**Supplement on Linear Coordinates, Vectors and Matrices Relative to a Basis, with Applications to Solving Systems of 1st Order Linear Differential Equations**

Though everyone most likely believes they understand vectors and matrices, there are some subtle aspects that warrant additional explanation. For example, if your concept of a vector in $\mathbf{R}^n$ is "an ordered n-tuple" or some similar definition, then this doesn't really hold up objectively. If you were to change units, for example, the components of the vector might be completely different but still represent the same vector. In physics, the acceleration due to Earth's gravity is a vector pointing downward, but is it 32 ft/sec$^2$ or 9.8 m/sec$^2$? The numerical value of the downward component can be different things depending on what coordinates you choose.

The same ambiguity applies to the description of functions. If we have a function represented as $y = f(x) = x^2$, think about what happens if you change coordinates by letting $x = u - 3$ and $y = 12v$. The former is a shift in the horizontal axis, and the latter is a linear change of scale. Substituting, we get $12v = (u-3)^2$ or $v = \frac{1}{12}(u-3)^2 = \tilde{f}(u)$. In the new coordinates, the function has a different algebraic representation (formula), but it still describes the same parabola. There's an interesting way to think about this in terms of the functions that determine the coordinate changes. If we write $x = h(u) = u - 3$ and $y = k(v) = 12v$, then $\tilde{f} = k^{-1} \circ f \circ h$. This can be schematically understood via the following diagram (where the variables are appended for guidance):

$$\begin{array}{ccc} \{\mathbf{R}, x\} & \xrightarrow{\ f\ } & \{\mathbf{R}, y\} \\ h \uparrow & & \uparrow k \\ \{\mathbf{R}, u\} & \xrightarrow{\ \tilde{f}\ } & \{\mathbf{R}, v\} \end{array}$$

A given function can, in fact, be represented in arbitrarily many different ways. We require only that the appropriate coordinate changes be understood and that the relationship between different representations be determined by a diagram such as the one above. If so, we'll say that $f$ and $\tilde{f}$ are <u>equivalent</u>.

This is somewhat simplified in the case where the coordinate change is the same in both the domain and range of a given function. If this change is given by a function $h$, we get the simpler relation $\tilde{f} = h^{-1} \circ f \circ h$ and the corresponding simplified schematic:

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{\ f\ } & \mathbf{R} \\ h \uparrow & & \uparrow h \\ \mathbf{R} & \xrightarrow{\ \tilde{f}\ } & \mathbf{R} \end{array}$$

Let's focus on how this plays out in the context of vectors and matrices. In our standard view of $\mathbf{R}^n$ we can think of a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + \cdots x_n\mathbf{e}_n$ where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, …, $\mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$. We refer to the numbers $\{x_1, x_2, \ldots, x_n\}$ as the <u>standard coordinates</u> of the vector. In Linear Algebra we'd say that these are the coordinates of the vector relative to the <u>standard basis</u> $\mathcal{E} = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$. Some Linear Algebra definitions are in order here. We call a subset $V$ of $\mathbf{R}^n$ a <u>subspace</u> if it is closed under vector addition and scalar multiplication. We call $\text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ the set of all vectors you can build out of a given set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ by scaling and adding, i.e. $\{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k : c_1, \ldots, c_k \text{ scalars}\}$. A single nonzero vector spans a line through the origin. Two nonparallel vectors in $\mathbf{R}^n$ span a plane through the origin. Any subspace is typically described by providing a spanning set. We say that a set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ are <u>linearly independent</u> if it's impossible to express any

one of them as a linear combination of the others. This is the same as saying that if $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ then necessarily $c_1 = \cdots = c_k = 0$. If a subspace $V = \text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$, and if $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ are linearly independent, we call this a basis for *V*. It can be shown that the number of vectors in a basis for a subspace *V* is always the same and this number is called the dimension of *V*. It is important to note that $\mathbf{R}^n$ itself is a subspace of dimension *n*. In addition to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ for $\mathbf{R}^n$, any collection $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ of *n* linearly independent vectors in $\mathbf{R}^n$ will form a basis for $\mathbf{R}^n$.

It can be easily shown that if $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis for $\mathbf{R}^n$, then any vector $\mathbf{x}$ in $\mathbf{R}^n$ can be expressed uniquely as $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{S}[\mathbf{x}]_{\mathcal{B}}$. The matrix $\mathbf{S} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}$ is called the change of basis matrix. It is necessarily invertible.

The vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ gives the coordinates of **x** relative to the basis $\mathcal{B}$.

Note that $\mathbf{x} = [\mathbf{x}]_{\mathcal{E}} = \mathbf{S}[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{x}$. These tell us how to change coordinates.

**Example**: In $\mathbf{R}^2$, the vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ form a basis $\mathcal{B}$ for $\mathbf{R}^2$. If we write $\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then for a vector such as $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$, we can calculate its coordinates relative to the basis $\mathcal{B}$ by $[\mathbf{x}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{x} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 13 \\ -11 \end{bmatrix} = \begin{bmatrix} 13/3 \\ -11/3 \end{bmatrix}$. You can verify that $\frac{13}{3}\mathbf{v}_1 - \frac{11}{3}\mathbf{v}_2 = \mathbf{x}$.

An *n* by *n* matrix **A** represents a linear function from $\mathbf{R}^n$ to $\mathbf{R}^n$, and matrix multiplication corresponds with composition of these linear functions, i.e. $(\mathbf{AB})\mathbf{x} = \mathbf{A}(\mathbf{Bx})$. We can use the facts that $\mathbf{x} = [\mathbf{x}]_{\mathcal{E}} = \mathbf{S}[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{x}$ in conjunction with our earlier observations about coordinate changes to define not only the idea of the coordinates of a vector relative to a basis, but also the idea of the matrix of a linear function relative to a basis. Specifically, if we let $\{\mathbf{R}^n, \mathcal{E}\}$ represent $\mathbf{R}^n$ with the coordinates of vectors expressed in terms of the standard basis $\mathcal{E}$, and let $\{\mathbf{R}^n, \mathcal{B}\}$ represent $\mathbf{R}^n$ with the coordinates of vectors expressed in terms of a different basis $\mathcal{B}$, and if we use the notation $[\mathbf{A}]_{\mathcal{B}}$ for the matrix representation of the linear function corresponding to matrix **A** but relative to the basis $\mathcal{B}$, then we have the following schematic:

$$\begin{array}{ccc} \{\mathbf{R}^n, \mathcal{E}\} & \xrightarrow{\mathbf{A}} & \{\mathbf{R}^n, \mathcal{E}\} \\ \mathbf{S}\uparrow & & \uparrow\mathbf{S} \\ \{\mathbf{R}^n, \mathcal{B}\} & \xrightarrow{[\mathbf{A}]_{\mathcal{B}}} & \{\mathbf{R}^n, \mathcal{B}\} \end{array}$$

From this we observe that $[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{AS}$. This is a extremely important result.

$[\mathbf{A}]_{\mathcal{B}}$ can also be calculated directly for a given basis $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ as $[\mathbf{A}]_{\mathcal{B}} = \begin{bmatrix} \uparrow & & \uparrow \\ [\mathbf{Av}_1]_{\mathcal{B}} & \cdots & [\mathbf{Av}_n]_{\mathcal{B}} \\ \downarrow & & \downarrow \end{bmatrix}$.

Let's look at what this means in the case of a matrix **A** for which we are able to find a basis $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ for $\mathbf{R}^n$ consisting of eigenvectors of **A** with corresponding eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Note that

$\left\{\begin{matrix} \mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1 \\ \vdots \\ \mathbf{A}\mathbf{v}_n = \lambda_n\mathbf{v}_n \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \mathbf{A}\mathbf{S}\mathbf{e}_1 = \lambda_1\mathbf{S}\mathbf{e}_1 \\ \vdots \\ \mathbf{A}\mathbf{S}\mathbf{e}_n = \lambda_1\mathbf{S}\mathbf{e}_n \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{e}_1 = \lambda_1\mathbf{e}_1 \\ \vdots \\ \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{e}_n = \lambda_n\mathbf{e}_n \end{matrix}\right\}$. If we interpret what this says about the matrix $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, it says that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ must be the diagonal matrix $\mathbf{D} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$, i.e. $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$. That is, the matrix of this linear function relative to this basis of eigenvalues is a diagonal matrix. This can also be observed directly. This is why we say that a matrix **A** is <u>diagonalizable</u> if it yields a basis $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ for $\mathbf{R}^n$ consisting of eigenvectors of **A**. In essence, when a basis of eigenvectors can be found, the matrix relative to that basis will be the simplest possible.

It is not always possible to find a basis of eigenvectors of a given matrix **A**.

**Application to Solving a System of 1st Order Linear Differential Equations**

The tool at the heart of these methods is diagonalization or, in the case where a matrix cannot be diagonalized, finding an appropriate change of basis relative to which the underlying linear transformation has the simplest possible matrix representation, i.e. Jordan Canonical Form. A second useful formalism is the use of "evolution matrices."

Suppose **S** is a change of basis matrix corresponding to either diagonalization or reduction to Jordan Canonical Form (the simplest possible form). We will have $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{B}$ in this case, where **B** is diagonal or otherwise in simplest form. We then calculate $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$, and substitution gives $\frac{d\mathbf{x}}{dt} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}\mathbf{x}$. Multiplying on the left by $\mathbf{S}^{-1}$ and using the basic calculus fact that $\frac{d}{dt}(\mathbf{M}\mathbf{x}) = \mathbf{M}\frac{d\mathbf{x}}{dt}$ for any (constant) matrix **M**, we have

$$\mathbf{S}^{-1}\frac{d\mathbf{x}}{dt} = \frac{d(\mathbf{S}^{-1}\mathbf{x})}{dt} = \mathbf{B}(\mathbf{S}^{-1}\mathbf{x}).$$

If we write $\mathbf{u} = \mathbf{S}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$, where $\mathcal{B}$ is the new, preferred basis, then in these new coordinates the system becomes $\frac{d\mathbf{u}}{dt} = \mathbf{B}\mathbf{u}$, but now the system will be much more straightforward to solve.

**The diagonalizable case**

In the case where **B** is a diagonal matrix with the eigenvalues of **A** on the diagonal, the system is just

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\mathbf{u} \text{ or } \left\{\begin{matrix} \frac{du_1}{dt} = \lambda_1 u_1 \\ \vdots \\ \frac{du_n}{dt} = \lambda_n u_n \end{matrix}\right\}.$$

This has the solution $\left\{\begin{matrix} u_1(t) = e^{\lambda_1 t}u_1(0) \\ \vdots \\ u_n(t) = e^{\lambda_n t}u_n(0) \end{matrix}\right\}$ or $\begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} u_1(0) \\ \vdots \\ u_n(0) \end{bmatrix}$.

If we use the shorthand notation $[e^{t\mathbf{B}}] = \text{Exp}(t\mathbf{B}) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$, sometimes referred to as the (time-varying) evolution matrix for the simplified system, we can succinctly write the solution as $\mathbf{u}(t) = [e^{t\mathbf{B}}]\mathbf{u}(0)$. To revert back to the original coordinates, we write $\mathbf{x} = \mathbf{S}\mathbf{u}$, so $\mathbf{x}(t) = \mathbf{S}\mathbf{u}(t) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{u}(0) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. If we denote the evolution matrix for the system in its original coordinates as $[e^{t\mathbf{A}}] = Exp(t\mathbf{A})$ where $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$, then the previous calculation gives the simple relation $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$.

In other words, the evolution matrices for the solution are in the same relationship as the matrices **A** and **B**, namely $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$. This pattern is very easy to remember, and this same pattern will again be the case where **B** is not diagonal but where the corresponding evolution matrix is still relatively easy to calculate.

$\boxed{\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1} \quad \Rightarrow \quad [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}}$, and the solution of the original system will be $\boxed{\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)}$.

**The complex eigenvalue case**

Suppose we want to solve a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where **A** is an 2 × 2 real matrix with a complex conjugate pair of eigenvalues $\lambda = a + ib$ and $\lambda = a - ib$. There are several reasonable ways to proceed, but they all come down to determining the evolution matrix $[e^{t\mathbf{A}}]$ so that we can solve for $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$.

First, put the system into (real) normal form.

Use the complex eigenvalue $\lambda = a + ib$ to find a complex eigenvector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$. If we change to the basis $\{\mathbf{v}, \mathbf{u}\}$ (note the reversal of order) then, using the change of basis matrix $\mathbf{S} = \begin{bmatrix} \mathbf{v} & \mathbf{u} \end{bmatrix}$, we'll get $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, a rotation-dilation matrix. Noting, as before, that $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1} \quad \Rightarrow \quad [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$, we need only to determine $[e^{t\mathbf{B}}]$.

Second, find the evolution matrix for the (real) normal form.

In fact, $[e^{t\mathbf{B}}] = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$, a time-varying rotation matrix with exponential scaling. This yields a trajectory that spirals out in the case where $\text{Re}(\lambda) = a > 0$ (look to the original vector field to see whether it's clockwise or counterclockwise), or a trajectory that spirals inward toward **0** in the case where $\text{Re}(\lambda) = a < 0$.

To derive this expression for $[e^{t\mathbf{B}}]$, make another coordinate change with complex eigenvectors starting with $\mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. We know this has the same eigenvalues of **A**, namely $\lambda = a + ib$ and $\lambda = a - ib$. Use $\lambda = a + ib$ to get the complex eigenvector $\mathbf{w} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. The eigenvalue $\lambda = a - ib$ will then give eigenvector $\widehat{\mathbf{w}} = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Using the (complex) change of basis matrix $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$, we have that $\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \mathbf{D} = \begin{bmatrix} a + ib & 0 \\ 0 & a - ib \end{bmatrix}$. It follows that:

$$[e^{t\mathbf{B}}] = \mathbf{P}[e^{t\mathbf{D}}]\mathbf{P}^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{(a+ib)t} & 0 \\ 0 & e^{(a-ib)t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} = e^{at}\begin{bmatrix} \frac{e^{ibt}+e^{-ibt}}{2} & -\frac{e^{ibt}-e^{-ibt}}{2i} \\ \frac{e^{ibt}-e^{-ibt}}{2i} & \frac{e^{ibt}+e^{-ibt}}{2} \end{bmatrix} = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}.$$

These calculations enable us to write down a closed form expression for the solution of this linear system, namely $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$ where $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1} = e^{at}\mathbf{S}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}\mathbf{S}^{-1}$. However, the more important result is the ability to qualitatively describe the trajectories for this system by knowing only the real part of the eigenvalues of the matrix **A** and the direction of the corresponding vector field (clockwise vs. counterclockwise).

**Repeated eigenvalues** (with geometric multiplicity less than the algebraic multiplicity)

Suppose we want to solve a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where **A** is a non-diagonalizable $2 \times 2$ real matrix with a repeated eigenvalue $\lambda$. In this case, we can always find a change of basis matrix **S** such that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. The eigenvalue $\lambda$ will produce one eigenvector $\mathbf{v}_1$, but not a second eigenvector. It is, however, always possible to find a second vector $\mathbf{v}_2$ such that $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2$ (simply solve the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$ or, equivalently, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_2 = -\mathbf{v}_1$ using the previously found eigenvector $\mathbf{v}_1$). As in the previous two cases, $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1} \Rightarrow [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$ and it comes down to finding $[e^{t\mathbf{B}}]$. This is perhaps most easily done by explicitly solving the corresponding differential equations. In the new coordinates, this system translates into $\left\{ \begin{array}{l} \frac{du_1}{dt} = \lambda u_1 + u_2 \\ \frac{du_2}{dt} = \lambda u_2 \end{array} \right\}$. The second equation is easily solved to get $u_2(t) = e^{\lambda t}u_2(0)$. We can guess a solution for the first equation of the form $u_1(t) = c_1 te^{\lambda t} + c_2 e^{\lambda t}$. Differentiating this and substituting into the first equation, we get $c_1(e^{\lambda t} + \lambda te^{\lambda t}) + c_2\lambda e^{\lambda t} = \lambda(c_1 te^{\lambda t} + c_2 e^{\lambda t}) + e^{\lambda t}u_2(0)$. Comparing like terms, we conclude that $c_1 = u_2(0)$. Substituting $t = 0$, we further conclude that $u_1(0) = c_2$. Putting these results together, we get $u_1(t) = u_2(0)te^{\lambda t} + u_1(0)e^{\lambda t} = e^{\lambda t}u_1(0) + te^{\lambda t}u_2(0)$. We therefore have that

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda t}u_1(0) + te^{\lambda t}u_2(0) \\ e^{\lambda t}u_2(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\mathbf{u}(0)$$

So, $[e^{t\mathbf{B}}] = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$ in this case and the solution is given by $\mathbf{x}(t) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1} = \mathbf{S}\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\mathbf{S}^{-1}\mathbf{x}(0)$.

An alternate method of deriving this was done in class.

Similar calculations enable us to deal with cases such as a repeated eigenvalue where the geometric multiplicity is 1 and the algebraic multiplicity is 3 (or even worse).

Finally, an actual system may exhibit several of these qualities – one or more complex pairs of eigenvalues, repeated eigenvalues, and distinct real eigenvalues. The Jordan Canonical Form of the matrix for such a system can be analyzed block by block and each of the above solutions applied within each block to determine the evolution matrix for the entire system.

**Exercise**: a) Find the general solution for the following system of differential equations:

$$\left\{ \begin{array}{l} \dot{x}_1 = 2x_1 - 4x_4 + 3x_5 \\ \dot{x}_2 = 2x_2 - 2x_3 + 2x_4 \\ \dot{x}_3 = x_2 - x_4 \\ \dot{x}_4 = -x_4 \\ \dot{x}_5 = -3x_4 + 2x_5 \end{array} \right\}$$

b) Find the solution in the case where $\mathbf{x}(0) = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$.