\mathbf{A})^{-1}$ is the value of the geometric series $1+k+k^2+\cdots$. Interpret this observation in terms of economics.