Math S-21b – Lecture #10 Notes

In today's lecture we'll finish up a few details on Least Square Approximate Solutions (see Lecture #9 notes), review inner product spaces, and then introduce the idea of an orthonormal set of functions. The primary application that we'll focus on is Fourier series. We'll cover this initially in the case of a continuous function on a closed interval, but the ideas can be extended to also represent discontinuous functions that are piecewise continuous. All results can then be used to describe periodic functions more generally. Much of the following notes are adapted from a course in differential equations.

Inner Products and Orthogonality

We are all familiar with the mutually perpendicular (orthogonal) unit vectors $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ in $\mathbf{R}^3$ and how we can express any vector $\mathbf{v}=\langle x,y,z\rangle$ in $\mathbf{R}^3$ as:

$$\mathbf{v}=\langle x,y,z\rangle = x\langle 1,0,0\rangle + y\langle 0,1,0\rangle + z\langle 0,0,1\rangle = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}\,.$$

The components of the vector are just the scalar projections of $\mathbf{v}$ in the directions of $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$, respectively. We find the scalar projection of a vector in any given direction by calculating its dot product with a unit vector $\mathbf{u}$ in the given direction, i.e. $\mathbf{v}\cdot\mathbf{u}$. So we can also express $\mathbf{v}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}=(\mathbf{v}\cdot\mathbf{i})\,\mathbf{i}+(\mathbf{v}\cdot\mathbf{j})\,\mathbf{j}+(\mathbf{v}\cdot\mathbf{k})\,\mathbf{k}$.

We could do the same thing with any set of three mutually orthogonal unit vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ in $\mathbf{R}^3$. That is, if $\mathbf{v} \in \mathbf{R}^3$ we could write $\mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{v} \cdot \mathbf{u}_3)\mathbf{u}_3$.

Just as we can add and scale vectors, we can also do this with functions. We add functions by adding their values and scale them by scaling their values. That is, $(f+g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$. We might speculate that if functions can be combined in a manner analogous to vectors (where we add respective values instead of respective components), perhaps there may be something analogous to the dot product that we could use to define notions such as orthogonality in spaces of functions.

Think about how the dot product of two vectors is calculated: We multiply the respective components of the two vectors and sum these products, i.e. in $\mathbf{R}^3$ we have $\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 \in \mathbf{R}$ and, more generally, in $\mathbf{R}^n$ we have $\mathbf{u} \cdot \mathbf{v} = \langle u_1, \cdots, u_n \rangle \cdot \langle v_1, \cdots, v_n \rangle = u_1 v_1 + \cdots + u_n v_n \in \mathbf{R}$. We also derive using the Law of Cosines that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. It is from this fact that we conclude that nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Vectors in $\mathbf{R}^3$ (or any $\mathbf{R}^n$) have just a finite list of components, whereas functions of a real variable have infinitely many values. If we think of the values of a function as analogous to the components of a vector, and if we use integration as analogous to a discrete sum, this suggests the following definition:

Definition: If f, g are functions defined on some interval $[a, b]$, we can define an inner product of these functions by $\langle f, g \rangle = \int_a^b f(t)g(t)dt$. Such an inner product will then satisfy the following properties (where defined for any functions f, g, h):

(1) $\langle g, f \rangle = \langle f, g \rangle$ (symmetric)

(2) $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$ and $\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$ (left and right distributive laws)

(3) $\langle cf, g \rangle = c \langle f, g \rangle = \langle f, cg \rangle$ for any scalar c

These properties are analogous to the algebraic properties of the dot product. We can use this inner product to define orthogonality.

Definition: We say that two (nonzero) functions f, g are orthogonal if $\langle f, g \rangle = 0$.

There is a 4th property of the dot product that doesn't quite work as simply in the context of functions. That is, for any vector $\mathbf{v}$, we have $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0$ only if $\mathbf{v} = \mathbf{0}$. The corresponding statement for functions is not generally true. It will always be the case that $\langle f, f \rangle = \int_a^b [f(t)]^2 \, dt \geq 0$, but this inner product could be equal to zero for a function that is not identically 0 with isolated discontinuities where the function takes on nonzero values. If we only consider continuous functions, then $\langle f, f \rangle = \int_a^b [f(t)]^2 \, dt = 0$ would imply that $f(t)$ is identically zero on the interval $[a,b]$. In any case, we can still define the **norm of a function *f*** by $\|f\|^2 = \langle f, f \rangle = \int_a^b [f(t)]^2 \, dt$ or $\|f\| = \sqrt{\langle f, f \rangle}$.

It's certainly possible that these integrals might not be defined, so we generally restrict the set of functions to those for which $\|f\|^2 = \langle f, f \rangle = \int_a^b [f(t)]^2 \, dt$ is finite. These are called **"square summable" functions**, and the set of all such functions for the interval $[a,b]$ is denoted by $L^2([a,b])$. It can be shown that in this space of functions, $|\langle f, g \rangle| \le \|f\| \|g\|$ (Cauchy-Schwartz Inequality) and $\|f + g\| \le \|f\| + \|g\|$ (Triangle Inequality).

Note: All of the above properties work just as well if we define the inner product as $\langle f, g \rangle = K \int_a^b f(t) g(t) dt$ for some nonzero constant *K*. This will alter the way in which the norm of a given function is defined, but using such a normalizing constant is often desirable when working with a particular interval. We'll start by considering functions defined on the interval $[-\pi, \pi]$ and choose our normalizing constant so that

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) g(t) dt .$$

Proposition: In $L^2([-\pi,\pi])$, the finite collection

$\mathcal{B}_n = \left\{\frac{1}{\sqrt{2}}, \cos t, \sin t, \cos 2t, \sin 2t, \cdots, \cos nt, \sin nt\right\}$ is an **orthonormal** set. That is, each function in $\mathcal{B}_n$ has norm 1 and any distinct pair has inner product equal to 0. We think of this set as consisting of mutually orthogonal unit elements. In linear algebra terminology, we would say that these $2n+1$ functions **span** a **subspace** (referred to as T_n) and that they form a **basis** for this subspace.

Proof: This is just a list of integral calculations. We'll calculate a few of them and just quote the rest (though you may want to try some integration techniques or consult an integral table to see why they are true). We have:

$$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\frac{1}{\sqrt{2}}\right)^2 dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\, dt = \frac{1}{\pi}(\tfrac{1}{2})2\pi = 1$$

$$\left\langle \frac{1}{\sqrt{2}}, \cos kt \right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \cos kt\, dt = \frac{1}{\pi\sqrt{2}} \frac{\sin kt}{k}\bigg|_{-\pi}^{\pi} = 0$$

$$\left\langle \frac{1}{\sqrt{2}}, \sin kt \right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \sin kt\, dt = 0 \text{ (integral of an odd function over a symmetric interval)}$$

$$\langle \cos kt, \cos kt \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 kt\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} [1+\cos(2kt)]\, dt = \frac{1}{2\pi}\left[t + \frac{\sin(2kt)}{2k}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}(2\pi) = 1$$

$$\left\langle \sin kt, \sin kt \right\rangle = \tfrac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 kt\, dt = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} [1 - \cos(2kt)]\, dt = \tfrac{1}{2\pi}\left[t - \frac{\sin(2kt)}{2k} \right]_{-\pi}^{\pi} = \tfrac{1}{2\pi}(2\pi) = 1$$

$$\left\langle \cos jt, \cos kt \right\rangle = \tfrac{1}{\pi}\int_{-\pi}^{\pi} \cos jt \cos kt\, dt = 0 \text{ for integers } j \neq k \qquad \text{(consult integral table)}$$

$$\left\langle \sin jt, \sin kt \right\rangle = \tfrac{1}{\pi}\int_{-\pi}^{\pi} \sin jt \sin kt\, dt = 0 \text{ for integers } j \neq k \qquad \text{(consult integral table)}$$

$$\left\langle \cos jt, \sin kt \right\rangle = \tfrac{1}{\pi}\int_{-\pi}^{\pi} \cos jt \sin kt\, dt = 0 \text{ for integers } j, k \qquad \text{(consult integral table)}$$

We can define the orthogonal projection of a function *f* onto the subspace T_n analogous to $\mathbf{R}^n$, namely:

$$f_n = \mathrm{Proj}_n(f) = \left\langle f, \tfrac{1}{\sqrt{2}} \right\rangle \tfrac{1}{\sqrt{2}} + \left\langle f, \cos t \right\rangle \cos t + \left\langle f, \sin t \right\rangle \sin t + \cdots + \left\langle f, \cos nt \right\rangle \cos nt + \left\langle f, \sin nt \right\rangle \sin nt$$

If we express these in terms of integrals, we get:

$$f_n = \tfrac{1}{2}\left[\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t) dt \right] + \left[\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos t\, dt \right] \cos t + \left[\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin t\, dt \right] \sin t + \cdots$$

$$\cdots + \left[\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt\, dt \right] \cos nt + \left[\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt\, dt \right] \sin nt$$

This function f_n is known as the ***n*th order Fourier approximation** of the function *f*.

This can be expressed more succinctly by defining the **Fourier coefficients** by:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\, dt, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\, dt$$

Then $f_n = \frac{a_0}{2} + a_1\cos t + b_1 \sin t + \cdots + a_n \cos nt + b_n \sin nt$ is the *n*th order Fourier approximation.

If, for any given *n*, we express $f = (f - f_n) + f_n$, we can think of $f_n \in T_n$ and $(f - f_n) \in T_n^{\perp}$ (known as the orthogonal complement of T_n). There is the analogue of the Pythagorean Theorem in this context that gives that $\| f \|^2 = \| f - f_n \|^2 + \| f_n \|^2$. With some careful analysis it can be shown that as *n* gets larger, the Fourier approximation converges in the sense that $\lim_{n\to\infty} \| f - f_n \|^2 = 0$, and this implies that $\boxed{\lim_{n\to\infty} \| f_n \|^2 = \| f \|^2}$ which can provide some very interesting results.

By letting $n \to \infty$ we produce the **Fourier Series** of *f* as $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nt + b_n \sin nt\right)$, and there's this accompanying theorem (proven elsewhere):

Theorem (Fourier): Suppose a function $f(t)$ is periodic with base period 2π and continuous except for a finite number of jump discontinuities. Then $f(t)$ may be represented by a (convergent) Fourier Series:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

where: $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt$, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt$.

The numbers $\{a_0, a_1, b_1, \cdots, a_n, b_n, \cdots\}$ are called the **Fourier coefficients** of the function $f(t)$.

This representation is an equality at all points of continuity of the function $f(t)$. At any point of discontinuity $t = a$, the series converges to the average of $f(a^-)$ and $f(a^+)$, i.e. the value $\frac{1}{2}[f(a^-) + f(a^+)]$.

Note: (a) If $f(t)$ is an even function [$f(-t) = f(t)$ for all *t*], then $b_n = 0$ for all *n* by basic facts from calculus.

(b) If $f(t)$ is an odd function [$f(-t) = -f(t)$ for all *t*], then $a_0 = 0$ and $a_n = 0$ for all *n* by basic facts from calculus.

Example (Square wave function): $f(t) = sq(t) = \begin{Bmatrix} -1 & t \in [-\pi, 0) \\ +1 & t \in [0, \pi) \end{Bmatrix}$, extended periodically for all t.

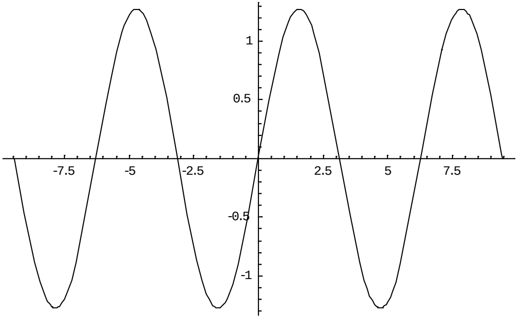
This function is periodic (with period 2π) and antisymmetric, i.e. an odd function. Therefore $a_0 = 0$ and $a_n = 0$ for all n. We calculate:

$$b_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \tfrac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin nt\,dt + \int_{0}^{\pi}\sin nt\,dt\right] = \tfrac{1}{\pi}\left[\left[\tfrac{\cos nt}{n}\right]_{-\pi}^{0} - \left[\tfrac{\cos nt}{n}\right]_{0}^{\pi}\right]$$

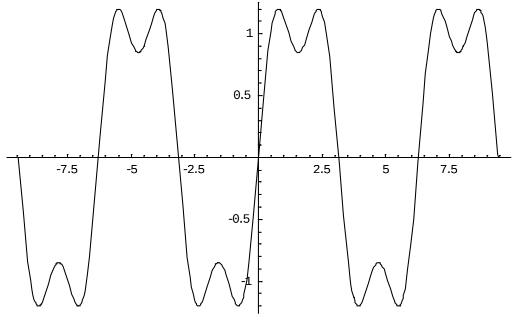
$$= \tfrac{1}{n\pi}\left[\left[1-(-1)^n\right] - \left[(-1)^n - 1\right]\right] = \begin{Bmatrix} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{Bmatrix}.$$

So $\boxed{sq(t) \sim \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n} = \frac{4}{\pi}[\sin t + \tfrac{1}{3}\sin 3t + \tfrac{1}{5}\sin 5t + \cdots]}$.

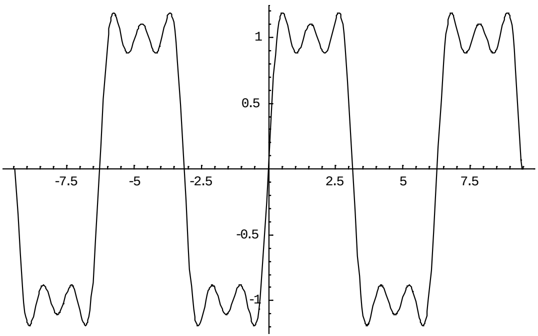
The nature of the convergence of this Fourier series toward the square wave function can be seen by graphing the partial sums:


1
0.5
-0.5
-1
-7.5
-5
-2.5
2.5
5
7.5

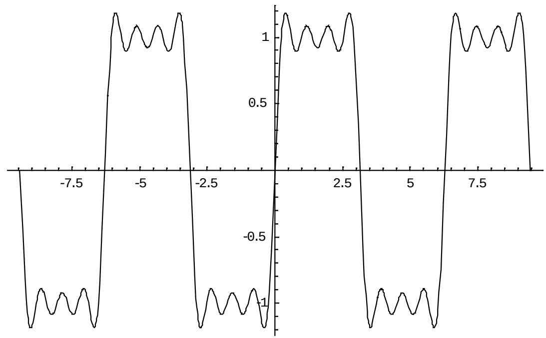
n = 1


1
0.5
-0.5
-1
-7.5
-5
-2.5
2.5
5
7.5

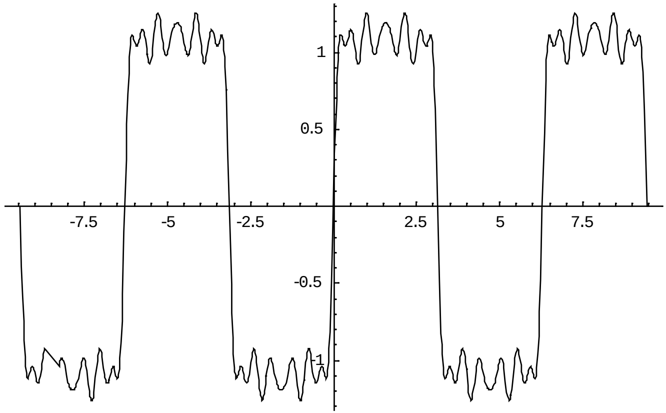
n = 3


1
0.5
-0.5
-1
-7.5
-5
-2.5
2.5
5
7.5

n = 5


1
0.5
-0.5
-1
-7.5
-5
-2.5
2.5
5
7.5

n = 7



n = 15

If we translate $\lim_{n\to\infty}\| f_n\|^2 = \| f\|^2$ for this function we get that $\| f\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} 1dt = 2$, and

$$\lim_{n\to\infty}\| f_n\|^2 = \frac{16}{\pi^2}\left[1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots\right] = \frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = 2.$$

Therefore $\boxed{\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}}$, <u>a curious fact</u>.

We can also apply the last statement in Fourier's Theorem by evaluating the square-wave function at $\pi/2$, a point of continuity, to get that $sq(\pi/2) = 1 = \frac{4}{\pi}[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots]$, so $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$, but the convergence is so abysmally slow as to be of no practical consequence – another curiosity.

Note: More generally, $\lim_{n\to\infty} \| f_n \|^2 = \dfrac{a_0{}^2}{2} + a_1{}^2 + b_1{}^2 + a_2{}^2 + b_2{}^2 + \cdots + a_n{}^2 + b_n{}^2 + \cdots = \| f \|^2$

Example (Sawtooth function): $f(t) = t$ on the interval $(-\pi, \pi]$, extended periodically for all t.

This is an odd function, so we conclude immediately that $a_0 = 0$ and $a_n = 0$ for all n. For the Fourier sine coefficients we do a little integration by parts:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t \sin nt \, dt = \frac{1}{\pi}\left[-\frac{t \cos nt}{n} + \cancel{\frac{\sin nt}{n^2}} \right]_{-\pi}^{\pi} = (-1)^{n+1}\frac{2}{n}$$

So $f(t) \sim \sum_{n=1}^{\infty} \left((-1)^{n+1} \frac{2}{n} \right) \sin nt$. In this case the fact that $\lim_{n\to\infty} \| f_n \|^2 = \| f \|^2$ translates into

$$\sum_{n=1}^{\infty} b_n{}^2 = \sum_{n=1}^{\infty} \frac{4}{n^2} = 4\sum_{n=1}^{\infty} \frac{1}{n^2} = \| f \|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \, dt = \frac{1}{3\pi}\left[t^3 \right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}.$$

So $4\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{2\pi^2}{3}$ or $\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}}$.

It's worth noting (from Calculus) that this is a *p*-series with $p=2$. There you may recall that we know that this series converges but *not necessarily what it converges to*. This curious result answers that question.

Tips & Tricks – Manipulation of Fourier series

<u>Different period</u>: We developed our Fourier series representation for functions with a standard period 2π and fundamental interval $[-\pi,\pi]$. If we instead have a function $f(t)$ with period $2L$ and fundamental interval $[-L,L]$, we can simply change variables to produce the corresponding Fourier series in this case. We let $u=\frac{\pi t}{L}$ (so $t=\frac{Lu}{\pi}$) and define $g(u)=f\left(\frac{Lu}{\pi}\right)$ with period 2π and fundamental interval $[-\pi,\pi]$. The Fourier series for $g(u)$ is then

$g(u)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nu+b_n\sin nu\right)$ and if we use the substitution $u=\frac{\pi t}{L}$ (and $du=\frac{\pi}{L}dt$), we'll have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)dt = \frac{1}{L}\int_{-L}^{L} g(\tfrac{\pi t}{L})\,dt = \frac{1}{L}\int_{-L}^{L} f(u)\,du = \frac{1}{L}\int_{-L}^{L} f(t)\,dt\,,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\cos nu\,du = \frac{1}{L}\int_{-L}^{L} g(\tfrac{\pi t}{L})\cos(\tfrac{n\pi t}{L})\,dt = \frac{1}{L}\int_{-L}^{L} f(t)\cos(\tfrac{n\pi t}{L})\,dt\,,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\sin nu\,du = \frac{1}{L}\int_{-L}^{L} g(\tfrac{\pi t}{L})\sin(\tfrac{n\pi t}{L})\,dt = \frac{1}{L}\int_{-L}^{L} f(t)\sin(\tfrac{n\pi t}{L})\,dt\,,$$ and we can write:

$$\boxed{f(t) = g(\tfrac{n\pi t}{L}) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(\tfrac{n\pi t}{L}) + b_n\sin(\tfrac{n\pi t}{L})\right)}$$

Fourier series can be differentiated or integrated term-by-term to produce other Fourier series:

Example: If we start with $sq(t) = \begin{Bmatrix} -1 & t\in[-\pi,0) \\ +1 & t\in[0,\pi) \end{Bmatrix} \sim \frac{4}{\pi}\sum_{n\text{ odd}} \frac{\sin nt}{n}$ and integrate term-by-term, we get $F(t) \sim \frac{4}{\pi}\sum_{n\text{ odd}} -\frac{\cos nt}{n^2} + C$. If we also insist that $F(0)=0$ and that $F(t)$ be continuous, we get that $-\frac{4}{\pi}\left(\sum_{n\text{ odd}}\frac{1}{n^2}\right) + C = -\frac{4}{\pi}\left(\frac{\pi^2}{8}\right) + C = -\frac{\pi}{2} + C = 0$, so $C = \frac{\pi}{2}$. This gives

$F(t) = |t| = \begin{cases} -t & t \in [-\pi, 0) \\ +t & t \in [0, \pi) \end{cases} \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nt}{n^2}$, extended periodically for all t, another kind of "sawtooth function". This series could also have been calculated directly using the formulas for the Fourier coefficients and some integration by parts.

Fourier series can be scaled, shifted, etc. to produce other Fourier series.

Example: Start with $sq(t) = \begin{cases} -1 & t \in [-\pi, 0) \\ +1 & t \in [0, \pi) \end{cases} \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}$.

Then $1 + sq(t) = \begin{cases} 0 & t \in [-\pi, 0) \\ 2 & t \in [0, \pi) \end{cases} \sim 1 + \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}$.

So $\frac{1}{2}\left[1 + sq(t)\right] = \begin{cases} 0 & t \in [-\pi, 0) \\ 1 & t \in [0, \pi) \end{cases} \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}$, extended periodically for all t, a different sort of square-wave function.

Example: Find the Fourier series for the function $f(t) = \cos(t - \pi/3)$.

Solution: This function is periodic with period 2π. There's no need to consider the formulas for the Fourier coefficients. Simply note that

$$f(t) = \cos(t - \pi/3) = \cos t \cos(\pi/3) + \sin t \sin(\pi/3) = \tfrac{1}{2}\cos t + \tfrac{\sqrt{3}}{2}\sin t.$$

A few applications

1) Variations of this idea provide a method for representing a periodic function (such as a musical sound) not as an analog signal but instead as a discrete list of coefficients. If the Fourier representation is truncated at a point beyond those frequencies that can be discerned by human hearing, this provides a recipe for creating a "digital" representation as a substitute for the analog original which can then be "read" by a device in order to recreate a digital version that is essentially indistinguishable from the original analog recording.

2) In the study of ordinary differential equations (ODEs), it's relatively straightforward to solve inhomogeneous linear ODEs such as $\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0 x = \cos(kt)$ or $\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0 x = \sin(kt)$. Using linear principles, we can then solve ODEs of the form $\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0 x = f(t)$ for a general periodic function $f(t)$ by representing $f(t)$ as a Fourier series, solving the problem term-by-term, and then reassembling the solutions as a linear combination to solve the original problem.

Notes by Robert Winters