General Linear Spaces (Vector Spaces) and Solutions of ODEs

Definition: A **vector space** V is a set, with addition and scaling of element defined for all elements of the set, that is closed under addition and scaling, contains a zero element (0), and satisfies the following axioms:
For all $f,g,h \in V$ and scalars c, c_1, c_2

(1) $(f+g)+h = f+(g+h)$
(2) $f+g = g+f$
(3) $f+0 = f$
(4) $f+(-f) = 0$
(5) $c(f+g) = cf+cg$
(6) $(c_1+c_2)f = c_1 f + c_2 f$
(7) $c_1(c_2 f) = (c_1 c_2) f$
(8) $1f = f$

We'll deal initially with the case where the scalars are *real numbers*. Such a vector space is called a **real vector space**. We will also use complex scalars in which case we'll call this a **complex vector space**.

Though we can give definitions and prove theorems about vector spaces in general, it's helpful to develop a library of examples to which we can refer.

1. $\mathbf{R}^n$ is a vector space. Indeed, the motivation for our definition and axioms is to define vector spaces to be spaces which a fundamentally like $\mathbf{R}^n$. All of the required axioms are familiar facts about vectors in $\mathbf{R}^n$.

2. Any subspace of $\mathbf{R}^n$ is a vector space. All of the axioms are inherited and every subspace contains the zero vector, and the definition of subspace ensures that a subspace is closed under addition and scaling.

3. The **complex numbers** $\mathbb{C} = \{a+bi : a,b \text{ are real numbers, } i^2 = -1\}$ can be viewed as a real vector space where addition is defined by $(a_1+b_1 i)+(a_2+b_2 i) = (a_1+a_2)+(b_1+b_2)i$ and scaling by a real number c is defined by $c(a+bi) = ca+cbi$. Note that we do not define multiplication of complex numbers in this context. The complex numbers contain the zero element $0 = 0+0i$ and all of the axioms follow from corresponding facts about real numbers.

4. $\mathbf{R}^{m\times n} = M(m,n) = \{m\times n \text{ matrices with real entries}\}$ is a real vector space with addition and scaling of matrices defined component-wise. The $m\times n$ zero matrix is the zero element and the axioms are all known properties of matrix algebra. Note that in this context we do not define the product of matrices.

5. $F(\mathbf{R},\mathbf{R}) = F(\mathbf{R}) = \{\text{functions } f:\mathbf{R}\to\mathbf{R} \text{ with domain } \mathbf{R}\}$ is a real vector space where addition of functions and scaling of functions are defined by pointwise by $(f+g)(x) = f(x)+g(x)$ and $(cf)(x) = c\,f(x)$. The zero element in this case is the function that is identically zero for all x. (This is quite different than just the real number 0.) Once again, the axioms all follow from familiar facts about real numbers.

6. $P_n = \{\text{real polynomials of degree} \le n\} = \{a_0 + a_1 x + \cdots + a_n x^n : a_0, a_1, \cdots, a_n \in \mathbf{R}\}$ is a real vector space. Note that we must allow all polynomials less than or equal to n because we might add two polynomials (or scale by 0) and get a polynomial of lesser degree.

7. $C^0(\mathbf{R},\mathbf{R}) = C^0(\mathbf{R}) = \{\text{continuous functions } f:\mathbf{R}\to\mathbf{R}\}$ is a real vector space. Closure follows from theorems of Calculus that the sum of continuous functions is continuous and a scalar multiple of a continuous function is also continuous. The zero function is clearly continuous, and the axioms are all easily verified.

Definition: A **subspace** W of a vector space V is a subset that is closed under addition and scaling of elements. That is, for any vectors $\mathbf{v}_1, \mathbf{v}_2 \in W$ and scalars c_1, c_2, it must be the case that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \in W$. We write $W \subseteq V$.

8. $C^1(\mathbf{R},\mathbf{R}) = C^1(\mathbf{R}) = \{\text{differentiable functions } f:\mathbf{R}\to\mathbf{R}\}$ is a real vector space. Closure follows from the theorems of Calculus that $(f+g)' = f'+g'$ and $(cf)' = c\,f'$. The zero function is clearly differentiable.

Note: All polynomials are differentiable, and a (hopefully) familiar theorem of Calculus tells us that differentiable functions must be continuous, so for any *n*, $P_n \subseteq C^1(\mathbf{R}) \subseteq C^0(\mathbf{R}) \subseteq F(\mathbf{R})$.

9. $C^k(\mathbf{R},\mathbf{R}) = C^k(\mathbf{R}) = \{\text{functions } f:\mathbf{R}\to\mathbf{R} \text{ that are at least } k \text{ times differentiable}\}$ is a real vector space. This follows similarly from Calculus theorems. The zero function is differentiable to all orders.

10. $C^\infty(\mathbf{R},\mathbf{R}) = C^\infty(\mathbf{R}) = \{\text{functions } f:\mathbf{R}\to\mathbf{R} \text{ that are differentiable to all orders}\}$ is a real vector space. This also follows from the Calculus theorems above. The zero function is differentiable to all orders.

Note: For any *n*, $P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_{n-1} \subseteq P_n \subseteq C^\infty(\mathbf{R}) \subseteq \cdots \subseteq C^k(\mathbf{R}) \subseteq \cdots \subseteq C^1(\mathbf{R}) \subseteq C^0(\mathbf{R}) \subseteq F(\mathbf{R})$. It's also important to note that when dealing with spaces of functions, these spaces are much "larger" that $\mathbf{R}^n$. To better understand this, we'll need some more definitions, but some of the important details will have to wait until you take course in analysis and topology.

Many of the definitions when working with vectors spaces are essentially the same as those in $\mathbf{R}^n$.

Definition: Given a collection of elements $\{f_1, f_2, \ldots, f_k\} \in V$, we define the **span** of these elements as:

$$\text{span}\{f_1,\ldots,f_k\} = \{c_1 f_1 + \ldots + c_k f_k \text{ where } c_1,\ldots c_k \text{ are scalars}\}.$$

Definition: A set of elements $\{f_1, f_2, \ldots, f_k\} \in V$ is called **linearly independent** if given any linear combination of the form $c_1 f_1 + \cdots + c_k f_k = 0$, this implies that $c_1 = \cdots = c_k = 0$. That is, there is no nontrivial way to combine these vectors to yield the zero element.

Definition: Given a subspace $W \subseteq V$, a collection of elements $\{f_1, f_2, \ldots, f_k\} \in W$ is called a **basis** of *W* if $\text{Span}\{f_1, f_2, \ldots, f_k\} = W$ and $\{f_1, f_2, \ldots, f_k\}$ are linearly independent.

A basis is a minimal spanning set and, as was the case in $\mathbf{R}^n$, if a basis for *W* consists of finitely many elements then any other basis will have the same number of elements, the **dimension** of *W*. It is important to note, however, that it will often be the case, especially in the case of function spaces, that a subspace might not be spanned by finitely many elements.

Coordinates relative to a basis

Definition: If $\mathcal{B} = \{f_1, \cdots, f_k\}$ is a basis for a finite dimensional vector space *V* (or a subspace of *V*), and if $f \in V$, then *f* can be expressed uniquely as $f = c_1 f_1 + \cdots + c_k f_k$ for scalars $\{c_1, \cdots, c_k\}$. These uniquely determined scalars are called the **coordinates of *f* relative to this basis**. If we express these coordinates as a column vector (effectively a vector in $\mathbf{R}^k$), we denote this coordinate vector by $[f]_{\mathcal{B}}$.

Definition: Given two vectors spaces *V* and *W*, a function $T: V \to W$ is called a **linear transformation** if for all elements $f_1, f_2 \in V$ and for all scalars c_1, c_2, *T* satisfies $T(c_1 f_1 + c_2 f_2) = c_1 T(f_1) + c_2 T(f_2)$. This can also be expressed by saying that *T* preserves addition and scalar multiplication. We call the input space *V* the **domain** of *T* and we call the output space *W* the **codomain**.

Definition: Suppose $T: V \to W$ is a linear transformation. We define:

$$\text{image}(T) = im(T) = \{T(f) : f \in V\} \subseteq W \quad \text{and} \quad \text{kernel}(T) = \ker(T) = \{f \in V : T(f) = 0\} \subseteq V$$

These are both subspaces. The argument is the same as we've seen before.

Definition: Given a linear transformation $T:V \to W$, we define: $\text{rank}(T) = \dim(\text{im } T)$ and $\text{nullity}(T) = \dim(\ker T)$ when these subspaces have finite dimension.

We can also state (without proof) the corresponding fact regarding the relationship between rank and nullity.

Rank-Nullity Theorem: If $T:V \to W$ is a linear transformation and V has finite dimension, then $\text{rank}(T) + \text{nullity}(T) = \dim(V)$.

Definition: A linear transformation $T:V \to W$ is called an **isomorphism** if it is one-to-one and onto its codomain. That is, for every $g \in W$ there is a unique $f \in V$ such that $T(f) = g$.

Application to Linear Ordinary Differential Equations

In any space consisting of differentiable functions, the differentiation operator $D(f) = f'$ is a linear transformation. This follows from the Calculus facts that $D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$ and $D(cf) = (cf)' = c\,f' = c\,D(f)$. The kernel of this linear transformation consists of the constant functions. This linear transformation is therefore NOT an isomorphism.

In any space of functions, another important linear transformation is multiplication by a fixed function g. That is, if we define $M_g(f) = gf$, then for any scalars c_1, c_2 and any functions f_1, f_2 we have:

$$M_g(c_1 f_1 + c_2 f_2) = g(c_1 f_1 + c_2 f_2) = c_1 g f_1 + c_2 g f_2 = c_1 M_g(f_1) + c_2 M_g(f_2)$$

If we combine the differentiation operator and various multiplication operators via addition and composition, we can build more complicated linear differential operators.

For example, $D^2 = D \circ D$ is a linear operator corresponding to taking the 2nd derivative.
Similarly, $D^k = D \circ D \circ \cdots \circ D$ corresponds with taking the kth derivative.

If $p(t)$ is a function, then $T = D + pI$ is a first order linear differential operator calculated by:

$$T(x(t)) = (D + pI)(x(t)) = \tfrac{dx}{dt} + p(t)x(t)$$

More generally, given functions $p_{n-1}(t), \ldots, p_1(t), p_0(t)$, we can form the linear differential operator $T = D^n + p_{n-1}D^{n-1} + \cdots + p_1 D + p_0 I$. When applied to a sufficiently differential function $x(t)$, this gives:

$$T(x(t)) = (D^n + p_{n-1}D^{n-1} + \cdots + p_1 D + p_0 I)(x(t)) = \tfrac{d^n x}{dt^n} + p_{n-1}(t)\tfrac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\tfrac{dx}{dt} + p_0(t)x(t)$$

Solving an nth order linear ODE of the form $\tfrac{d^n x}{dt^n} + p_{n-1}(t)\tfrac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\tfrac{dx}{dt} + p_0(t)x(t) = q(t)$ can thus be seen as solving $T(x(t)) = q(t)$. In the case of a homogeneous linear ODE, the becomes simply $T(x(t)) = 0$, i.e. we are trying to find the kernel of the linear differential operator.

Perhaps the greatest benefit of this formulation is that if we can find several (linearly independent) homogeneous solutions $x_1(t), \ldots x_k(t)$ [that is, $T(x_1(t)) = T(x_2(t)) = \cdots = T(x_k(t)) = 0$], then for any scalars $c_1, \ldots c_k$, we have (by linearity) $T(c_1 x_1(t) + \cdots + c_k x_k(t)) = c_1 T(x_1(t)) + \cdots + c_k T(x_k(t)) = 0$. That is, any linear combination of solutions will also be a solution [Principle of Superposition].

Linear ODEs with constant coefficients

An important (and easily solvable) case is where all of the coefficient functions are constant, i.e. an operator of the form $T = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 I$ where $a_{n-1}, \ldots, a_1, a_0$ are constants. We'll first consider the homogeneous case, i.e. solving a homogeneous linear ODE with constant coefficients of the form $\tfrac{d^n x}{dt^n} + a_{n-1}\tfrac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1\tfrac{dx}{dt} + a_0 x = 0$. We can generate solutions to such an ODE by (initially) considering exponential functions of the form $x(t) = e^{rt}$ where r is constant. Substituting this into the ODE gives:

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1\frac{dx}{dt} + a_0 x = r^n e^{rt} + a_{n-1}r^{n-1}e^{rt} + \cdots + a_1 r e^{rt} + a_0 e^{rt} = (r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0)e^{rt} = 0$$

The polynomial $P(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0$ is called the characteristic polynomial of this ODE. Any root of this polynomial will yield an exponential solution to the ODE, and the Fundamental Theorem of Algebra guarantees *n* roots, though some may be complex and some may be repeated. The roots of the characteristic polynomial are called the characteristic roots or characteristic values of the ODE. Our strategy will be to try to create a basis out of such exponential solutions from which ALL homogeneous solutions can be constructed.

First Order Example: If we solve $\frac{dx}{dt} - kx = 0$, we have the characteristic polynomial $P(r) = r - k = 0$ and characteristic value $r = k$. This yields the exponential solution e^{kt}. By linearity, $x(t) = ce^{kt}$ will also be a solution and, in fact, this yields ALL homogeneous solutions. That is $\{e^{kt}\}$ is a basis for all such solutions.

Second Order Example: Suppose we want to solve the ODE $\boxed{\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0}$. The corresponding linear differential operator is $T = D^2 + 3D + 2I$ and the characteristic polynomial is $P(r) = r^2 + 3r + 2 = (r+2)(r+1)$. This yields the characteristic values $r = -2$ and $r = -1$ and exponential solutions e^{-2t} and e^{-t}. By linearity, any function of the form $x(t) = c_1 e^{-2t} + c_2 e^{-t}$ will also be a homogeneous solution for any scalars c_1, c_2. Does this give ALL solutions? That is, does $\{e^{-2t}, e^{-t}\}$ constitute a basis for all solutions?

We can show that this is actually the case by thinking of $T = D^2 + 3D + 2I = (D + 2I) \circ (D + I)$, i.e. as a composition of first-order linear operators. We are seeking a solution $x(t)$ such that $(D+2I)[(D+I)(x(t))] = 0$. If we let $y(t) = (D+I)(x(t))$, then we can first solve $(D+2I)[y(t))] = \dot{y} + 2y = 0$ to get all solutions of the form $y(t) = c_1 e^{-2t}$. Next, for any such solution we'll have $(D+I)(x(t)) = \dot{x} + x = c_1 e^{-2t}$. This is an inhomogeneous first-order ODE. The homogeneous solutions are all of the form $x_h(t) = c_2 e^{-t}$ and we can use the Method of Undetermined Coefficients to produce a particular solution to the inhomogeneous equation, i.e. $x_p(t) = Ae^{-2t}$ for some scalar *A*. Substitution gives $\dot{x} + x = (-2A + A)e^{-2t} = -Ae^{-2t} = c_1 e^{-2t}$, so $A = -c_1$. We then add the homogeneous and particular solutions to get that all solutions must be of the form $x(t) = c_2 e^{-t} - c_1 e^{-2t}$. If we now simply rename the coefficients we see that all solutions are of the form $x(t) = c_1 e^{-2t} + c_2 e^{-t}$, i.e. $\{e^{-2t}, e^{-t}\}$ does constitute a basis for all solutions. (It's straightforward to show that these functions are linearly independent.)

The above argument works the same in the case of an *n*th order linear homogeneous ODE with constant coefficients – as along as all the roots of the characteristic polynomial are distinct. It's even valid in the case of complex roots (though it's generally preferable to use Euler's Formula to re-express the exponential solutions in terms of sine and cosine functions). Repeated roots have to be handled differently. An example will illustrate what should be done in that case.

Repeated root example: Suppose we want to solve the ODE $\boxed{\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0}$. The corresponding linear differential operator is $T = D^2 + 4D + 4I$ and the characteristic polynomial is $P(r) = r^2 + 4r + 4 = (r+2)^2$. This yields only the characteristic value $r = -2$ with corresponding exponential solutions e^{-2t}. Does $x(t) = c_1 e^{-2t}$ yield ALL homogeneous solutions, i.e. does $\{e^{-2t}\}$ constitute a basis for all solutions? The answer is NO. If we proceed in the same manner as the previous example, we write $(D+2I)[(D+2I)(x(t))] = 0$ and let $y(t) = (D+2I)(x(t))$. We solve $(D+2I)[y(t))] = \dot{y} + 2y = 0$ to get $y(t) = c_1 e^{-2t}$ and then seek solutions of $(D+2I)(x(t)) = \dot{x} + 2x = c_1 e^{-2t}$. In this case, it may not be so clear how to use Undetermined Coefficients to guess a particular solution to this inhomogeneous equation. We can use Variation of Parameters to find solutions. If we see a solution of the form $x_p(t) = v(t)e^{-2t}$, the method gives that $v'(t) = \frac{c_1 e^{-2t}}{e^{-2t}} = c_1$, so $v(t) = c_1 t + c_2$ (we actually don't need to include the $+c_2$ if we're just seeking a particular solution). So we can choose $x_p(t) = c_1 te^{-2t}$. The homogeneous solutions are all of the form $x_h(t) = c_2 e^{-2t}$. Therefore, ALL solutions are of the form $x(t) = c_1 te^{-2t} + c_2 e^{-2t}$. That is, $\{e^{-2t}, te^{-2t}\}$ forms a basis for ALL solutions.

This can be generalized for any repeated root of any multiplicity. For example, a similar calculation shows that for the homogeneous linear ODE $(D+3I)^3(x(t)) = 0$, $\{e^{-3t}, te^{-3t}, t^2 e^{-3t}\}$ gives a basis for all solutions. That is, all solutions are expressible (uniquely) in the form $x(t) = c_1 e^{-3t} + c_2 te^{-3t} + c_3 t^2 e^{-3t}\}$

As long as we can easily factor the characteristic polynomial, the above example illustrate how we can simply generate a basis for all homogeneous solutions for in the case of constant coefficients.

Example: Find all solutions to the ODE: $\frac{d^4x}{dt^4} + 2\frac{d^3x}{dt^3} + 5\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 4x = 0$.

Solution: The characteristic polynomial in the case is $P(r) = r^4 + 2r^3 + 5r^2 + 8r + 4 = (r+1)^2(r^2+4) = 0$. The characteristic values are $r = -1$ (with multiplicity 2), $r = 2i$, and $r = -2i$. From these we get a basis for all solutions $\{e^{-t}, te^{-t}, e^{2it}, e^{-2it}\}$. We could express all solutions in the form $x(t) = c_1 e^{-t} + c_2 te^{-t} + c_3 e^{2it} + c_4 e^{-2it}$ where we necessarily have to allow for complex coefficients. However, if we note (using Euler's Formula) that $e^{2it} = \cos(2t) + i\sin(2t)$ and $e^{-2it} = \cos(2t) - i\sin(2t)$ we can easily see that we could also (perhaps more simply) use $\{e^{-t}, te^{-t}, \cos 2t, \sin 2t\}$ as a basis for all solutions, i.e. all solutions may be expressed in the form $x(t) = c_1 e^{-t} + c_2 te^{-t} + c_3 \cos 2t + c_4 \sin 2t$ where all the coefficients are real scalars.

Our next several tasks will be to consider inhomogeneous equations and initial value problems. This will necessitate a slightly more nuanced view of what we mean by linearly independent functions for the purpose of finding unique solutions to well-posed initial value problems. This will involve what is known as the *Wronskian* associated with a given basis of homogeneous solutions.

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