Math S-21b – Lecture #13 Notes

We continue the discussion of eigenvalues, eigenvectors, and diagonalizability of matrices. We want to know what conditions will assure that a matrix can be diagonalized and what the obstructions are to this being possible. We'll look at some algebraic invariants of a linear transformation, namely its trace and determinant, and relate these to the eigenvalues of a matrix representing the transformation. We'll discuss what the presence of a complex conjugate pair of eigenvalues means in terms of invariant subspaces and how the transformation acts within such a subspace.

Summary of results

If $\mathbf{A}$ is an $n \times n$ matrix, we call a vector $\mathbf{v}$ an **eigenvector** of $\mathbf{A}$ if $T(\mathbf{v}) = \mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ. This scalar λ is called the eigenvalue associated with the eigenvector. The existence of an eigenvector depends upon whether there are any solutions to the equation $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$, and this is only possible if the matrix $\lambda\mathbf{I} - \mathbf{A}$ is not invertible. A necessary and sufficient condition for this is that $p_A(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = 0$. This will always be an *n*th degree polynomial in λ called the **characteristic polynomial** of $\mathbf{A}$. So $\lambda\mathbf{I} - \mathbf{A}$ will have a nontrivial kernel if and only if λ is a root of this characteristic polynomial. The eigenvalues are therefore the roots of the characteristic polynomial. The set of all eigenvalues of a matrix $\mathbf{A}$ is called the **spectrum of A**. By the Fundamental Theorem of Algebra, this can always be factored as a product of linear factors and irreducible quadratic factors. The linear factors yield real roots, and the irreducible quadratic factors yield (by the quadratic formula) complex conjugate pairs of roots.

If an eigenvalue λ occurs as a repeated root of the characteristic polynomial of $\mathbf{A}$, we refer to the multiplicity of the root as the **algebraic multiplicity** of the eigenvalue. For any eigenvalue λ of $\mathbf{A}$, the subspace $\ker(\lambda\mathbf{I} - \mathbf{A})$ is called the **eigenspace** of λ, or E_λ. The **geometric multiplicity** is $\dim[\ker(\lambda\mathbf{I} - \mathbf{A})]$, i.e. the number of linearly independent eigenvectors associated with this eigenvalue.

If, for an $n\times n$ matrix $\mathbf{A}$, we are able to construct a basis $\mathscr{B}=\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ for all of $\mathbf{R}^n$ consisting of eigenvectors, we call this an eigenbasis and say that the matrix is **diagonalizable**.

If $\mathbf{A}$ is diagonalizable with eigenbasis $\mathscr{B}=\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ and

$$\mathbf{S}=\begin{bmatrix}\uparrow & & \uparrow\\ \mathbf{v}_1 & \cdots & \mathbf{v}_n\\ \downarrow & & \downarrow\end{bmatrix},\ \text{then}\ \left\{\begin{matrix}\mathbf{A}\mathbf{v}_1=\lambda_1\mathbf{v}_1\\ \vdots\\ \mathbf{A}\mathbf{v}_n=\lambda_n\mathbf{v}_n\end{matrix}\right\}\Rightarrow\left[\mathbf{A}\right]_{\mathscr{B}}=\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\mathbf{D}=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}.$$

If a matrix $\mathbf{A}$ is diagonalizable and we write $\left[\mathbf{A}\right]_{\mathscr{B}}=\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\mathbf{D}$ for some change of basis matrix $\mathbf{S}$, then $\mathbf{A}=\mathbf{S}\mathbf{D}\mathbf{S}^{-1}$ and $\mathbf{A}^t=(\mathbf{S}\mathbf{D}\mathbf{S}^{-1})(\mathbf{S}\mathbf{D}\mathbf{S}^{-1})\cdots(\mathbf{S}\mathbf{D}\mathbf{S}^{-1})=\mathbf{S}\mathbf{D}^t\mathbf{S}^{-1}$ where

$$\mathbf{D}^t=\begin{bmatrix}\lambda_1^{\,t} & & 0\\ & \ddots & \\ 0 & & \lambda_n^{\,t}\end{bmatrix}.$$

We proved that **eigenvectors corresponding to distinct eigenvalues are linearly independent** which yielded the corollary that if $\mathbf{A}$ is an $n\times n$ matrix with distinct, real eigenvalues, then $\mathbf{A}$ must be diagonalizable. This also means that for a matrix $\mathbf{A}$ to fail to be diagonalizable, its spectrum must contain either repeated eigenvalues, complex eigenvalues, or possibly both.

It is possible for a matrix with repeated eigenvalues to still be diagonalizable. In the case where all of the eigenvalues of a matrix are real but with some multiplicity, as long as $GM(\lambda_i)=AM(\lambda_i)$ for each eigenvalue λ_i (that is, the geometric and algebraic multiplicities are the same), the matrix will still be diagonalizable. **So the only obstructions to being able to diagonalize an $n\times n$ matrix are the existence of complex eigenvalues or having a repeated eigenvalue where its geometric multiplicity is strictly less than its algebraic multiplicity.**

Example: Find the eigenvalues and eigenvectors for the matrix $\mathbf{A}=\begin{bmatrix}-2 & -3 & 3\\ -3 & -2 & 3\\ -3 & -3 & 4\end{bmatrix}$ and determine whether this matrix is diagonalizable. Find an expression for $\mathbf{A}^t\mathbf{x}_0$ for the vector $\mathbf{x}_0=\begin{bmatrix}6\\ 1\\ 2\end{bmatrix}$.

Solution: We first write $\lambda\mathbf{I}-\mathbf{A}=\begin{bmatrix}\lambda+2 & 3 & -3\\ 3 & \lambda+2 & -3\\ 3 & 3 & \lambda-4\end{bmatrix}$. The characteristic polynomial is then:

$$\begin{aligned}p_A(\lambda)=\det(\lambda\mathbf{I}-\mathbf{A})&=(\lambda+2)(\lambda^2-2\lambda+1)-3(3\lambda-3)-3(3-3\lambda)\\ &=\lambda^3-3\lambda+2=(\lambda-1)^2(\lambda+2)=0\end{aligned}$$

This gives the eigenvalues $\lambda_1=1$ with algebraic multiplicity 2, and $\lambda_2=-2$ with algebraic multiplicity 1.

Taking $\lambda_1=1$, we seek eigenvectors: $\left[\begin{array}{ccc|c}3 & 3 & -3 & 0\\ 3 & 3 & -3 & 0\\ 3 & 3 & -3 & 0\end{array}\right]\to\left[\begin{array}{ccc|c}1 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$. Because this has rank 1, its kernel will have dimension 2, so the geometric multiplicity will be 2. For eigenvectors we can parameterize the kernel as $\left\{\begin{array}{l}x_1=-s+t\\ x_2=s\\ x_3=t\end{array}\right\}$, $\mathbf{x}=s\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$, so the eigenspace is spanned by $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$. Taking $\lambda_2=-2$, we seek eigenvectors: $\left[\begin{array}{ccc|c}0&3&-3&0\\3&0&-3&0\\3&3&-6&0\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$. This has rank 2 and the geometric multiplicity is 1. For eigenvectors we can parameterize the kernel as $\left\{\begin{array}{l}x_1=t\\x_2=t\\x_3=t\end{array}\right\}$, $\mathbf{x}=t\begin{bmatrix}1\\1\\1\end{bmatrix}$, so the eigenspace is spanned by $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$.

Taken together, $\mathscr{B}=\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ is a basis of eigenvectors, so the matrix is diagonalizable even though there was a repeated eigenvalue.

If we let $\mathbf{S}=\begin{bmatrix}-1&1&1\\1&0&1\\0&1&1\end{bmatrix}$, then we can calculate $\mathbf{S}^{-1}=\begin{bmatrix}-1&0&1\\-1&-1&2\\1&1&-1\end{bmatrix}$ and $\mathbf{S}^{-1}\mathbf{AS}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\end{bmatrix}=\mathbf{D}$.

We can then write $\mathbf{A}=\mathbf{SDS}^{-1}$ and $\mathbf{A}^t=\mathbf{SD}^t\mathbf{S}^{-1}$, so:

$$\mathbf{A}^t\mathbf{x}_0=\mathbf{SD}^t\mathbf{S}^{-1}\mathbf{x}_0=\begin{bmatrix}-1&1&1\\1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&(-2)^t\end{bmatrix}\begin{bmatrix}-1&0&1\\-1&-1&2\\1&1&-1\end{bmatrix}\begin{bmatrix}6\\1\\2\end{bmatrix}$$
$$=\begin{bmatrix}-1&1&(-2)^t\\1&0&(-2)^t\\0&1&(-2)^t\end{bmatrix}\begin{bmatrix}-4\\-3\\5\end{bmatrix}=\begin{bmatrix}1+5(-2)^t\\-4+5(-2)^t\\-3+5(-2)^t\end{bmatrix}.$$

Proposition: If two matrices $\mathbf{A}$ and $\mathbf{B}$ are similar, i.e. if $\mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$ for some invertible matrix $\mathbf{S}$, then they have the same characteristic polynomial and therefore the same eigenvalues with the same algebraic multiplicities.

Proof: $p_{\mathbf{B}}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{B}) = \det(\lambda\mathbf{I} - \mathbf{S}^{-1}\mathbf{AS}) = \det(\mathbf{S}^{-1}\lambda\mathbf{IS} - \mathbf{S}^{-1}\mathbf{AS}) = \det[\mathbf{S}^{-1}(\lambda\mathbf{I} - \mathbf{A})\mathbf{S}]$

$$= \det(\mathbf{S}^{-1})\det(\lambda\mathbf{I} - \mathbf{A})\det(\mathbf{S}) = \det(\lambda\mathbf{I} - \mathbf{A}) = p_{\mathbf{A}}(\lambda).$$

Proposition: If two matrices $\mathbf{A}$ and $\mathbf{B}$ are similar, then they have the same eigenvalues with the same algebraic multiplicities <u>and</u> the same geometric multiplicities.

Proof: $\lambda\mathbf{I} - \mathbf{B} = \lambda\mathbf{I} - \mathbf{S}^{-1}\mathbf{AS} = \mathbf{S}^{-1}\lambda\mathbf{IS} - \mathbf{S}^{-1}\mathbf{AS} = \mathbf{S}^{-1}(\lambda\mathbf{I} - \mathbf{A})\mathbf{S}$. Therefore, if $\mathbf{v} \in \ker(\lambda\mathbf{I} - \mathbf{B})$, we'll have $(\lambda\mathbf{I} - \mathbf{B})\mathbf{v} = \mathbf{0}$, so $\mathbf{S}^{-1}(\lambda\mathbf{I} - \mathbf{A})\mathbf{Sv} = \mathbf{0} \Rightarrow (\lambda\mathbf{I} - \mathbf{A})\mathbf{Sv} = \mathbf{0} \Rightarrow \mathbf{Sv} \in \ker(\lambda\mathbf{I} - \mathbf{A})$. Similarly any $\mathbf{w} \in \ker(\lambda\mathbf{I} - \mathbf{A})$ will give $\mathbf{S}^{-1}\mathbf{w} \in \ker(\lambda\mathbf{I} - \mathbf{B})$. Because $\mathbf{S}$ is invertible this correspondence is an isomorphism, so the subspaces must have the same dimension. Therefore, if λ is an eigenvalue of $\mathbf{A}$ (and hence $\mathbf{B}$), then its geometric multiplicity will be the same for both $\mathbf{A}$ and $\mathbf{B}$. The eigenvectors, however, will not be the same.

Trace and determinant

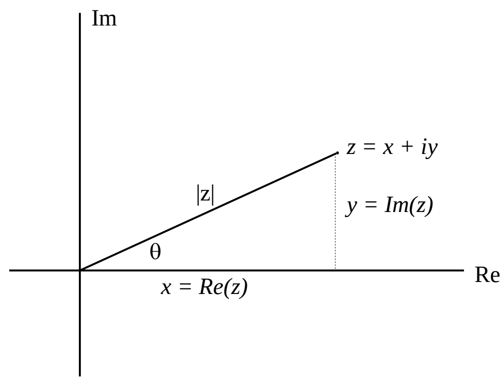
If two matrices $\mathbf{A}$ and $\mathbf{B}$ are similar, we have already shown that $\det\mathbf{A} = \det\mathbf{B}$. A homework exercise also shows that $\text{trace}\,\mathbf{A} = \text{trace}\,\mathbf{B}$, where for an $n \times n$ matrix $\mathbf{A}$ its trace is the sum of its diagonal elements $\sum_{i=1}^{n} a_{ii}$. In the case of a diagonalizable matrix, this means that its trace must be the sum of its eigenvalues, and its determinant must be the product of its eigenvalues. In fact, these statements are true generally:

Theorem: If $\mathbf{A}$ is any $n\times n$ matrix with eigenvalues $\{\lambda_1,\cdots,\lambda_n\}$, including any repeated eigenvalues, then $\operatorname{tr}(\mathbf{A})=\lambda_1+\cdots+\lambda_n$ and $\det(\mathbf{A})=\lambda_1\cdots\lambda_n$.
Proof: This is easy to show for 2×2 matrices. The general case is left as an exercise.

Complex eigenvalues
If an $n\times n$ matrix $\mathbf{A}$ has any complex eigenvalues, we will not be able to produce any *real* eigenvectors. However, as in the case of real eigenvalues, the algebra can still formally proceed in the same manner. Our goal will be to produce basis vectors associated with any complex eigenvalues such that the matrix relative to this basis has a simple, if not diagonal, canonical form. In order to do this, we have to temporarily wander off into the world of complex numbers, complex eigenvalues, and complex eigenvectors.

You should not attempt to visualize a vector whose components are complex numbers. This is merely an algebraically consistent extension of the idea of real vectors and real matrices where all the rules of linear algebra are still in effect. This temporary excursion will yield real vectors relative to which the matrix acts in an easy-to-describe fashion, namely as a rotation-dilation, i.e. it rotates vectors in a 2-dimensional (invariant) subspace and scales them by the modulus of the complex eigenvalue.

Basics of complex numbers
First, we need a few basic definitions associated with complex numbers. A complex number $z=x+iy$, where i^2 = -1 can be viewed in vector-like terms in the complex plane as shown in this diagram to the right. We define:



$$\text{modulus}(z) = \text{mod}(z) = |z| = \sqrt{x^2 + y^2} \qquad \text{argument}(z) = \arg(z) = \theta = \tan^{-1}\left(\tfrac{y}{x}\right).$$

We add complex numbers by adding their respective real and imaginary parts, in much the same way as vector addition was defined. We multiply complex numbers via the distributive law and the fact that $i^2 = -1$. For example:

$$(3+2i)(-1-4i) = -3-2i-12i-8i^2 = (-3+8)+i(-2-12) = 5-14i$$

If we note that $x = |z|\cos\theta$ and $y = |z|\sin\theta$, then we can write
$z = x+iy = |z|\cos\theta + i|z|\sin\theta = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$ (polar form, using **Euler's Formula**).

A short calculation shows that when we multiply two complex numbers, we <u>multiply their moduli and add their arguments</u>. Specifically, $z_1 z_2 = |z_1|e^{i\theta_1}|z_2|e^{i\theta_2} = |z_1||z_2|e^{i(\theta_1+\theta_2)}$. You may want to try this out with some simple complex numbers to convince yourself of this fact.

The ***complex conjugate*** of $z = x+iy$ is defined to be $\overline{z} = x - iy$. In the complex plane, z and $\overline{z}$ are reflections of each other across the real axis. It's not hard to show that $\overline{z_1 z_2} = \overline{z}_1\overline{z}_2$.

The **Fundamental Theorem of Algebra** guarantees that, at least in theory, any polynomial of degree *n* can be factored into *n* linear factors and will therefore produce *n* roots, including multiplicity. Some roots may have multiplicity greater than 1 and some of the roots may be complex. It is also the case that for a polynomial with all real coefficients, any complex roots will necessarily occur in complex conjugate pairs λ and $\overline{\lambda}$ (this follows from the quadratic formula, resulting from an irreducible quadratic factor).

Let $\mathbf{A}$ be a matrix which has a complex conjugate pair of eigenvalues λ and $\bar{\lambda}$. We can proceed just as in the case of real eigenvalues and find a complex vector $\mathbf{v}$ such that $(\lambda\mathbf{I}-\mathbf{A})\mathbf{v}=\mathbf{0}$. The components of such a vector $\mathbf{v}$ will have complex numbers for its components. If we write $\lambda = a+ib$, and decompose $\mathbf{v}$ into its real and imaginary vector components as $\mathbf{v}=\mathbf{x}+i\mathbf{y}$ (where $\mathbf{x}$ and $\mathbf{y}$ and real vectors), we can calculate that:

$$\text{(1)} \qquad \mathbf{Av}=\mathbf{Ax}+i\mathbf{Ay}=\lambda\mathbf{v}=(a+ib)(\mathbf{x}+i\mathbf{y})=(a\mathbf{x}-b\mathbf{y})+i(b\mathbf{x}+a\mathbf{y})$$

If we define the vector $\hat{\mathbf{v}}=\mathbf{x}-i\mathbf{y}$ and use the easy-to-prove fact that for a matrix $\mathbf{A}$ with all real entries we'll have $\overline{\mathbf{Av}} = \mathbf{A}\hat{\mathbf{v}} = \overline{\lambda\mathbf{v}} = \bar{\lambda}\hat{\mathbf{v}}$, we see that $\hat{\mathbf{v}}=\mathbf{x}-i\mathbf{y}$ will also be an eigenvector with eigenvalue $\bar{\lambda}$, and:

$$\text{(2)} \qquad \overline{\mathbf{Av}}=\mathbf{A}\hat{\mathbf{v}} = \mathbf{Ax}-i\mathbf{Ay}=\bar{\lambda}\hat{\mathbf{v}}=(a-ib)(\mathbf{x}-i\mathbf{y})=(a\mathbf{x}-b\mathbf{y})-i(b\mathbf{x}+a\mathbf{y})$$

The true value of this excursion into the world of complex numbers and complex vectors is seen when we add and subtract equation (1) and (2). We get:

$$2\mathbf{Ax}=2(a\mathbf{x}-b\mathbf{y})$$

$$2i\mathbf{Ay}=2i(b\mathbf{x}+a\mathbf{y})$$

After cancellation of the factors of 2 and 2*i* in the respective equations and rearranging, we get:

$$\boxed{\begin{aligned}\mathbf{Ay}&=a\mathbf{y}+b\mathbf{x}\\ \mathbf{Ax}&=-b\mathbf{y}+a\mathbf{x}\end{aligned}}$$

Note that we are now back in the "real world": all vectors and scalars in the above equations are real. If we use the two vectors **y** and **x** as basis vectors associated with the two complex conjugate eigenvalues, grouped together in the full basis $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, we'll produce a 2×2 (Jordan) block in the matrix $[\mathbf{A}]_{\mathcal{B}}$ of the form:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2}\begin{bmatrix} a\big/\sqrt{a^2+b^2} & -b\big/\sqrt{a^2+b^2} \\ b\big/\sqrt{a^2+b^2} & a\big/\sqrt{a^2+b^2} \end{bmatrix} = |\lambda|\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = |\lambda|\mathbf{R}_\theta$$

where $\mathbf{R}_\theta$ is the rotation matrix corresponding to the angle $\theta = \arg(\lambda)$.

In other words, the Jordan block associated with the basis vectors $\{\mathbf{y}, \mathbf{x}\}$ is a **rotation-dilation matrix where the angle of rotation is the same as the argument of the complex eigenvalue and where the scaling factor is just the modulus (magnitude) of the complex eigenvalue**. Again, the very nature of the complex eigenvalues tells us much about the way the matrix acts, at least if we choose the right basis with which to view things.

Example: Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. You'll recognize this as the matrix corresponding to counterclockwise rotation in the plane through an angle of 90°. The characteristic polynomial is $\lambda^2 + 1 = 0$, with complex eigenvalues $\lambda = \pm i$. Note that with $\lambda = +i$, we have $\arg(\lambda) = 90°$ and $\text{modulus}(\lambda) = |\lambda| = 1$. The preceding discussion says that this matrix is similar to a rotation-dilation matrix which does no scaling and which rotates by an angle of 90°. But this should come as no surprise at all. The given matrix is already in the form of exactly this rotation-dilation matrix, i.e. Jordan form.

Example: Consider the matrix $\mathbf{A}=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}$. We have $\lambda\mathbf{I}-\mathbf{A}=\begin{bmatrix}\lambda-2 & 1\\ -3 & \lambda-2\end{bmatrix}$, and the characteristic polynomial is $\lambda^2-4\lambda+7=0$. Its eigenvalues are $\lambda=2+i\sqrt{3}$ and $\bar{\lambda}=2-i\sqrt{3}$.

If we substitute $\lambda=2+i\sqrt{3}$ into $\lambda\mathbf{I}-\mathbf{A}=\begin{bmatrix}\lambda-2 & 1\\ -3 & \lambda-2\end{bmatrix}$, we get that if $\mathbf{v}=\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ is to be an eigenvector, we must have $\begin{bmatrix}i\sqrt{3} & 1\\ -3 & i\sqrt{3}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. This means that $(i\sqrt{3})\alpha+\beta=0$. (The second equation is redundant, even though this might not immediately appear to be the case.) One choice for α and β is $\alpha=1$, $\beta=-i\sqrt{3}$. This gives us the complex eigenvector $\mathbf{v}=\begin{bmatrix}1\\ -i\sqrt{3}\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}+i\begin{bmatrix}0\\ -\sqrt{3}\end{bmatrix}=\mathbf{x}+i\mathbf{y}$. Using $\mathscr{B}=\{\mathbf{y},\mathbf{x}\}=\left\{\begin{bmatrix}0\\ -\sqrt{3}\end{bmatrix},\begin{bmatrix}1\\ 0\end{bmatrix}\right\}$ as a basis, and calling $\mathbf{S}=\begin{bmatrix}0 & 1\\ -\sqrt{3} & 0\end{bmatrix}$, we have that $[\mathbf{A}]_{\mathscr{B}}=\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\begin{bmatrix}2 & -\sqrt{3}\\ \sqrt{3} & 2\end{bmatrix}=\sqrt{7}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=|\lambda|R_\theta$ where $\mathbf{R}_\theta$ is the rotation matrix corresponding to the angle $\theta=\arg(\lambda)=\tan^{-1}(\sqrt{3}/2)\cong 40.89°$.

If we have need to consider the powers $\mathbf{A}^t$ for any positive integer power t, we'll have that $\mathbf{A}=\mathbf{S}[\mathbf{A}]_{\mathscr{B}}\mathbf{S}^{-1}$ and $\mathbf{A}^t=\mathbf{S}\left([\mathbf{A}]_{\mathscr{B}}\right)^t\mathbf{S}^{-1}=\mathbf{S}|\lambda|^t\mathbf{R}_{t\theta}\mathbf{S}^{-1}=|\lambda|^t\mathbf{S}\mathbf{R}_{t\theta}\mathbf{S}^{-1}$. That is, except for the change of basis, $\mathbf{A}^t$ corresponds to rotation through the angle $t\theta$ and scaling by the factor $|\lambda|^t$.

We'll look at a few more detailed examples next time. We'll also take up the discussion of the Spectral Theorem. **Notes by Robert Winters**