**Math E-21c – Ordinary Differential Equations – Lecture #9**

## Generalized Functions, Distributions, and Laplace Transform Methods

**The Main Idea**: Beginning with a **linear *n*th order ODE with initial conditions** (an initial value problem), we'll transform this into an **algebraic equation**, solve this equation, and then **transform back** in order to produce a solution to the initial value problem. We will only be concerned with the solution for $t > 0$.

**Big Idea #1: Generalized functions**, a.k.a. "a function is only as good as how it is integrated" - in particular, delta functions and step functions.

**Big Idea #2**: We'll devise a systematic way of formally solving an ODE with such inputs, and then use integration (convolution) to produce solutions to any given initial value problem.

### Heaviside functions, box functions, and delta functions

The Heaviside function [named for Oliver Heaviside (1850–1925)] is $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$. For our purposes it really doesn't matter how it is defined at $t = 0$, because it's not relevant when integrating this function. We can also define translated Heaviside functions $u_a(t) = u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$. These functions can be scaled and added to represent functions corresponding to "switching on and off". For example, we can represent the function $f(t) = \begin{cases} 0 & t < 3 \\ 4 & 3 < t < 5 \\ 0 & t > 5 \end{cases} = 4[u_3(t) - u_5(t)]$. This is called a **box function**. We can combine box functions as necessary, e.g. $g(t) = \begin{cases} 0 & t < 3 \\ 4 & 3 < t < 5 \\ 1 & 5 < t < 6 \\ 0 & t > 6 \end{cases} = 4[u_3(t) - u_5(t)] + 1[u_5(t) - u_6(t)] = 4u_3(t) - 3u_5(t) - u_6(t)$.

The Heaviside function is constant everywhere except at $t = 0$, and because it has a jump discontinuity there we usually just say that it's not differentiable at $t = 0$. However, we could heuristically observe that by considering points immediately to the left and right of the discontinuity any continuous approximation to this function would have to have a very large slope in the vicinity of $t = 0$. We might at least try to express this by saying that $u'(t) = \begin{cases} 0 & t < 0 \\ \infty & t = 0 \\ 0 & t > 0 \end{cases} = \delta(t)$, the so-called delta function, but this doesn't really make much sense in terms of traditional functions. We may, however, still be able to make sense out of this if we take the view that "*a function is only as good as how it is integrated*." Similarly, $\dot{u}(t-a) = \delta(t-a)$, a translated delta function.

### Generalized functions

You can heuristically think of the step function $u(t)$ as any nice smooth function which is 0 for $t < -\varepsilon$ and 1 for $t > \varepsilon$, where $\varepsilon$ is a positive number which is much smaller than any time scale we care about in the context we are studying at the moment. Similarly, a good way for you to visualize the "delta function" (defined below) is to think of it as a function which is zero everywhere except in the immediate neighborhood of $t = 0$ and which has integral 1. We showed that the delta functions $\delta(t)$ and $\delta(t-a)$ can be viewed as the "functions you integrate against" in order to evaluate a function at respectively $t = 0$ and at any $t = a$.

That is, $\boxed{\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)}$ and $\boxed{\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a)}$.



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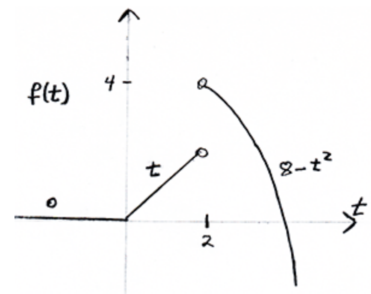
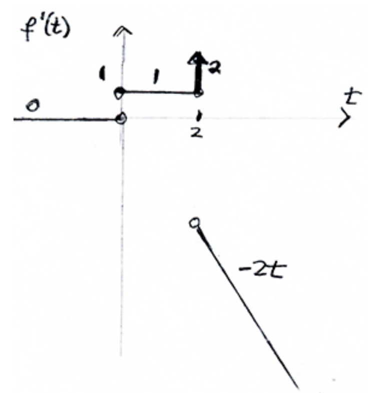
**You can also take a sequential approach to make sense of this in terms of limits**, i.e. if you successively approximate the delta function by a sequence of continuous functions $f_k(t)$ where the support (domain where it's nonzero) gets narrower $[-\varepsilon_k, +\varepsilon_k]$ and the values grow reciprocally in such a way that at each step the integral is always $\int_{-\infty}^{+\infty} f_k(t)dt = \int_{-\varepsilon_k}^{+\varepsilon_k} f_k(t)dt = 1$ (we call such functions probability densities), then you can show that $\lim_{k\to\infty}\left[\int_{-\infty}^{+\infty} g(t)\,f_k(t)dt\right] = g(0)$.

**Note**: The Fundamental Theorem of Calculus as well as all the usual rules of differentiation also apply to generalized derivatives, so we actually have a "generalized calculus" for dealing with these generalized functions or distributions (though it may take a while getting used to it). Basically, we extend the usual rules of differentiation to generalized functions together with the fact that $\dot{u}(t-a) = \delta(t-a)$.

**Example**: Consider the piecewise-defined function $f(t) = \begin{cases} 0 & t<0 \\ t & 0<t<2 \\ 8-t^2 & t>2 \end{cases}$.




It's continuous but not differentiable at $t=0$, and it has a jump discontinuity at $t=2$. We can also express this function in terms of box functions and Heaviside functions by "switching functions on and off." That is, we can write
$f(t) = t[u(t)-u(t-2)] + (8-t^2)u(t-2)$.




We can differentiate this using the usual rules of Calculus together with the fact that $u'(t) = \delta(t)$ and $u'(t-a) = \delta(t-a)$. This gives:

$$f'(t) = t[\delta(t)-\delta(t-2)] + 1[u(t)-u(t-2)] + (8-t^2)\delta(t-2) - 2t\cdot u(t-2)$$
$$= t\cdot\delta(t) + (8-t-t^2)\cdot\delta(t-2) + 1[u(t)-u(t-2)] - 2t\cdot u(t-2)$$

While this is formally correct, it's not the simplest way to express this. If we embrace the notion that a delta function only has meaning as an integrand <u>and</u> that it has a value of 0 everywhere except at a single point <u>and</u> that only the value at that point of the function by which it is multiplied is relevant, we can greatly simplify this expression. Specifically, if we look at the term $t\cdot\delta(t)$ we see that only the value of the function $t$ at 0 is relevant, and that value is 0, so effectively $t\cdot\delta(t)=0$. Similarly, if we look at the term $(8-t-t^2)\cdot\delta(t-2)$ and note that only the value at $t=2$ is relevant, we can write $(8-t-t^2)\cdot\delta(t-2) = 2\cdot\delta(t-2)$, so $f'(t) = 2\cdot\delta(t-2) + 1[u(t)-u(t-2)] - 2t\cdot u(t-2)$. Note that this is the sum of the "singular part" $2\cdot\delta(t-2)$ and the "regular part" $1[u(t)-u(t-2)] - 2t\cdot u(t-2)$. The graph is shown and it's worth noting that except for the arrow indicating the delta function with "weight" 2, this is exactly what we have obtained by simply differentiation the given piece-wise defined function.

A function $f(t)$ is "**regular**" or "piecewise smooth" if it can be broken into pieces each having all higher derivatives and such that at each breakpoint $f^{(n)}(a-)$ and $f^{(n)}(a+)$ exist. A "singularity function" is a linear combination of shifted delta functions. A "**generalized function**" $f(t)$ is a sum $f(t) = f_r(t) + f_s(t)$ of a regular function and a singularity function. Any regular function $f(t)$ has a "**generalized derivative**" $f'(t)$, with regular part $f_r{}'(t)$ the regular derivative of $f(t)$ wherever it exists, and singular part $f_s{}'(t)$ given by a sum of terms $(f(a+)-f(a-))\delta(t-a)$ as $a$ runs over the discontinuities of $f$.

Now, to get back to **the Main Idea**, how can we solve a linear differential equation $[p(D)]x(t)=q(t)$ by transforming it into an algebraic equation, solving that algebraic equation, and then transforming back to produce a solution to an initial value problem? As we will only be concerned with forward time, we'll presume that $q(t)$ satisfies $q(t)=0$ for $t<0$.

**The Laplace Transform**

**Definition**: The ***Laplace transform*** of a function $f(t)$ is defined by

$$\boxed{\mathcal{L}[f(t)]=F(s)=\int_{0-}^{\infty}e^{-st}f(t)dt}$$

where the new (complex) variable $s$ is such that its real part $\operatorname{Re}(s)\gg 0$ (the integral would otherwise not converge). Note that the lower limit of the integral indicates that $t=0$ is included and is intended to address potential discontinuities and delta functions.

We will liberally make use of the convention that a function of $t$ will be represented by a lower case name and its Laplace transform by the corresponding upper case name, e.g. $\mathcal{L}[x(t)]=X(s)$.

**Linearity**

Because the Laplace transform is defined as an integral, it's easy to see that:

$$\mathcal{L}[af(t)+bg(t)]=a\mathcal{L}[f(t)]+b\mathcal{L}[g(t)]=aF(s)+bG(s).$$

Specifically:

$$\mathcal{L}[af(t)+bg(t)]=\int_{0-}^{\infty}e^{-st}[af(t)+bg(t)]dt=a\int_{0-}^{\infty}e^{-st}f(t)dt+b\int_{0-}^{\infty}e^{-st}g(t)dt$$
$$=a\mathcal{L}[f(t)]+b\mathcal{L}[g(t)]=aF(s)+bG(s)$$

This will permit us to transform a differential equation term-by-term (and transform back as well).

**Inverse transform**: $F(s)$ essentially determines $f(t)$ for $t\geq 0$. This will generally allow us to produce solutions to a given Initial Value Problem by simply recognizing, term by term, a solution by identifying which functions gave rise to each term of the transformed differential equation.

**Some Calculations**

1) For our purposes, since we are only concerned with $t\geq 0$, the constant function $f(t)=1$ and the Heaviside function $u(t)=\begin{cases}0 & t<0\\ 1 & t>0\end{cases}$ are indistinguishable. Thus $\mathcal{L}[1]=\mathcal{L}[u(t)]=\int_{0-}^{\infty}e^{-st}\cdot 1\,dt=\left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=\infty}=0+\frac{1}{s}=\frac{1}{s}$

Here we used the fact that for $s>0$, $\lim_{t\to\infty}\left[e^{-st}\right]=0$. Indeed, this is still the case even if we permit $s$ to be complex with positive real part, i.e. $\operatorname{Re}(s)>0$.

2) If $f(t)=t$, we calculate its Laplace Transform as

$$F(s)=\mathcal{L}[t]=\int_{0-}^{\infty}te^{-st}dt=\left[\frac{te^{-st}}{-s}\right]_{t=0-}^{t=\infty}+\frac{1}{s}\int_{0-}^{\infty}e^{-st}dt=0+\frac{1}{s}\mathcal{L}[1]=\frac{1}{s^2}$$

3) If $f(t)=t^2$, we calculate $F(s)=\mathcal{L}[t^2]=\int_{0-}^{\infty}t^2e^{-st}dt=\left[\frac{t^2e^{-st}}{-s}\right]_{t=0-}^{t=\infty}+\frac{2}{s}\int_{0-}^{\infty}te^{-st}dt=0+\frac{2}{s}\mathcal{L}[t]=\frac{2}{s}\cdot\frac{1}{s^2}=\frac{2}{s^3}$

4) ***s*-derivative rule**: $\boxed{\mathcal{L}[tf(t)]=-F'(s)}$. We can establish this by noting that if $F(s)=\mathcal{L}[f(t)]=\int_{0-}^{\infty}e^{-st}f(t)dt$, then $F'(s)=\frac{d}{ds}\int_{0-}^{\infty}e^{-st}f(t)dt=-\int_{0-}^{\infty}e^{-st}t\,f(t)dt=-\mathcal{L}[t\,f(t)]$, so $\mathcal{L}[tf(t)]=-F'(s)$.

From this we see that $\mathcal{L}[t^2]=\mathcal{L}[t\cdot t]=-\frac{d}{ds}\mathcal{L}[t]=-\frac{d}{ds}\left[\frac{1}{s^2}\right]=\frac{2}{s^3}$; $\mathcal{L}[t^3]=\mathcal{L}[t\cdot t^2]=-\frac{d}{ds}\mathcal{L}[t^2]=-\frac{d}{ds}\left[\frac{2}{s^3}\right]=\frac{3!}{s^4}$;

$\mathcal{L}[t^4]=\mathcal{L}[t\cdot t^3]=-\frac{d}{ds}\mathcal{L}[t^3]=-\frac{d}{ds}\left[\frac{3!}{s^4}\right]=\frac{4!}{s^5}$; and so on. Generally, $\boxed{\mathcal{L}[t^n]=\frac{n!}{s^{n+1}}}$.

This, together with linearity, enables us to calculate the Laplace transform of any polynomial function.

5) If $f(t)=e^{at}$ is an exponential function (really $f(t)=u(t)e^{at}$ since we are only concerned with $t\geq 0$),

$\mathcal{L}[e^{at}]=\int_{0-}^{\infty}e^{-st}e^{at}dt=\int_{0-}^{\infty}e^{-(s-a)t}dt=\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_{t=0-}^{t=\infty}=\frac{1}{s-a}$, so $\boxed{\mathcal{L}[e^{at}]=\frac{1}{s-a}}$.

6) ***s*-shift rule**: $\boxed{\mathcal{L}[e^{rt}f(t)]=F(s-r)}$. To establish this, we calculate

$\mathcal{L}[e^{rt}f(t)]=\int_{0-}^{\infty}e^{-st}e^{rt}f(t)dt=\int_{0-}^{\infty}e^{-(s-r)t}f(t)dt=F(s-r)$ simply by noting the substitution.

7) **Transforming derivatives**: For any generalized function, $\boxed{\mathcal{L}[f'(t)]=sF(s)-f(0-)}$ where $f(0-)$ represents the initial value of $f(t)$. The unusual notation is there because we will be dealing with discontinuous and generalized functions where we may need to distinguish left-hand from right-hand limits. We can establish this ***t*-derivative rule** by noting that $\mathcal{L}[f'(t)]=\int_{0-}^{\infty}e^{-st}f'(t)dt$. If we use Integration by Parts with $u=e^{-st}$ and $dv=f'(t)dt$, we get $du=-se^{-st}dt$ and $v=f(t)$, so:

$$\mathcal{L}[f'(t)]=\int_{0-}^{\infty}e^{-st}f'(t)dt=\left[e^{-st}f(t)\right]_{t=0-}^{t=\infty}+s\int_{0-}^{\infty}e^{-st}f(t)dt=[0]+sF(s)=sF(s)-f(0-)$$

For second derivatives, note that $f''(t)=\frac{d}{dt}f'(t)$, so we can apply the above result to get that

$\mathcal{L}[f''(t)]=s\mathcal{L}[f'(t)]-f'(0-)=s(sF(s)-f(0-))-f'(0-)=s^2F(s)-s\cdot f(0-)-f'(0-)$, so

$\boxed{\mathcal{L}[f''(t)]=s^2F(s)-s\,f(0-)-f'(0-)}$.

Continuing, we get that $\boxed{\mathcal{L}[f'''(t)]=s^3F(s)-s^2\,f(0-)-s\,f'(0-)-f''(0-)}$, and so on.

Generally, $\boxed{\mathcal{L}[f^{(n)}(t)]=s^nF(s)-s^{n-1}f(0-)-s^{n-2}f'(0-)-\cdots-f^{(n-1)}(0-)}$.

8) **Transforming the delta function**: One of our most fundamental transforms is $\boxed{\mathcal{L}[\delta(t)]=1}$. This is relatively easy to see once you're comfortable with the integral formalisms concerning the delta function and how they relate to evaluation. Specifically, $\mathcal{L}[\delta(t)]=\int_{0-}^{\infty}e^{-st}\delta(t)dt=e^0=1$ since this is really just evaluation of the function $e^{-st}$ at $t=0$.

9) **Transforming sines and cosines**: $\boxed{\mathcal{L}[\cos(\omega t)]=\frac{s}{s^2+\omega^2}}$ and $\boxed{\mathcal{L}[\sin(\omega t)]=\frac{\omega}{s^2+\omega^2}}$

We can derive each of these independently, but if we use Euler's Formula and linearity we have that:

$$\mathcal{L}[e^{i\omega t}]=\mathcal{L}[\cos(\omega t)+i\sin(\omega t)]=\mathcal{L}[\cos(\omega t)]+i\mathcal{L}[\sin(\omega t)]\text{, and}$$

$$\mathcal{L}[e^{i\omega t}]=\frac{1}{s-i\omega}\left[\frac{s+i\omega}{s+i\omega}\right]=\frac{s+i\omega}{s^2+\omega^2}=\left(\frac{s}{s^2+\omega^2}\right)+i\left(\frac{\omega}{s^2+\omega^2}\right)$$

Taking real and imaginary parts separately we get that: $\mathcal{L}[\cos(\omega t)]=\frac{s}{s^2+\omega^2}$ and $\mathcal{L}[\sin(\omega t)]=\frac{\omega}{s^2+\omega^2}$.

We'll add to this list as we go and as the need arises.

**Unit Impulse Response, weight function, transfer function**

Unit impulse response refers to the solution of the ODE $p(D)[x(t)] = \delta(t)$ with rest initial conditions. The solution is also known as the ***weight function*** for the given differential operator $p(D)$. It is the simplest to tackle algebraically and we'll use it soon along with convolution to solve Initial Value Problems. We generally denote the unit impulse response (weight function) by $w(t)$. It's Laplace Transform $W(s)$ is called the ***transfer function***.

**Unit Step Response**

Unit step response refers to the solution of the ODE $p(D)[x(t)] = u(t)$ with rest initial conditions. It is a bit more algebraically complicated to solve than the unit impulse response but is still relatively simple. We generally denote the unit step response by $v(t)$.

It is worth noting that because these differential operators are time-invariant (constant coefficients), we can use the generalized derivative to differentiate both sides of $p(D)[x(t)] = u(t)$ to get $D \circ p(D)[v(t)] = p(D)[D(v(t))] = p(D)[\dot{v}(t)] = D[u(t)] = \delta(t)$, so $p(D)[\dot{v}(t)] = \delta(t)$. Therefore $\dot{v}(t) = w(t)$.

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**Example 1**: Find the unit impulse response and the unit step response for the operator $p(D) = D + 3I$.

**Solution**: For the unit impulse response we solve $\dot{w} + 3w = \delta(t)$ with rest initial conditions. Transforming both sides gives $p(s)W(s) = (s+3)W(s) = 1$, so $W(s) = \dfrac{1}{p(s)} = \dfrac{1}{s+3}$. This is just $\mathcal{L}(e^{-3t})$, so $\boxed{w(t) = e^{-3t}}$.

For the unit step response we solve $\dot{v} + 3v = u(t)$ with rest initial conditions. Transforming both sides gives $p(s)V(s) = (s+3)V(s) = \dfrac{1}{s}$, so $V(s) = \dfrac{1}{s(s+3)} = \tfrac{1}{3}\left(\dfrac{1}{s} - \dfrac{1}{s+3}\right)$. It follows that $\boxed{v(t) = \tfrac{1}{3}(1 - e^{-3t})}$.

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**Example 2**: Find the unit impulse response for the operator $p(D) = D^2 + \omega^2$ where $\omega$ is a given positive constant (natural frequency for a harmonic oscillator).

**Solution**: For the unit impulse response we solve $\ddot{w} + \omega^2 w = \delta(t)$ with rest initial conditions. Transforming both sides gives $p(s)W(s) = (s^2 + \omega^2)W(s) = 1$, so $W(s) = \dfrac{1}{p(s)} = \dfrac{1}{s^2 + \omega^2}$. Adjusting the coefficients to write this as $W(s) = \dfrac{1}{\omega}\left(\dfrac{\omega}{s^2 + \omega^2}\right)$ we deduce from our table of transforms that $\boxed{w(t) = \tfrac{1}{\omega}\sin(\omega t)}$.

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**Example 3**: Find the unit impulse response for the operator $p(D) = D^2 + 3D + 2I$, i.e. find the response for the Initial Value Problem $\ddot{x} + 3\dot{x} + 2x = \delta(t), \quad x(0) = 0, \quad \dot{x}(0) = 0 \quad$ (rest initial conditions).

**Solution**: Transforming both sides of this ODE gives $(s^2 + 3s + 2)X(s) = p(s)X(s) = 1$ where $p(s) = s^2 + 3s + 2$ is the characteristic polynomial of the operator. A quick calculation of the partial fractions decomposition gives:

$$X(s) = \frac{1}{p(s)} = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} = W(s)$$

So the unit impulse response (weight function) is $x(t) = w(t) = e^{-t} - e^{-2t}$.

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Now let's move on to some Initial Value Problems other than Unit Impulse Response.

**Example 3**: Solve the IVP $\frac{dx}{dt} + 3x = 3\cos 2t$ with initial value $x(0-) = 2$ (the $0-$ is just for emphasis here).

**Solution**: First, it should be emphasized that for a problem like this our previous methods work well and there is no particular need to use Laplace transform methods. That said, we proceed with two different approaches.

**Laplace Direct**: For this we simply transform both sides of the equation mindful of the need to incorporate the initial condition as we transform the derivative. This gives:

$$sX(s)-2+3X(s)=(s+3)X(s)-2=\frac{3s}{s^2+4}\text{, so } (s+3)X(s)=2+\frac{3s}{s^2+4}=\frac{2s^2+3s+8}{s^2+4}.$$

Therefore $X(s)=\dfrac{2s^2+3s+8}{(s+3)(s^2+4)}=\dfrac{A}{s+3}+\dfrac{Bs+C}{s^2+4}$.

Clearing fractions gives $2s^2+3s+8=A(s^2+4)+(s+3)(Bs+C)$

There are several good ways to proceed. First, if we choose convenient points we might first choose $s=-3$ to quickly conclude that $17=13A$, so $\boxed{A=\frac{17}{13}}$. You might think the well has run dry, but we are free to use complex numbers. If we choose $s=2i$ (and as we'll see we won't even have to separately consider its complex conjugate) we get $-8+6i+8=6i=(3+2i)(2Bi+C)=(-4B+3C)+i(6B+2C)$. We can equate both real and imaginary parts to conclude that $-4B+3C=0$ and $6B+2C=6$. These give $\boxed{B=\frac{9}{13}}$ and $\boxed{C=\frac{12}{13}}$.

Thus $\boxed{X(s)=\frac{17}{13}\left(\dfrac{1}{s+3}\right)+\frac{9}{13}\left(\dfrac{s}{s^2+4}\right)+\frac{6}{13}\left(\dfrac{2}{s^2+4}\right)}$. So $\boxed{x(t)=\frac{17}{13}e^{-3t}+\frac{9}{13}\cos 2t+\frac{6}{13}\sin 2t}$.

Alternatively, we could simply multiply out and collect terms to get $2s^2+3s+8=(A+B)s^2+(3B+C)s+(4A+3C)$ and then use your favorite linear algebra method to derive the same results as above.

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## $\mathbf{ZIR+ZSR}$

Given an *n*-th order linear ODE $p(D)[x(t)]=f(t)$ with initial conditions $x(t_0)=x_0$ and $\dot{x}(t_0)=\dot{x}_0$, …, $x^{(n-1)}(t_0)={x_0}^{(n-1)}$, we refer to the case where $x(t_0)=0$ and $\dot{x}(t_0)=0$, …, $x^{(n-1)}(t_0)=0$ as the **zero state**. If we solve $p(D)[x(t)]=f(t)$ for the zero state, we refer to this solution $x_{ZSR}(t)$ as the **zero state response (ZSR)**.

If we seek homogeneous solutions to the ODE $p(D)[x(t)]=0$ with initial conditions $x(t_0)=x_0$ and $\dot{x}(t_0)=\dot{x}_0$, …, $x^{(n-1)}(t_0)={x_0}^{(n-1)}$, this will have a unique solution $x_{ZIR}(t)$ called the **zero input response (ZIR)**.

The general solution to the ODE $p(D)[x(t)]=f(t)$ will be $x(t)=x_h(t)+x_p(t)$ for some particular solution $x_p(t)$ and homogeneous solutions $x_h(t)$, and we would then use the initial conditions to determine any unknown coefficients. However, note that the zero state response (ZSR) is a particular solution and the zero input response is a (single) homogeneous solution. If we let $x(t)=x_{ZIR}(t)+x_{ZSR}(t)$, note that:

$$\left\{\begin{array}{c} x(t_0)=x_{ZIR}(t_0)+x_{ZSR}(t_0)=x_{ZIR}(t_0)+0=x_{ZIR}(t_0)=x_0 \\ \dot{x}(t_0)=\dot{x}_{ZIR}(t_0)+\dot{x}_{ZSR}(t_0)=\dot{x}_{ZIR}(t_0)+0=\dot{x}_{ZIR}(t_0)=\dot{x}_0 \\ \vdots \\ x^{(n-1)}(t_0)={x_{ZIR}}^{(n-1)}(t_0)+{x_{ZSR}}^{(n-1)}(t_0)={x_{ZIR}}^{(n-1)}(t_0)+0={x_{ZIR}}^{(n-1)}(t_0)={x_0}^{(n-1)} \end{array}\right\}$$

so $x(t)=x_h(t)+x_p(t)$ satisfies the initial value problem (IVP) without the need to introduce any additional constants. That is, $x(t)=\mathbf{ZIR}+\mathbf{ZSR}$.

This observation is very helpful when solving initial value problems using Laplace Transform methods – specifically when we use the Unit Impulse Response together with ***convolution*** to solve for the zero state response (ZSR). More on that later.

**Previous example using ZSR+ZIR** (not really recommended here): If we first solve $\frac{dx}{dt}+3x=3\cos 2t$ with rest initial conditions we get $(s+3)X(s)=\dfrac{3s}{s^2+4}$ and

$X(s)=\dfrac{3s}{(s+3)(s^2+4)}=-\frac{9}{13}\left(\dfrac{1}{s+3}\right)+\frac{9}{13}\left(\dfrac{s}{s^2+4}\right)+\frac{6}{13}\left(\dfrac{2}{s^2+4}\right)$. So $\boxed{x_{ZSR}(t)=-\frac{9}{13}e^{-3t}+\frac{9}{13}\cos 2t+\frac{6}{13}\sin 2t}$. Next we seek the zero input response, so we solve $\frac{dx}{dt}+3x=0$ with $x(0)=2$. This quickly gives $\boxed{x_{ZIR}(t)=2e^{-3t}}$. Combining these gives $\boxed{x(t)=\frac{17}{13}e^{-3t}+\frac{9}{13}\cos 2t+\frac{6}{13}\sin 2t}$.

**Example 4**: Solve the Initial Value Problem $\ddot{x}+3\dot{x}+2x=2e^{-t},\quad x(0)=0,\quad \dot{x}(0)=0$ (rest initial conditions).

**Old Faithful Solution**: The homogeneous equation $\ddot{x}+3\dot{x}+2x=0$ is easy to solve. Its characteristic polynomial is $p(s)=s^2+3s+2=(s+2)(s+1)$ which yields the two roots $s=-2$ and $s=-1$. This gives the two independent solutions $e^{-2t}$ and $e^{-t}$, and all homogeneous solutions are of the form $x_h(t)=c_1e^{-2t}+c_2e^{-t}$. Note that both of these homogeneous solutions are transient in the sense that they decay exponentially as $t$ increases.

Next, we need to find a particular solution $x_p(t)$ that satisfies the inhomogeneous differential equation. One look at the right-hand-side and we see that the Exponential Response Formula (ERF) won't work – there is resonance. We can, however, use the Resonant Response Formula to get the particular solution $x_p(t)=\dfrac{2te^{-t}}{p'(-1)}=\dfrac{2te^{-t}}{1}=2te^{-t}$, so the general solution is $x(t)=x_h(t)+x_p(t)=c_1e^{-2t}+c_2e^{-t}+2te^{-t}$. Its derivative is $\dot{x}(t)=-2c_1e^{-2t}-c_2e^{-t}-2te^{-t}+2e^{-t}$. Substituting the (rest) initial conditions gives $\left\{\begin{matrix} x(0)=c_1+c_2=0 \\ \dot{x}(0)=-2c_1-c_2+2=0\end{matrix}\right\}$, and these can be solved to give $c_1=2, c_2=-2$, so the solution is $\boxed{x(t)=2e^{-2t}-2e^{-t}+2te^{-t}}$.

**Solving directly by Laplace transform**: We calculated the following Laplace transforms:

(1) $\mathcal{L}(e^{kt})=\dfrac{1}{s-k}$ with region of convergence $\text{Re}(s)>k$, so $\mathcal{L}(e^{-2t})=\dfrac{1}{s+2}$.

(2) If the Laplace transform of $x(t)$ is $X(s)$, then the Laplace transforms of its derivatives are $\mathcal{L}(\dot{x}(t))=sX(s)-x(0-)$ and $\mathcal{L}(\ddot{x}(t))=s^2X(s)-s\,x(0-)-\dot{x}(0-)$. In the case of rest initial conditions $x(0-)=\dot{x}(0-)=0$, these are greatly simplified and, in fact $\mathcal{L}(p(D)x)=p(s)X(s)$. Specifically, $\mathcal{L}(\ddot{x}+3\dot{x}+2x)=s^2X(s)+3sX(s)+2X(s)=(s^2+3s+2)X(s)=p(s)X(s)$.

If we now transform the entire differential equation, we get $(s^2+3s+2)X(s)=\dfrac{2}{s+1}$.

We then solve for $X(s)=\dfrac{2}{(s+1)(s^2+3s+2)}=\dfrac{2}{(s+2)(s+1)^2}=\dfrac{A}{s+2}+\dfrac{B}{s+1}+\dfrac{C}{(s+1)^2}$.

There are many good ways to find the unknowns $A$, $B$, and $C$. For example, if we multiply through by the common denominator to clear fractions, we get $2=A(s+1)^2+B(s+1)(s+2)+C(s+2)$. Plugging in the specific values $s=-2$ and $s=-1$ quickly yields that $A=2$ and $C=2$. Plugging in, for example, $s=0$ and using the values for $A$ and $C$ then yields $B=-2$. So:

$$X(s)=\frac{2}{s+2}-\frac{2}{s+1}+\frac{2}{(s+1)^2}=2\left(\frac{1}{s+2}\right)-2\left(\frac{1}{s+1}\right)+2\left(\frac{1}{(s+1)^2}\right).$$

Consulting our table of common Laplace transforms, we see that $\frac{1}{s+2}=\mathcal{L}(e^{-2t})$, $\frac{1}{s+1}=\mathcal{L}(e^{-t})$, and $\frac{1}{(s+1)^2}=\mathcal{L}(te^{-t})$, so transforming back (using linearity) gives $\boxed{x(t)=2e^{-2t}-2e^{-t}+2te^{-t}}$.

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**Example**: Solve the Initial Value Problem $\ddot{x}+3\dot{x}+2x=2e^{-t},\quad x(0)=3,\quad \dot{x}(0)=1$ (rest initial conditions).

**Laplace Direct Solution**: Transforming both sides gives

$$(s^2X-s\cdot 3-1)+3(sX-3)+2X=(s^2+3s+2)X-3s-10=\frac{2}{s+1}$$

$$\Rightarrow\quad (s^2+3s+2)X=3s+10+\frac{2}{s+1}=\frac{3s^2+13s+12}{s+1}\quad\Rightarrow\quad X(s)=\frac{3s^2+13s+12}{(s+1)^2(s+2)}$$

$$X(s)=\frac{3s^2+13s+12}{(s+2)(s+1)^2}=\frac{A}{s+2}+\frac{B}{s+1}+\frac{C}{(s+1)^2}\quad\Rightarrow\quad 3s^2+13s+12=A(s+1)^2+B(s+1)(s+2)+C(s+2)$$

Substitution the values $s=-1$ and $s=-2$ quickly yields that $C=2$ and $A=-2$; and $s=0$ yields that $B=5$.

So $X(s)=\frac{3s^2+13s+12}{(s+2)(s+1)^2}=\frac{-2}{s+2}+\frac{5}{s+1}+\frac{2}{(s+1)^2}$ and consulting the Laplace transform table we see that it must be the case that $\boxed{x(t)=-2e^{-2t}+5e^{-t}+2te^{-t}}$.

Note: Having non-rest initial conditions tends to complicate the algebra somewhat.

**ZSR+ZIR solution**: We already found the Zero State Solution in Example 4, i.e. $\boxed{x_{ZSR}(t)=2e^{-2t}-2e^{-t}+2te^{-t}}$.
For the Zero Input Response we solve the problem $\{\ddot{x}+3\dot{x}+2x=0,\quad x(0)=3,\quad \dot{x}(0)=1\}$. This gives the homogeneous solutions $x(t)=c_1e^{-2t}+c_2e^{-t}$ and differentiation gives $\dot{x}(t)=-2c_1e^{-2t}-c_2e^{-t}$. Substitution of the initial conditions gives $\left\{\begin{array}{l} x(0)=\quad c_1+c_2=3 \\ \dot{x}(0)=-2c_1-c_2=1\end{array}\right\}\quad\Rightarrow\quad c_1=-4,\ c_2=7\quad\Rightarrow\quad \boxed{x_{ZIR}(t)=-4e^{-2t}+7e^{-t}}$.

Combining these gives $x(t)=x_{ZSR}(t)+x_{ZIR}(t)=\left(2e^{-2t}-2e^{-t}+2te^{-t}\right)+\left(-4e^{-2t}+7e^{-t}\right)=\boxed{-2e^{-2t}+5e^{-t}+2te^{-t}}$.

**Looking ahead:**

**Definition (Convolution)**: Given two functions $w(t)$ and $f(t)$, we define $\boxed{(f*w)(t)=\int_0^t f(\tau)w(t-\tau)d\tau}$.

It's a straightforward exercise to show that the convolution product is commutative, i.e. $f*w=w*f$.

We'll show that the solution to the differential equation $p(D)x=f(t)$ will have a Zero State Response given by $(f*w)(t)$. This is also known as ***Green's Formula***.

We'll show this by making use of box functions, delta functions, linearity of the Laplace Transform, and Riemann Sums to develop the convolution integral.

**Note**: When applying the convolution method to solving $p(D)x=f(t)$ for more general initial conditions, the solution will be $x(t)=\mathbf{ZSR}+\mathbf{ZIR}$, where **ZSR** is the **zero state response** and **ZIR** is the **zero input response**.

### *Properties of the Laplace transform*

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} e^{-st} f(t)dt$ for $\text{Re}(s) \gg 0$.

1. Linearity: $\mathcal{L}[af(t)+bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s)+bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. $s$-shift rule: $\mathcal{L}[e^{rt} f(t)] = F(s-r)$.

4. $t$-shift rule: $\mathcal{L}[f(t-a)] = e^{-as}F(s)$ if $a \ge 0$ and $f(t)=0$ for $t<0$.

   This may also be expressed as $\mathcal{L}[f_a(t)] = e^{-as}F(s)$ where $f_a(t) = u(t-a)f(t-a) = \left\{\begin{matrix} f(t-a) & \text{if } t>a \\ 0 & \text{if } t<a \end{matrix}\right\}$.

5. $s$-derivative rule: $\mathcal{L}[t f(t)] = -F'(s)$.

6. $t$-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0-)$

   $\mathcal{L}[f''(t)] = s^2F(s) - sf(0-) - f'(0-)$

   $\mathcal{L}[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0-) - s^{n-2}f'(0-) - \cdots - f^{(n-1)}(0-)$

7. Convolution rule: $\mathcal{L}[f(t)*g(t)] = F(s)G(s)$, $(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s)$, $w(t)$ the unit impulse response.

   If $q(t)$ is regarded as the input signal in $p(D)x = q(t)$, $W(s) = \frac{1}{p(s)}$.

### *Formulas for the Laplace transform*

$\mathcal{L}[1] = \frac{1}{s}$

$\mathcal{L}[\delta(t)] = 1$

$\mathcal{L}[\delta(t-a)] = \mathcal{L}[\delta_a(t)] = e^{-as}$

$\mathcal{L}[u(t-a)] = \mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$

$\mathcal{L}[e^{at}] = \frac{1}{s-a}$

$\mathcal{L}[t] = \frac{1}{s^2}$

$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$

$\mathcal{L}[u(t-a)f(t-a)] = e^{-as}F(s)$

$\mathcal{L}[u(t-a)f(t)] = e^{-as}\mathcal{L}[f(t+a)]$

$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$

$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$

$\mathcal{L}[t\cos(\omega t)] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$

$\mathcal{L}[t\sin(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2}$

$\mathcal{L}[e^{zt}\cos(\omega t)] = \frac{s-z}{(s-z)^2+\omega^2}$

$\mathcal{L}[e^{zt}\sin(\omega t)] = \frac{\omega}{(s-z)^2+\omega^2}$

**Notes by Robert Winters**



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