Math E-21c – Linear nth Order ODE Cookbook

Study guide

1. **Linear Models**. A linear differential equation is one of the form $a_n(t)x^{(n)} + \cdots + a_1(t)x' + a_0(t)x = q(t)$. The $a_k(t)$ are coefficients functions. The left side models a <u>system</u>, $q(t)$ arises from an input <u>signal</u>, and solutions $x(t)$ provide the system <u>response</u>. In this course we mainly focus on the time-invariant case where the coefficient functions are all constant. In this case the equation can be written in terms of the characteristic polynomial $p(s) = a_n s^n + \cdots + a_1 s + a_0$ as $p(D)x = q(t)$. However, some of the ideas developed are also applicable to the more general case, e.g. variation of parameters for finding particular solutions.

   <u>Spring system</u>: If $m\ddot{x} + c\dot{x} + kx = F_{ext}(t)$ with $m > 0$, $b, k \geq 0$, and an external driving force $F_{ext}(t)$, we can rewrite this as $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}F_{ext}(t)$ with characteristic polynomial $p(s) = s^2 + \frac{c}{m}s + \frac{k}{m}$. The system response $x(t)$ gives the position of the mass. If driven directly, $q(t) = \frac{1}{m}F_{ext}(t)$. If driven through the spring, $q(t) = \frac{k}{m}y(t)$ (where $y(t)$ is the position of the far end of the spring). If driven through the dashpot, $q(t) = \frac{c}{m}\dot{y}$ (where $y(t)$ = position of far end of dashpot).
   [**Note**: We did not go into this level of fine detail in class about *how* the system was driven.]

2. **Homogeneous Equations**. The "mode" e^{rt} solves $p(D)x = 0$ exactly when $p(r) = 0$. If *r* is a double root one needs te^{rt} also (and t^2e^{rt}, etc. if the root has greater multiplicity). The general solution is a linear combination of these. If the coefficients are real and if the roots are complex, i.e. $r = a \pm bi$ with $b \neq 0$, then $e^{at}\cos bt$ and $e^{at}\sin bt$ are independent real solutions. If all roots have negative real part then all solutions decay to zero as $t \to \infty$ and are called *transients*. In the spring case with $p(s) = s^2 + \frac{c}{m}s + \frac{k}{m}$ with $m > 0$ and $b, k \geq 0$, the characteristic roots are $s = \dfrac{-c \pm \sqrt{c^2 - 4km}}{2m}$. The equation is *overdamped* if the roots are real and distinct ($c^2 - 4km > 0$), *underdamped* if the roots are complex ($c^2 - 4km < 0$), and *critically damped* if there is just one (repeated) root ($c^2 - 4km = 0$). In the underdamped case the general solution is $Ae^{-ct/2m}\cos(\omega_d t - \phi)$ where $\omega_d = \frac{1}{2m}\sqrt{4km - c^2}$ is the *damped circular frequency* and ϕ is a phase angle.

3. **Linearity**. In addition to earlier observations about linearity, we also have the following <u>superposition principle</u>: if $p(D)x_1 = q_1(t)$ and $p(D)x_2 = q_2(t)$, then $x = c_1x_1 + c_2x_2$ solves $p(D)x = c_1q_1(t) + c_2q_2(t)$ (where c_1, c_2 are constants).
   <u>1st Consequence</u>: The general solution to $p(D)x = q(t)$ is $x_h + x_p$ where x_h is the general solution to the homogeneous equation $p(D)x = 0$ and x_p is any particular solution to the inhomogeneous equation $p(D)x = q(t)$.
   <u>2nd Consequence</u>: If a particular solution to an equation of the form $p(D)x = c_1q_1(t) + c_2q_2(t)$ is needed where $q_1(t)$ and $q_2(t)$ are dissimilar functions (e.g. polynomial and trigonometric), we can separately solve $p(D)x_1 = q_1(t)$ and $p(D)x_2 = q_2(t)$ for particular solutions x_1 and x_2, and then put them together to get a particular solution $x = c_1x_1 + c_2x_2$ to the equation $p(D)x = c_1q_1(t) + c_2q_2(t)$.

4. **Exponential Response formula**: If $p(r) \neq 0$ then $\dfrac{ae^{rt}}{p(r)}$ solves $p(D)x = ae^{rt}$. If $p(r) = 0$ but $p'(0) \neq 0$ then $\dfrac{ate^{rt}}{p'(r)}$ solves $p(D)x = ae^{rt}$. If $p(r) = p'(r) = 0$ but $p''(0) \neq 0$ then $\dfrac{at^2e^{rt}}{p''(r)}$ solves $p(D)x = ae^{rt}$, etc. These latter cases are known as the **Resonant Response Formula(s)**.

5. **Complex Replacement**: If $p(s)$ has real coefficients then solutions of $p(D)x = Ae^{rt}\cos(\omega t)$ are real parts of solutions of $p(D)x = Ae^{(r+i\omega)t}$. Solutions to $p(D)x = Ae^{rt}\sin(\omega t)$ may be found from the imaginary parts. This is a particularly useful method when used in conjunction with the Exponential Response Formula.

6. **Undetermined Coefficients (and reduction of order)**: With $p(s) = a_n s^n + \cdots + a_1 s + a_0$, if $a_0 \neq 0$ then $p(D)x = b_k t^k + \cdots + b_1 t + b_0$ has a polynomial (particular) solution, which has degree at most *k*. If a_k is the first nonzero coefficient (for example, in the equation $\dddot{x} + 3\ddot{x} = t^5$ we would have $k = 2$), you can make the substitution $u = x^{(k)}$ and proceed ("reduction of order") to determine $u(t)$. For a particular solution $x_p(t)$ you can take any constants of integration to be zero.

7. **Exponential Shift Rule**: To solve $p(D)x = q(t)e^{rt}$, try $x = u(t)e^{rt}$. This leads to a different equation for $u(t)$ with right hand side $q(t)$. You can then use a method like Undetermined Coefficients or Complex Substitution to find $u(t)$ and thus find the particular solution $x_p(t) = u(t)e^{rt}$. This procedure can be formalized as the Exponential Shift Rule. Specifically, suppose we wish to solve an ODE of the form $[p(D)]x(t) = e^{rt}q(t)$ where $p(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 I$ is a linear differential operator with constant coefficients. If $u(t)$ is a solution of the ODE $[p(D + rI)]u(t) = q(t)$, then $x(t) = e^{rt}u(t)$ will solve $[p(D)]x(t) = e^{rt}q(t)$.

8. **Variation of Parameters**: When other simpler methods are unavailable to find a particular solution to a linear ODE of the form $a_n(t)x^{(n)} + \cdots + a_1(t)x' + a_0(t)x = R(t)$, and if you have found a full complement of independent homogeneous solutions $x_1(t), \cdots, x_n(t)$, then you can try a solution of the form $x = v_1 x_1 + \cdots v_n x_n$ where $v_1(t), \cdots, v_n(t)$ are undetermined functions. By imposing additional conditions on the derivatives, you can then solve a system of equations for $\dot{v}_1(t), \cdots, \dot{v}_n(t)$ and integrate to find $v_1(t), \cdots, v_n(t)$.

   In the 2nd order case, with the ODE $\ddot{x} + p_1(t)\dot{x} + p_0(t)x = R(t)$, this leads to the system of equations $\left\{ \begin{matrix} x_1\dot{v}_1 + x_2\dot{v}_2 = 0 \\ \dot{x}_1\dot{v}_1 + \dot{x}_2\dot{v}_2 = R(t) \end{matrix} \right\}$ or, in matrix form, $\begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix}\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ R(t) \end{bmatrix}$ and $\dot{v}_1 = -\frac{x_2 R}{W}$, $\dot{v}_2 = \frac{x_1 R}{W}$ where $W = W(t) = \begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} = x_1\dot{x}_2 - x_2\dot{x}_1$ is the Wronskian determinant. It should be emphasized that if other, simpler methods can be used to find a particular solution, you may wish to try those first.

9. **Time Invariance**: If $p(D)x = q(t)$, then $y = x(t-a)$ solves $p(D)y = q(t-a)$. This allows you to solve the simpler, more standard ODE $p(D)x = q(t)$ first and then substitute to get the desired solution.

10. **Frequency Response**: An input signal $y(t)$ determines $q(t)$ in $p(D)x = q(t)$, e.g. a diffusion problem where $p(D)x = k\,y(t) = q(t)$. With $y = y_{cx} = e^{i\omega t}$, an exponential system response has the form $H(\omega)e^{i\omega t}$ for some complex number $H(\omega)$, calculated using ERF. (If ERF fails then the complex gain is infinite.) Then with $y = A\cos(\omega t)$, $x_p = g\cos(\omega t - \phi)$ where $g = |H(\omega)|$ is the gain and $\phi = -Arg(H(w))$ is the phase lag. By time invariance the gain and phase lag are the same for any sinusoidal input signal of frequency ω.