Math E-21c – Ordinary Differential Equations – Lecture #8

Theorem (Fourier): Suppose a function $f(t)$ is periodic with base period 2π and continuous except for a finite number of jump discontinuities. Then $f(t)$ may be represented by a (convergent) Fourier Series:

$$f(t) \sim \tfrac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

where: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$.

The numbers $\{a_0, a_1, b_1, \cdots, a_n, b_n, \cdots\}$ are called the **Fourier coefficients** of the function $f(t)$.

This representation is an equality at all points of continuity of the function $f(t)$. At any point of discontinuity $t = a$, the series converges to the average of $f(a^-)$ and $f(a^+)$, i.e. the value $\frac{1}{2}[f(a^-) + f(a^+)]$.

Miscellaneous Fourier Facts

We performed these calculations last time for the **square-wave function** $f(t) = sq(t) = \begin{Bmatrix} -1 & t \in [-\pi, 0) \\ +1 & t \in [0, \pi) \end{Bmatrix}$, extended periodically for all *t*, and derived that $\boxed{sq(t) \sim \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n} = \frac{4}{\pi}[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots]}$.

If we translate $\lim_{n\to\infty}\|f_n\|^2 = \|f\|^2$ for this function we get that $\|f\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} 1dt = 2$, and

$\lim_{n\to\infty}\|f_n\|^2 = \frac{16}{\pi^2}\left[1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots\right] = \frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = 2$. Therefore $\boxed{\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}}$, a curious fact.

We can also apply the last statement in Fourier's Theorem by evaluating the square-wave function at $\pi/2$, a point of continuity, to get that $sq(\pi/2) = 1 = \frac{4}{\pi}[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots]$, so $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$, but the convergence is so abysmally slow as to be of no practical consequence – another curiosity.

Note: In terms of the Fourier coefficients for a periodic function f of period 2π, the statement that

$\lim_{n\to\infty}\|f_n\|^2 = \|f\|^2$ translates into: $\boxed{\frac{a_0^{\,2}}{2} + a_1^{\,2} + b_1^{\,2} + a_2^{\,2} + b_2^{\,2} + \cdots + a_n^{\,2} + b_n^{\,2} + \cdots = \|f\|^2}$

Example (Sawtooth function): $f(t) = t$ on the interval $(-\pi, \pi]$, extended periodically for all *t*.

This is an odd function, so we conclude immediately that $a_0 = 0$ and $a_n = 0$ for all *n*. For the Fourier sine coefficients we do a little integration by parts: $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt\,dt = \frac{1}{\pi}\left[-\frac{t\cos nt}{n} + \frac{\sin nt}{n^2}\right]_{-\pi}^{\pi} = (-1)^{n+1}\frac{2}{n}$

So $f(t) \sim \sum_{n=1}^{\infty}\left((-1)^{n+1}\frac{2}{n}\right)\sin nt$. In this case the fact that $\lim_{n\to\infty}\|f_n\|^2 = \|f\|^2$ translates into

$\sum_{n=1}^{\infty} b_n^{\,2} = \sum_{n=1}^{\infty}\frac{4}{n^2} = 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \|f\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{1}{3\pi}\left[t^3\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$. So $4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{3}$ or $\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$.

It's worth noting (from Calculus) that this is a *p*-series with $p = 2$. There you may recall that we know that this series converges but *not necessarily what it converges to*. This curious result answers that question.

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Of greater relevance to us is how Fourier series representations can be applied to the solution of differential equations.

Harmonic Response to Periodic Inputs

If we couple the Fourier series representation of a periodic input with linearity, we can produce series representations to linear time-independent (LTI) differential equations.

Example: Find the general solution to the differential equation $\ddot{x}+4x=sq(t)$, where $sq(t)$ is the square-wave function.

Solution: The system corresponds to a harmonic oscillator. The characteristic polynomial is $p(s)=s^2+4$ with characteristic roots $s=\pm 2i$ and the homogeneous solutions are of the form $x_h(t)=c_1\cos 2t+c_2\sin 2t$.

For a particular solution, we use linearity. Using the Fourier series representation $sq(t)\sim\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{2n+1}$, we individually solve $\ddot{x}+4x=\sin(2n+1)t$ for each *n*. To do this we use complex replacement and solve $\ddot{z}+4z=e^{i(2n+1)t}$ using the Exponential Response Formula (ERF). We have $p(i(2n+1))=4-(2n+1)^2$, so $\frac{e^{i(2n+1)t}}{4-(2n+1)^2}=\frac{\cos(2n+1)t+i\sin(2n+1)t}{4-(2n+1)^2}$ is a solution, and we extract its imaginary part to get $\frac{\sin(2n+1)t}{4-(2n+1)^2}$.

Using linearity for the ODE $\ddot{x}+4x=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{2n+1}$, we appropriately scale the individual terms and sum to get the particular solution $x_p(t)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{(2n+1)[4-(2n+1)^2]}$. If we expand this to show the first few terms, we have $x_p(t)=\frac{4}{\pi}\left[\frac{1}{3}\sin t-\frac{1}{15}\sin 3t-\frac{1}{105}\sin 5t-\frac{1}{315}\sin 7t-\frac{1}{693}\sin 9t-\cdots\right]$. Note how the amplitudes of the higher frequencies decrease rapidly. As always, the general solution is $x(t)=x_h(t)+x_p(t)$.

More generally, we could solve $\ddot{x}+\omega^2x=sq(t)$ to get $x_p(t)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{(2n+1)[\omega^2-(2n+1)^2]}$. This will usually yield a convergent series, but we have a problem in the case where ω is an odd integer since one term of the series will “blow up” in that case. This is a case of resonance and we’ll look at that case shortly.

Harmonic response with resonance

One of the more interesting aspects of using Fourier Series is analyzing how a linear time-independent ODE with a periodic signal yields a response that exhibits **resonance**. The basic idea is that if we expand a periodic signal in a Fourier Series, it’s sometimes that a single term in the series may be responsible for resonance. The signal may be composed of a whole range of frequencies, but one of them may produce resonance that may be the dominant feature of the response.

Suppose we wish to solve the ODE $\ddot{x}+\omega^2x=sq(t)\sim\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{2n+1}$, where $sq(t)$ is the square-wave function. We previously observed that this would yield the series solution:

$$x_p(t)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{(2n+1)[\omega^2-(2n+1)^2]}$$

There is a catch, however. All of the terms in the series make sense unless ω is an odd integer. If this is the case, then all but one of the terms in the series will continue to make sense, but we’ll have to treat the one term where $\omega=2n+1$ differently. Let’s consider a specific example.

Example: Find a particular solution to the ODE $\ddot{x}+9x=sq(t)$.

In this case, all of the terms in the above series are as stated, but we have to deal with the $n=1$ term separately since $\omega=3$. For this one term we separately solve the ODE $\ddot{x}+9x=\frac{4}{3\pi}\sin 3t$. If we use complex replacement and later extract the imaginary part, we'll be solving the ODE $\ddot{z}+9z=\frac{4}{3\pi}e^{3it}$.

Since the characteristic polynomial is $p(s)=s^2+9$ and $s=3i$ is a characteristic root, we must use the Resonant Response Formula, i.e. $z=\dfrac{\frac{4}{3\pi}te^{3it}}{p'(3i)}$. Since $p'(s)=2s$ and $p'(3i)=6i$, we have the (complex) solution:

$$z=\frac{\frac{4}{3\pi}te^{3it}}{6i}=\tfrac{2}{9\pi}t(-i)[\cos 3t+i\sin 3t]=\tfrac{2}{9\pi}t[\sin 3t-i\cos 3t].$$

Extracting the imaginary part gives $x_3(t)=-\frac{2}{9\pi}t\cos 3t$. This term can then be added into the previous sum to replace the $n=1$ term. Note, however, that this term is oscillatory but its amplitude grows linearly in time. This is exactly the sort of thing we would expect when the system has resonance – even if it is caused by just one resonant frequency embedded among others.

Tips & Tricks – Manipulation of Fourier series

Different period: We developed our Fourier series representation for functions with a standard period 2π and fundamental interval $[-\pi,\pi]$. If we instead have a function $f(t)$ with period $2L$ and fundamental interval $[-L,L]$, we can simply change variables to produce the corresponding Fourier series in this case. We let $u=\frac{\pi t}{L}$ (so $t=\frac{Lu}{\pi}$) and define $g(u)=f\left(\frac{Lu}{\pi}\right)$ with period 2π and fundamental interval $[-\pi,\pi]$. The Fourier series for $g(u)$ is then

$g(u)\sim\dfrac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nu+b_n\sin nu\right)$ and if we use the substitution $u=\frac{\pi t}{L}$ (and $du=\frac{\pi}{L}dt$), we'll have

$$a_0=\tfrac{1}{\pi}\int_{-\pi}^{\pi}g(u)dt=\tfrac{1}{L}\int_{-L}^{L}g(\tfrac{\pi t}{L})\,dt=\tfrac{1}{L}\int_{-L}^{L}f(u)\,du=\tfrac{1}{L}\int_{-L}^{L}f(t)\,dt,$$

$$a_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi}g(u)\cos nu\,du=\tfrac{1}{L}\int_{-L}^{L}g(\tfrac{\pi t}{L})\cos(\tfrac{n\pi t}{L})\,dt=\tfrac{1}{L}\int_{-L}^{L}f(t)\cos(\tfrac{n\pi t}{L})\,dt,$$

$$b_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi}g(u)\sin nu\,du=\tfrac{1}{L}\int_{-L}^{L}g(\tfrac{\pi t}{L})\sin(\tfrac{n\pi t}{L})\,dt=\tfrac{1}{L}\int_{-L}^{L}f(t)\sin(\tfrac{n\pi t}{L})\,dt,$$ and we can write:

$$\boxed{f(t)=g(\tfrac{n\pi t}{L})\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos(\tfrac{n\pi t}{L})+b_n\sin(\tfrac{n\pi t}{L})\right)}$$

Fourier series can be differentiated or integrated term-by-term to produce other Fourier series:

Example: If we start with $sq(t)=\begin{Bmatrix}-1 & t\in[-\pi,0)\\ +1 & t\in[0,\pi)\end{Bmatrix}\sim\frac{4}{\pi}\sum_{n\text{ odd}}\frac{\sin nt}{n}$ and integrate term-by-term, we get $F(t)\sim\frac{4}{\pi}\sum_{n\text{ odd}}-\frac{\cos nt}{n^2}+C$. If we also insist that $F(0)=0$ and that $F(t)$ be continuous, we get that $-\frac{4}{\pi}\left(\sum_{n\text{ odd}}\frac{1}{n^2}\right)+C=-\frac{4}{\pi}\left(\frac{\pi^2}{8}\right)+C=-\frac{\pi}{2}+C=0$, so $C=\frac{\pi}{2}$. This gives

$F(t)=|t|=\begin{Bmatrix}-t & t\in[-\pi,0)\\ +t & t\in[0,\pi)\end{Bmatrix}\sim\frac{\pi}{2}-\frac{4}{\pi}\sum_{n\text{ odd}}\frac{\cos nt}{n^2}$, extended periodically for all *t*, a "sawtooth function".

This series could also have been calculated directly using the formulas for the Fourier coefficients and some integration by parts.

Fourier series can be scaled, shifted, etc. to produce other Fourier series

Example #1: Start with $sq(t)=\begin{cases}-1 & t\in[-\pi,0)\\ +1 & t\in[0,\pi)\end{cases} \sim \frac{4}{\pi}\sum_{n\text{ odd}}\frac{\sin nt}{n}$.

Then $1+sq(t)=\begin{cases}0 & t\in[-\pi,0)\\ 2 & t\in[0,\pi)\end{cases} \sim 1+\frac{4}{\pi}\sum_{n\text{ odd}}\frac{\sin nt}{n}$.

So $\frac{1}{2}\left[1+sq(t)\right]=\begin{cases}0 & t\in[-\pi,0)\\ 1 & t\in[0,\pi)\end{cases} \sim \frac{1}{2}+\frac{2}{\pi}\sum_{n\text{ odd}}\frac{\sin nt}{n}$, extended periodically for all *t*, a different sort of square-wave function.

Example #2: Find the Fourier series for the function $f(t)=\cos(t-\pi/3)$.
Solution: This function is periodic with period 2π. There's no need to consider the formulas for the Fourier coefficients. Simply note that $f(t)=\cos(t-\pi/3)=\cos t\cos(\pi/3)+\sin t\sin(\pi/3)=\frac{1}{2}\cos t+\frac{\sqrt{3}}{2}\sin t$.

Generalized Functions, Distributions, and Transform Methods

Our next task is to address the situation of linear *n*th order ODEs with **discontinuous and/or non-differentiable inputs**. The method we'll develop (**Laplace Transform**) will be applicable to other types of inputs, but it's especially relevant when dealing with discontinuous inputs and inputs defined only by numerical data.

The Main Idea: Beginning with a **linear *n*th order ODE with initial conditions** (an initial value problem), we'll transform this into an **algebraic equation**, solve this equation, and then **transform back** in order to produce a solution to the initial value problem. We will only be concerned with the solution for $t>0$.

Big Idea #1: Generalized functions, a.k.a. "a function is only as good as how it is integrated" - in particular, delta functions and step functions.

Big Idea #2: We'll devise a systematic way of formally solving an ODE with such inputs, and then use integration (convolution) to produce solutions to any given initial value problem.

Suppose that $g(t)$ is a function with ***compact support***, i.e. it vanishes outside some closed and bounded interval. We would like to consider two functions $f_1(t)$ and $f_2(t)$ to be **equivalent *in the sense of measurement*** if for all functions $g(t)$ with compact support, they integrate in the same way, i.e. $\int_{-\infty}^{+\infty} f_1(t)g(t)dt=\int_{-\infty}^{+\infty} f_2(t)g(t)dt$. Said differently, $\int_{-\infty}^{+\infty}[f_1(t)-f_2(t)]g(t)dt=0$ for all functions $g(t)$ with compact support. It's not hard to see that for continuous functions this means that necessarily $f_1(t)=f_2(t)$ for all t, but we're really interested in what this means for *discontinuous* functions and functions with *impulses*, i.e. "delta functions".

Heaviside functions, box functions, and delta functions

The Heaviside function [named for Oliver Heaviside (1850–1925)] is $u(t)=\begin{cases}0 & t<0\\ 1 & t>0\end{cases}$. For our purposes it really doesn't matter how it is defined at $t=0$, because it's not relevant when integrating this function. We can also define translated Heaviside functions $u_a(t)=u(t-a)=\begin{cases}0 & t<a\\ 1 & t>a\end{cases}$. These functions can be scaled and added to represent functions corresponding to "switching on and off". For example, we can represent the function $f(t)=\begin{Bmatrix} 0 & t<3 \\ 4 & 3<t<5 \\ 0 & t>5 \end{Bmatrix}=4[u_3(t)-u_5(t)]$. This is called a **box function**. We can combine box functions as necessary, e.g. $g(t)=\begin{Bmatrix} 0 & t<3 \\ 4 & 3<t<5 \\ 1 & 5<t<6 \\ 0 & t>6 \end{Bmatrix}=4[u_3(t)-u_5(t)]+1[u_5(t)-u_6(t)]=4u_3(t)-3u_5(t)-u_6(t)$.

The Heaviside function is constant everywhere except at $t=0$, and because it has a jump discontinuity there we usually just say that it's not differentiable at $t=0$. However, we could heuristically observe that by considering points immediately to the left and right of the discontinuity any continuous approximation to this function would have to have a very large slope in the vicinity of $t=0$. We might at least try to express this by saying that $u'(t)=\begin{Bmatrix} 0 & t<0 \\ \infty & t=0 \\ 0 & t>0 \end{Bmatrix}=\delta(t)$, the so-called delta function, but this doesn't really make much sense in terms of traditional functions. We may, however, still be able to make sense out of this if we take the view that "*a function is only as good as how it is integrated*." Similarly, $\dot{u}(t-a)=\delta(t-a)$, a translated delta function.

Digression – Linear functionals and measurement

One of the most common things we do in vector calculus is finding the component or scalar projection of a vector in $\mathbf{R}^n$ in a given direction. The tool used to accomplish this task is the dot product. If $\mathbf{u}$ is a unit vector we have that $\text{comp}_{\mathbf{u}}(\mathbf{v})=\mathbf{v}\cdot\mathbf{u}$. This is, in fact, a linear function from $\mathbf{R}^n$ to $\mathbf{R}$, i.e. $\mathbf{v}\in\mathbf{R}^n \;\rightarrow\; \mathbf{v}\cdot\mathbf{u}\in\mathbf{R}$. These are called **linear functional**s. Indeed, the standard components of a vector $\mathbf{v}=\langle v_1,v_2,v_3\rangle$ in $\mathbf{R}^3$ are "measured" by noting that $v_1=\mathbf{v}\cdot\mathbf{i}$, $v_2=\mathbf{v}\cdot\mathbf{j}$, and $v_3=\mathbf{v}\cdot\mathbf{k}$ using the standard unit vectors as a basis for $\mathbf{R}^3$.

If we let $L(\mathbf{v})=\text{comp}_{\mathbf{u}}(\mathbf{v})=\mathbf{v}\cdot\mathbf{u}$, i.e. the component of **v** in the direction of the unit vector **u**, we see that $L(c_1\mathbf{v}_1+c_2\mathbf{v}_2)=(c_1\mathbf{v}_1+c_2\mathbf{v}_2)\cdot\mathbf{u}=c_1\mathbf{v}_1\cdot\mathbf{u}+c_2\mathbf{v}_2\cdot\mathbf{u}=c_1L(\mathbf{v}_1)+c_2L(\mathbf{v}_2)$, so L is linear.

The Fourier coefficients are just the "measure" of how much of a given periodic function is associated with each "mode". It's really no different than calculating the components of a vector in specific directions.

$$a_0=\tfrac{1}{\pi}\int_{-\pi}^{\pi}f(t)dt \qquad a_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos nt\,dt \qquad b_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt\,dt$$

In this case, each of the calculations of Fourier coefficients takes a (periodic) function and produces a real number, e.g. $a_n=L(f)=\tfrac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos nt\,dt$.

Note that just as was the case with vectors and dot products, L acts linearly, i.e.

$$L(c_1f_1+c_2f_2)=\tfrac{1}{\pi}\int_{-\pi}^{\pi}[c_1f_1(t)+c_2f_2(t)]\cos nt\,dt$$

$$=c_1\left(\tfrac{1}{\pi}\int_{-\pi}^{\pi}f_1(t)\cos nt\,dt\right)+c_2\left(\tfrac{1}{\pi}\int_{-\pi}^{\pi}f_2(t)\cos nt\,dt\right)=c_1L(f_1)+c_2L(f_2)$$

So ***L* is also a linear functional**, though in this case *it takes functions and produces real numbers*.

There is, however, one linear functional, arguably the simplest imaginable one, that we don't usually think of in terms of integration (though maybe we should), namely ***evaluation***. Specifically, if $f(t)$ is a function, we can, for any specific value $t=a$, consider $L_a(f)=f(a)$. It's quite simple to see that

$$L_a(c_1f_1+c_2f_2)=(c_1f_1+c_2f_2)(a)=c_1f_1(a)+c_2f_2(a)=c_1L_a(f_1)+c_2L_a(f_2),$$

so L_a is a linear functional. It is not defined in terms of integration, but we will find it useful to do so nonetheless.

Generalized functions

You can heuristically think of the step function $u(t)$ as any nice smooth function which is 0 for $t < -\varepsilon$ and 1 for $t > \varepsilon$, where ε is a positive number which is much smaller than any time scale we care about in the context we are studying at the moment. Similarly, a good way for you to visualize the "delta function" (defined below) is to think of it as a function which is zero everywhere except in the immediate neighborhood of $t = 0$ and which has integral 1. As we'll see, we can also think of the delta functions $\delta(t)$ and $\delta(t-a)$ as the "function you integrate against" in order to evaluate a function at respectively $t = 0$ and at any $t = a$. That is, $\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$ and $\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$. *How can we make sense of this?*

Making the most of integration by parts

In first-year Calculus we learned that for differentiable functions $u(t)$ and $v(t)$ the Product Rule applies, i.e. $\frac{d}{dt}[u(t)v(t)] = u(t)v'(t) + v(t)u'(t)$.

On any finite interval $[a,b]$ we can integrate both sides of the Product Rule this and apply the Fundamental Theorem of Calculus to get that $u(b)v(b) - u(a)v(a) = \int_a^b \frac{d}{dt}[u(t)v(t)]\,dt = \int_a^b u(t)v'(t)dt + \int_a^b v(t)u'(t)dt$. Though we often think of Integration by Parts as defined formally by $\int u dv = uv - \int v du$, the stated result is really what this means, i.e. we can say that $\int_a^b u(t)v'(t)dt = \left[uv\right]_a^b - \int_a^b v(t)u'(t)dt$.

If one of these functions has compact support, i.e. if it vanishes outside of some closed, bounded (compact) interval, then we can extend the result to the entire real line and simplify the statement considerably (since the value of the product of the functions will vanish outside some interval. Specifically, if $g(t)$ has compact support and if $f(t)$ is any function, we can say that $\boxed{\int_{-\infty}^{+\infty} f'(t)g(t)dt = -\int_{-\infty}^{+\infty} f(t)g'(t)dt}$. We can actually use this to define a derivative $f'(t)$ in a generalized way, i.e. a ***generalized derivative***. It rests on the notion that functions can be understood by *how they are integrated* and not just by *how they are evaluated*. These ***generalized functions*** are also known as ***distributions***.

Perhaps the most important illustration of this is the generalized derivative of the Heaviside function $u(t)$. We formally called this the delta function $u'(t) = \delta(t)$ even though it didn't really make sense at the point of discontinuity $t = 0$ for the Heaviside function. However, we can say that if $u'(t) = \delta(t)$ then for any function $g(t)$ with compact support:

$$\int_{-\infty}^{+\infty} \delta(t)g(t)dt = \int_{-\infty}^{+\infty} u'(t)g(t)dt = -\int_{-\infty}^{+\infty} u(t)g'(t)dt = -\int_{0}^{+\infty} g'(t)dt = -[0 - g(0)] = g(0).$$

That is, if we "integrate a function against the delta function", this is simply evaluation of that function at 0. It is really this property that defines the delta function as a generalized function.

Similarly, we can do the same for the translated Heaviside function $u_a(t) = u(t-a)$ to conclude that its generalized derivative $\delta_a(t) = \delta(t-a)$ is such that for any function $g(t)$ with compact support:

$$\int_{-\infty}^{+\infty} g(t)\delta_a(t)dt = \int_{-\infty}^{+\infty} g(t)\delta(t-a)dt = g(a).$$

You can also take a sequential approach to make sense of this in terms of limits, i.e. if you successively approximate the delta function by a sequence of continuous functions $f_k(t)$ where the support (domain where it's nonzero) gets narrower $[-\varepsilon_k, +\varepsilon_k]$ and the values grow reciprocally in such a way that at each step the integral is always $\int_{-\infty}^{+\infty} f_k(t)dt = \int_{-\varepsilon_k}^{+\varepsilon_k} f_k(t)dt = 1$ (we call such functions probability densities), then you can show that $\lim_{k\to\infty}\left[\int_{-\infty}^{+\infty} g(t)\, f_k(t)dt\right] = g(0)$.

Note: The Fundamental Theorem of Calculus as well as all the usual rules of differentiation also apply to generalized derivatives, so we actually have a "generalized calculus" for dealing with these generalized functions or distributions (though it may take a while getting used to it). Basically, we extend the usual rules of differentiation to generalized functions together with the fact that $\dot{u}(t-a) = \delta(t-a)$.

A function $f(t)$ is "**regular**" or "piecewise smooth" if it can be broken into pieces each having all higher derivatives and such that at each breakpoint $f^{(n)}(a-)$ and $f^{(n)}(a+)$ exist. A "singularity function" is a linear combination of shifted delta functions. A "**generalized function**" $f(t)$ is a sum $f(t) = f_r(t) + f_s(t)$ of a regular function and a singularity function. Any regular function $f(t)$ has a "**generalized derivative**" $f'(t)$, with regular part $f_r{}'(t)$ the regular derivative of $f(t)$ wherever it exists, and singular part $f_s{}'(t)$ given by a sum of terms $(f(a+) - f(a-))\delta(t-a)$ as a runs over the discontinuities of f.

Now, to get back to **the Main Idea**, how can we solve a linear differential equation $[p(D)]x(t) = q(t)$ by transforming it into an algebraic equation, solving that algebraic equation, and then transforming back to produce a solution to an initial value problem? As we will only be concerned with forward time, we'll presume that $q(t)$ satisfies $q(t) = 0$ for $t < 0$.

The Laplace Transform

Definition: The ***Laplace transform*** of a function $f(t)$ is defined by

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} e^{-st} f(t)dt$$

where the new (complex) variable s is such that its real part $\text{Re}(s) \gg 0$ (the integral would otherwise not converge). Note that the lower limit of the integral indicates that $t = 0$ is included and is intended to address potential discontinuities and delta functions.

We will liberally make use of the convention that a function of t will be represented by a lower case name and its Laplace transform by the corresponding upper case name, e.g. $\mathcal{L}[x(t)] = X(s)$.

Linearity

Because the Laplace transform is defined as an integral, it's easy to see that:

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s) + bG(s).$$

Specifically:

$$\mathcal{L}[af(t)+bg(t)] = \int_{0-}^{\infty} e^{-st}[af(t)+bg(t)]dt = a\int_{0-}^{\infty} e^{-st} f(t)dt + b\int_{0-}^{\infty} e^{-st} g(t)dt$$
$$= a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s) + bG(s)$$

This will permit us to transform a differential equation term-by-term (and transform back as well).

Inverse transform: $F(s)$ essentially determines $f(t)$ for $t \ge 0$. This will generally allow us to produce solutions to a given Initial Value Problem by simply recognizing, term by term, a solution by identifying which functions gave rise to each term of the transformed differential equation.

Some Calculations

1) For our purposes, since we are only concerned with $t \geq 0$, the constant function $f(t) = 1$ and the Heaviside function $u(t) = \begin{Bmatrix} 0 & t < 0 \\ 1 & t > 0 \end{Bmatrix}$ are indistinguishable. Thus $\mathcal{L}[1] = \mathcal{L}[u(t)] = \int_{0-}^{\infty} e^{-st} \cdot 1\, dt = \left[\frac{e^{-st}}{-s} \right]_{t=0}^{t=\infty} = 0 + \frac{1}{s} = \frac{1}{s}$

Here we used the fact that for $s > 0$, $\lim_{t \to \infty} \left[e^{-st} \right] = 0$. Indeed, this is still the case even if we permit s to be complex with positive real part, i.e. $\mathrm{Re}(s) > 0$.

2) If $f(t) = t$, we calculate its Laplace Transform as

$$F(s) = \mathcal{L}[t] = \int_{0-}^{\infty} te^{-st} dt = \left[\frac{te^{-st}}{-s} \right]_{t=0-}^{t=\infty} + \frac{1}{s} \int_{0-}^{\infty} e^{-st} dt = 0 + \frac{1}{s} \mathcal{L}[1] = \frac{1}{s^2}$$

3) If $f(t) = t^2$, we calculate $F(s) = \mathcal{L}[t^2] = \int_{0-}^{\infty} t^2 e^{-st} dt = \left[\frac{t^2 e^{-st}}{-s} \right]_{t=0-}^{t=\infty} + \frac{2}{s} \int_{0-}^{\infty} te^{-st} dt = 0 + \frac{2}{s} \mathcal{L}[t] = \frac{2}{s} \cdot \frac{1}{s^2} = \frac{2}{s^3}$

4) ***s*-derivative rule**: $\boxed{\mathcal{L}[t f(t)] = -F'(s)}$. We can establish this by noting that if $F(s) = \mathcal{L}[f(t)] = \int_{0-}^{\infty} e^{-st} f(t) dt$, then $F'(s) = \frac{d}{ds} \int_{0-}^{\infty} e^{-st} f(t) dt = -\int_{0-}^{\infty} e^{-st} t\, f(t) dt = -\mathcal{L}[t\, f(t)]$, so $\mathcal{L}[t f(t)] = -F'(s)$.

From this we see that $\mathcal{L}[t^2] = \mathcal{L}[t \cdot t] = -\frac{d}{ds} \mathcal{L}[t] = -\frac{d}{ds} \left[\frac{1}{s^2} \right] = \frac{2}{s^3}$; $\mathcal{L}[t^3] = \mathcal{L}[t \cdot t^2] = -\frac{d}{ds} \mathcal{L}[t^2] = -\frac{d}{ds} \left[\frac{2}{s^3} \right] = \frac{3!}{s^4}$; $\mathcal{L}[t^4] = \mathcal{L}[t \cdot t^3] = -\frac{d}{ds} \mathcal{L}[t^3] = -\frac{d}{ds} \left[\frac{3!}{s^4} \right] = \frac{4!}{s^5}$; and so on. Generally, $\boxed{\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}}$.

This, together with linearity, enables us to calculate the Laplace transform of any polynomial function.

5) If $f(t) = e^{at}$ is an exponential function (really $f(t) = u(t)e^{at}$ since we are only concerned with $t \geq 0$),

$$\mathcal{L}[e^{at}] = \int_{0-}^{\infty} e^{-st} e^{at} dt = \int_{0-}^{\infty} e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_{t=0-}^{t=\infty} = \frac{1}{s-a}, \text{ so } \boxed{\mathcal{L}[e^{at}] = \frac{1}{s-a}}.$$

6) ***s*-shift rule**: $\boxed{\mathcal{L}[e^{rt} f(t)] = F(s-r)}$. To establish this, we calculate

$\mathcal{L}[e^{rt} f(t)] = \int_{0-}^{\infty} e^{-st} e^{rt} f(t) dt = \int_{0-}^{\infty} e^{-(s-r)t} f(t) dt = F(s-r)$ simply by noting the substitution.

7) **Transforming derivatives**: For any generalized function, $\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0-)}$ where $f(0-)$ represents the initial value of $f(t)$. The unusual notation is there because we will be dealing with discontinuous and generalized functions where we may need to distinguish left-hand from right-hand limits. We can establish this ***t*-derivative rule** by noting that $\mathcal{L}[f'(t)] = \int_{0-}^{\infty} e^{-st} f'(t) dt$. If we use Integration by Parts with $u = e^{-st}$ and $dv = f'(t)dt$, we get $du = -se^{-st}dt$ and $v = f(t)$, so:

$$\mathcal{L}[f'(t)] = \int_{0-}^{\infty} e^{-st} f'(t) dt = \left[e^{-st} f(t) \right]_{t=0-}^{t=\infty} + s\int_{0-}^{\infty} e^{-st} f(t) dt = [0] + sF(s) = sF(s) - f(0-)$$

For second derivatives, note that $f''(t) = \frac{d}{dt} f'(t)$, so we can apply the above result to get that $\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0-) = s(sF(s) - f(0-)) - f'(0-) = s^2 F(s) - s \cdot f(0-) - f'(0-)$, so $\boxed{\mathcal{L}[f''(t)] = s^2 F(s) - s\, f(0-) - f'(0-)}$.

Continuing, we get that $\boxed{\mathcal{L}[f'''(t)] = s^3F(s) - s^2 f(0-) - s f'(0-) - f''(0-)}$, and so on.

Generally, $\boxed{\mathcal{L}[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0-) - s^{n-2}f'(0-) - \cdots - f^{(n-1)}(0-)}$.

8) **Transforming the delta function**: One of our most fundamental transforms is $\boxed{\mathcal{L}[\delta(t)] = 1}$. This is relatively easy to see once you're comfortable with the integral formalisms concerning the delta function and how they relate to evaluation. Specifically, $\mathcal{L}[\delta(t)] = \int_{0-}^{\infty} e^{-st}\delta(t)dt = e^0 = 1$ since this is really just evaluation of the function e^{-st} at $t = 0$.

9) **Transforming sines and cosines**: $\boxed{\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}}$ and $\boxed{\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}}$

We can derive each of these independently, but if we use Euler's Formula and linearity we have that:

$$\mathcal{L}[e^{i\omega t}] = \mathcal{L}[\cos(\omega t) + i\sin(\omega t)] = \mathcal{L}[\cos(\omega t)] + i\mathcal{L}[\sin(\omega t)], \text{ and}$$

$$\mathcal{L}[e^{i\omega t}] = \frac{1}{s-i\omega}\left[\frac{s+i\omega}{s+i\omega}\right] = \frac{s+i\omega}{s^2+\omega^2} = \left(\frac{s}{s^2+\omega^2}\right) + i\left(\frac{\omega}{s^2+\omega^2}\right)$$

Taking real and imaginary parts separately we get that

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \text{ and } \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}.$$

We'll add to this list as we go and as the need arises.

Example: Solve the Initial Value Problem $\ddot{x} + 3\dot{x} + 2x = 2e^{-t}, \quad x(0) = 0, \quad \dot{x}(0) = 0$.

Old Faithful Solution: The homogeneous equation $\ddot{x} + 3\dot{x} + 2x = 0$ is easy to solve. Its characteristic polynomial is $p(s) = s^2 + 3s + 2 = (s+2)(s+1)$ which yields the two roots $s = -2$ and $s = -1$. This gives the two independent solutions e^{-2t} and e^{-t}, and all homogeneous solutions are of the form $x_h(t) = c_1e^{-2t} + c_2e^{-t}$. Note that both of these homogeneous solutions are <u>transient</u> in the sense that they decay exponentially as t increases.

Next, we need to find a particular solution $x_p(t)$ that satisfies the inhomogeneous differential equation. One look at the right-hand-side and we see that the Exponential Response Formula (ERF) won't work – there is resonance. We can, however, use the Resonant Response Formula to get the particular solution $x_p(t) = \frac{2te^{-t}}{p'(-1)} = \frac{2te^{-t}}{1} = 2te^{-t}$, so the general solution is $x(t) = x_h(t) + x_p(t) = c_1e^{-2t} + c_2e^{-t} + 2te^{-t}$. Its derivative is $\dot{x}(t) = -2c_1e^{-2t} - c_2e^{-t} - 2te^{-t} + 2e^{-t}$. Substituting the (rest) initial conditions gives $\left\{\begin{matrix} x(0) = c_1 + c_2 = 0 \\ \dot{x}(0) = -2c_1 - c_2 + 2 = 0 \end{matrix}\right\}$, and these can be solved to give $c_1 = 2, c_2 = -2$, so the solution is $\boxed{x(t) = 2e^{-2t} - 2e^{-t} + 2te^{-t}}$.

Solving directly by Laplace transform: We calculated the following Laplace transforms:

(1) $\mathcal{L}(e^{kt}) = \frac{1}{s-k}$ with region of convergence $\mathrm{Re}(s) > k$, so $\mathcal{L}(e^{-2t}) = \frac{1}{s+2}$.

(2) If the Laplace transform of $x(t)$ is $X(s)$, then the Laplace transforms of its derivatives are $\mathcal{L}(\dot{x}(t)) = sX(s) - x(0-)$ and $\mathcal{L}(\ddot{x}(t)) = s^2X(s) - s\,x(0-) - \dot{x}(0-)$. In the case of rest initial conditions $x(0-) = \dot{x}(0-) = 0$, these are greatly simplified and, in fact $\mathcal{L}(p(D)x) = p(s)X(s)$. Specifically, $\mathcal{L}(\ddot{x} + 3\dot{x} + 2x) = s^2X(s) + 3sX(s) + 2X(s) = (s^2 + 3s + 2)X(s) = p(s)X(s)$.

If we now transform the entire differential equation, we get $(s^2+3s+2)X(s)=\dfrac{2}{s+1}$.

We then solve for $X(s)=\dfrac{2}{(s+1)(s^2+3s+2)}=\dfrac{2}{(s+2)(s+1)^2}=\dfrac{A}{s+2}+\dfrac{B}{s+1}+\dfrac{C}{(s+1)^2}$.

There are many good ways to find the unknowns *A*, *B*, and *C*. For example, if we multiply through by the common denominator to clear fractions, we get $2=A(s+1)^2+B(s+1)(s+2)+C(s+2)$. Plugging in the specific values $s=-2$ and $s=-1$ quickly yields that $A=2$ and $C=2$. Plugging in, for example, $s=0$ and using the values for *A* and *C* then yields $B=-2$. So:

$$X(s)=\frac{2}{s+2}-\frac{2}{s+1}+\frac{2}{(s+1)^2}=2\left(\frac{1}{s+2}\right)-2\left(\frac{1}{s+1}\right)+2\left(\frac{1}{(s+1)^2}\right).$$

Consulting our table of common Laplace transforms, we see that $\dfrac{1}{s+2}=\mathcal{L}(e^{-2t})$, $\dfrac{1}{s+1}=\mathcal{L}(e^{-t})$, and $\dfrac{1}{(s+1)^2}=\mathcal{L}(te^{-t})$, so transforming back (using linearity) gives $\boxed{x(t)=2e^{-2t}-2e^{-t}+2te^{-t}}$.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} e^{-st} f(t)dt$ for $\text{Re}(s) \gg 0$.

1. Linearity: $\mathcal{L}[af(t)+bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s)+bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{rt} f(t)] = F(s-r)$.

4. t-shift rule: $\mathcal{L}[f(t-a)] = e^{-as}F(s)$ if $a \geq 0$ and $f(t)=0$ for $t<0$.

   This may also be expressed as $\mathcal{L}[f_a(t)] = e^{-as}F(s)$ where $f_a(t) = u(t-a)f(t-a) = \begin{Bmatrix} f(t-a) & \text{if } t>a \\ 0 & \text{if } t<a \end{Bmatrix}$.

5. s-derivative rule: $\mathcal{L}[t f(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0-)$

$$\mathcal{L}[f''(t)] = s^2F(s) - sf(0-) - f'(0-)$$

$$\mathcal{L}[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0-) - s^{n-2}f'(0-) - \cdots - f^{(n-1)}(0-)$$

7. Convolution rule: $\mathcal{L}[f(t)*g(t)] = F(s)G(s)$, $(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s)$, $w(t)$ the unit impulse response.

   If $q(t)$ is regarded as the input signal in $p(D)x = q(t)$, $W(s) = \frac{1}{p(s)}$.

Formulas for the Laplace transform

$\mathcal{L}[1] = \dfrac{1}{s}$

$\mathcal{L}[\delta(t)] = 1$

$\mathcal{L}[\delta(t-a)] = \mathcal{L}[\delta_a(t)] = e^{-as}$

$\mathcal{L}[u(t-a)] = \mathcal{L}[u_a(t)] = \dfrac{e^{-as}}{s}$

$\mathcal{L}[e^{at}] = \dfrac{1}{s-a}$

$\mathcal{L}[t] = \dfrac{1}{s^2}$

$\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$

$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$

$\mathcal{L}[u(t-a)f(t-a)] = e^{-as}F(s)$

$\mathcal{L}[u(t-a)f(t)] = e^{-as}\mathcal{L}[f(t+a)]$

$\mathcal{L}[\cos(\omega t)] = \dfrac{s}{s^2+\omega^2}$

$\mathcal{L}[\sin(\omega t)] = \dfrac{\omega}{s^2+\omega^2}$

$\mathcal{L}[t\cos(\omega t)] = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

$\mathcal{L}[t\sin(\omega t)] = \dfrac{2\omega s}{(s^2+\omega^2)^2}$

$\mathcal{L}[e^{zt}\cos(\omega t)] = \dfrac{s-z}{(s-z)^2+\omega^2}$

$\mathcal{L}[e^{zt}\sin(\omega t)] = \dfrac{\omega}{(s-z)^2+\omega^2}$

Notes by Robert Winters