A Worked Example of a 2nd Order Linear System with Sinusoidal Input Signal

Problem: Solve the differential equation: $\ddot{x}+3\dot{x}+2x=2\cos(3t), \quad x(0)=2, \quad \dot{x}(0)=3$

Solution: First, the homogeneous equation $\ddot{x}+3\dot{x}+2x=0$ is easy to solve. If we seek an exponential solution $x=e^{rt}$, we have $\ddot{x}+3\dot{x}+2x=r^2e^{rt}+3re^{rt}+2e^{rt}=(r^2+3r+2)e^{rt}=0$. The characteristic polynomial is $p(r)=r^2+3r+2=(r+2)(r+1)$ and yields the two roots $r=-2$ and $r=-1$. This gives the two independent solutions e^{-2t} and e^{-t}, and all homogeneous solutions are of the form $x_h(t)=c_1e^{-2t}+c_2e^{-t}$. Note that both of these homogeneous solutions are <u>transient</u> in the sense that they decay exponentially as t increases.

Next, we need to find a particular solution $x_p(t)$ that satisfies the inhomogeneous differential equation. Once found, we can express the general solution as $x(t)=x_h(t)+x_p(t)$. There are several ways of finding a particular solution.

<u>Method 1</u>: Undetermined Coefficients with trigonometric functions

Examination of the differential equation suggests that we might be able to find a solution of the form $x_p(t)=a\cos(3t)+b\sin(3t)$. We then have to calculate some derivatives, substitute them into the differential equation, and determine which coefficients give the correct right-hand-side for the differential equation.

$$\begin{aligned} x_p(t) &= a\cos(3t)+b\sin(3t) \\ \dot{x}_p(t) &= 3b\cos(3t)-3a\sin(3t) \\ \ddot{x}_p(t) &= -9a\cos(3t)-9b\sin(3t) \end{aligned}$$

So $\ddot{x}_p+3\dot{x}_p+2x_p=(-7a+9b)\cos(3t)+(-9a-7b)\sin(3t)=2\cos(3t)$, and we must then have:

$$\begin{Bmatrix} -7a+9b=2 \\ -9a-7b=0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} a=-\frac{7}{65} \\ b=\frac{9}{65} \end{Bmatrix} \Rightarrow \boxed{x_p(t)=-\frac{7}{65}\cos(3t)+\frac{9}{65}\sin(3t)}$$

We can also express this in terms of a single sinusoidal function with a phase lag, i.e. in the form $A\cos(3t-\phi)$ where $A=\sqrt{(-\frac{7}{65})^2+(\frac{9}{65})^2}=\frac{\sqrt{130}}{65}=\frac{2}{\sqrt{130}}$ and where ϕ is the angle in the 2nd quadrant determined by $\tan\phi=-\frac{9}{7}$, i.e. $\phi\cong 127.87°\cong 2.23$ radians.

<u>Method 2</u>: Undetermined Coefficients and complex replacement

We can simultaneously solve both $\ddot{x}+3\dot{x}+2x=2\cos(3t)$ and $\ddot{y}+3\dot{y}+2y=2\sin(3t)$ by letting $z=x+iy$ and using linearity and Euler's formula with $e^{3it}=\cos(3t)+i\sin(3t)$. By solving $\ddot{z}+3\dot{z}+2z=2e^{3it}$, we'll get both solutions by extracting the real and imaginary parts. We only want the real part in this case.

If we choose to solve $\ddot{z}+3\dot{z}+2z=2e^{3it}$ using undetermined coefficients, the natural choice is to seek a solution of the form $z=Ae^{3it}$. Its derivatives are $\dot{z}=3iAe^{3it}$ and $\ddot{z}=-9Ae^{3it}$. Substitution into the differential equation gives $\ddot{z}+3\dot{z}+2z=(-9+9i+2)Ae^{3it}=(-7+9i)Ae^{3it}=2e^{3it}$, so $(-7+9i)A=2$ and $A=\frac{2}{-7+9i}$.

At this point, we have a choice in how to proceed. We could do the following calculation to remove the complex number from the denominator and apply Euler's formula express the real and imaginary parts in terms of sines and cosines:

$$\begin{aligned} z &= \frac{2}{-7+9i}e^{3it}=\frac{2}{-7+9i}\left(\frac{-7-9i}{-7-9i}\right)e^{3it}=\frac{1}{65}(-7-9i)(\cos(3t)+i\sin(3t)) \\ &= \frac{1}{65}[(-7\cos(3t)+9\sin(3t))+i(-9\cos(3t)-7\sin(3t))] \end{aligned}$$

The real part is then $x_p(t)=-\frac{7}{65}\cos(3t)+\frac{9}{65}\sin(3t)$ which coincides with what we obtained above.

Alternatively, we could have expressed the divisor $-7+9i$ in polar form, i.e. $\sqrt{130}\,e^{i\phi}$ where ϕ is the angle in the 2nd quadrant determined by $\tan\phi=-\frac{9}{7}$, i.e. $\phi\cong 127.87°\cong 2.23$ radians. Then the solution is expressed as $z=\frac{2}{-7+9i}e^{3it}=\frac{2e^{3it}}{\sqrt{130}e^{i\phi}}=\frac{2}{\sqrt{130}}e^{i(3t-\phi)}=\frac{2}{\sqrt{130}}[\cos(3t-\phi)+i\sin(3t-\phi)]$ and the real part gives $x_p=\frac{2}{\sqrt{130}}\cos(3t-\phi)$.

<u>Method 3</u>: Exponential Response Formula
Whenever we have an input signal (right-hand-side of the linear differential equation) that is either exponential or sinusoidal (in which case it can be expressed either as the real part or the imaginary part of a complex exponential) and if the system has constant coefficients, we have a very simple method of finding a particular solution, namely the **Exponential Response Formula**. The only problematic case is when the frequency of the input signal coincides with a root of the characteristic polynomial.

Specifically, if the differential equation is of the form $[p(D)]x(t)=ae^{\omega t}$ where $p(r)$ is the characteristic polynomial and where the input signal frequency ω can be either real or complex, we calculate that with $x(t)=Ae^{\omega t}$, we have $[p(D)](Ae^{\omega t})=Ap(\omega)e^{\omega t}=ae^{\omega t}$, so $Ap(\omega)=a$ and $A=\frac{a}{p(\omega)}$, and therefore $\boxed{x(t)=\frac{a}{p(\omega)}e^{\omega t}}$ gives a particular solution to the differential equation. This is the Exponential Response Formula. It's very simple to apply.

In our example, the characteristic polynomial is $p(r)=r^2+3r+2$, and with complex replacement we can take the input signal frequency to be $\omega=3i$ for the differential equation $\ddot{x}+3\dot{x}+2x=2e^{3it}$. We calculate $p(\omega)=(3i)^2+3(3i)+2=-9+9i+2=-7+9i$, so the particular (complex) solution is $\frac{2}{-7+9i}e^{3it}$ and this gives $\frac{2e^{3it}}{\sqrt{130}e^{i\phi}}=\frac{2}{\sqrt{130}}e^{i(3t-\phi)}=\frac{2}{\sqrt{130}}[\cos(3t-\phi)+i\sin(3t-\phi)]$. The real part, corresponding to a solution to $\ddot{x}+3\dot{x}+2x=2\cos(3t)$, is then $x_p(t)=\frac{2}{\sqrt{130}}[\cos(3t-\phi)]$, as before.

Putting it all together with the initial conditions
We can combine the homogeneous and particular solutions to get the general solution, and then use the initial conditions to determine any unknown constants and determine the unique solution to the initial value problem. How simple this is depends, to some degree, on the chosen format of the particular solution. Specifically, had we chosen the format with sines and cosines, we would have the general solution and its derivative:

$$x(t)=x_h(t)+x_p(t)=c_1e^{-2t}+c_2e^{-t}-\tfrac{7}{65}\cos(3t)+\tfrac{9}{65}\sin(3t)$$
$$\dot{x}(t)=-2c_1e^{-2t}-c_2e^{-t}+\tfrac{21}{65}\sin(3t)+\tfrac{27}{65}\cos(3t)$$

Substitution the initial conditions $x(0)=2,\ \dot{x}(0)=3$ gives

$$\begin{Bmatrix} x(0)=c_1+c_2-\frac{7}{65}=2 \\ \dot{x}(0)=-2c_1-c_2+\frac{27}{65}=3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} c_1+\ c_2=\frac{137}{65} \\ -2c_1-c_2=\frac{168}{65} \end{Bmatrix} \Rightarrow \quad c_1=-\frac{61}{13}, c_2=\frac{34}{5}$$

The solution is therefore $x(t)=-\frac{61}{13}e^{-2t}+\frac{34}{5}e^{-t}-\frac{7}{65}\cos(3t)+\frac{9}{65}\sin(3t)$. It should be emphasized that in this example the homogeneous solutions were transients, so after a should while, the solution essentially coincides with the particular solution $x_p(t)=\frac{2}{\sqrt{130}}[\cos(3t-\phi)]$. In this format, we see that though the input had an amplitude of 2, the response has an amplitude of $\frac{2}{\sqrt{130}}$, so the **gain** is $g=\frac{1}{\sqrt{130}}$ and the **lag** is ϕ as described earlier. It is often helpful to note that $x_p(t)=\frac{2}{\sqrt{130}}[\cos 3(t-\phi/3)]$, so the **time lag** is $\phi/3$.