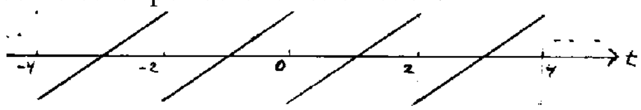
Ordinary Differential Equations – Lecture #7

Periodic Inputs and Fourier Series

The solution of an ODE of the form $[p(D)]x(t) = a\cos kt$ or $[p(D)]x(t) = a\sin kt$ is now relatively straightforward through the use of complex replacement, the Exponential Response Formula, and, when needed, the Resonant Response Formula. How might we solve an ODE of the form $[p(D)]x(t) = f(t)$ where $f(t)$ is some other periodic function such:



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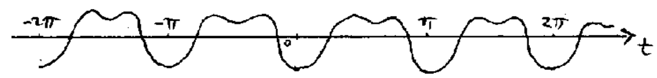


The way we'll handle this is to successively approximate any such periodic function as a sum of trigonometric functions, solve term-by-term, and then reassemble a solution using linearity (superposition). The approximation method involved **Fourier Series**.

Definition: A function $f(t)$ is called **periodic** with period T if $f(t+nT) = f(t)$ for all t and all integers n. We say that T is the **base period** if it is the least such $T > 0$.

Examples: The functions $\sin t$ and $\cos t$ are both periodic with base period 2π. The functions $\sin \omega t$ and $\cos \omega t$ are both periodic with base period $\frac{2\pi}{\omega}$.

Note: Any constant function is also periodic, but with no base period.

For the sake of simplicity, we'll begin by considering periodic functions with base period 2π. We will later rescale to adapt our methods to other base periods. Our methods will be based on the following theorem:

Theorem (Fourier): Suppose a function $f(t)$ is periodic with base period 2π and continuous except for a finite number of jump discontinuities. Then $f(t)$ may be represented by a (convergent) Fourier Series:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

where: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt$.

The numbers $\{a_0, a_1, b_1, \cdots, a_n, b_n, \cdots\}$ are called the **Fourier coefficients** of the function $f(t)$.

This representation is an equality at all points of continuity of the function $f(t)$. At any point of discontinuity $t = a$, the series converges to the average of $f(a^-)$ and $f(a^+)$, i.e. the value $\frac{1}{2}[f(a^-) + f(a^+)]$.

Note: (a) If $f(t)$ is an even function [$f(-t) = f(t)$ for all t], then $b_n = 0$ for all n by basic facts from calculus.

(b) If $f(t)$ is an odd function [$f(-t) = -f(t)$ for all t], then $a_0 = 0$ and $a_n = 0$ for all n by basic facts from calculus.

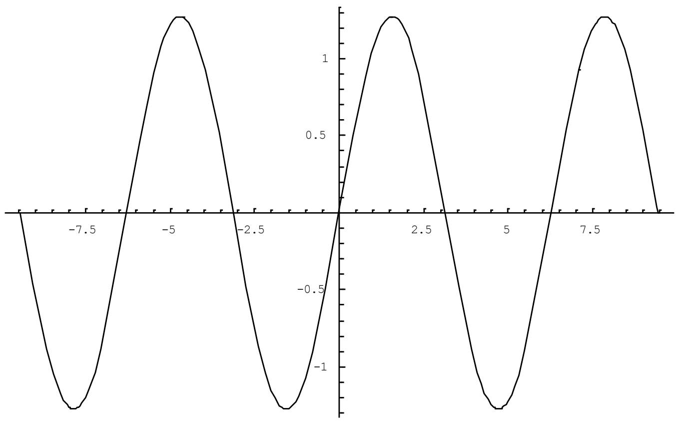
Example (Square wave function): $f(t) = sq(t) = \begin{Bmatrix} -1 & t \in [-\pi, 0) \\ +1 & t \in [0, \pi) \end{Bmatrix}$, extended periodically for all t.

This function is periodic (with period 2π) and antisymmetric, i.e. an odd function. Therefore $a_0 = 0$ and $a_n = 0$ for all n. We calculate $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin nt\, dt + \int_{0}^{\pi}\sin nt\, dt\right] = \frac{1}{\pi}\left[\left[\frac{\cos nt}{n}\right]_{-\pi}^{0} - \left[\frac{\cos nt}{n}\right]_{0}^{\pi}\right]$

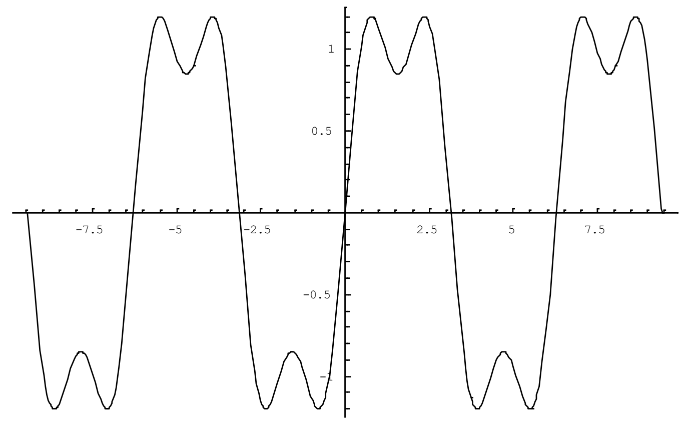
$$= \frac{1}{n\pi}\left[\left[1-(-1)^n\right] - \left[(-1)^n - 1\right]\right] = \begin{Bmatrix} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{Bmatrix}.$$

So $\boxed{sq(t) \sim \frac{4}{\pi}\sum_{n\ \mathrm{odd}} \frac{\sin nt}{n} = \frac{4}{\pi}[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots]}$.

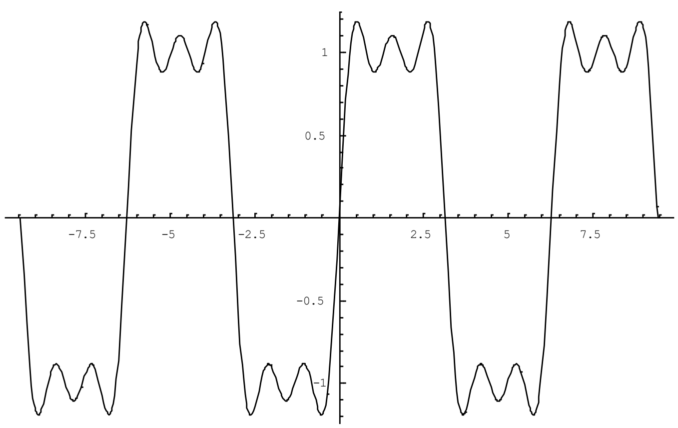
The nature of the convergence of this Fourier series toward the square wave function can be seen by graphing the partial sums:



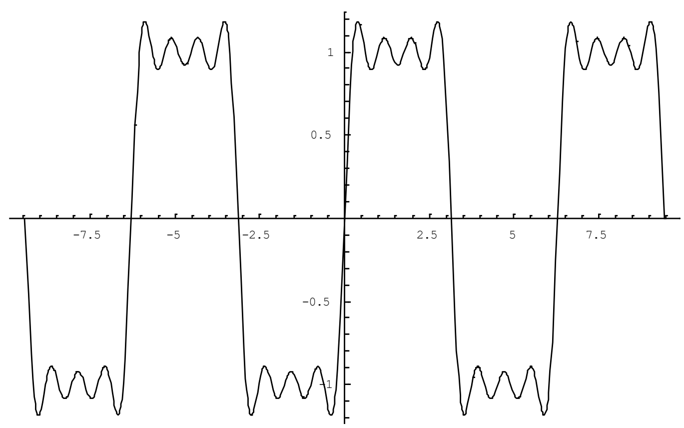
n = 1



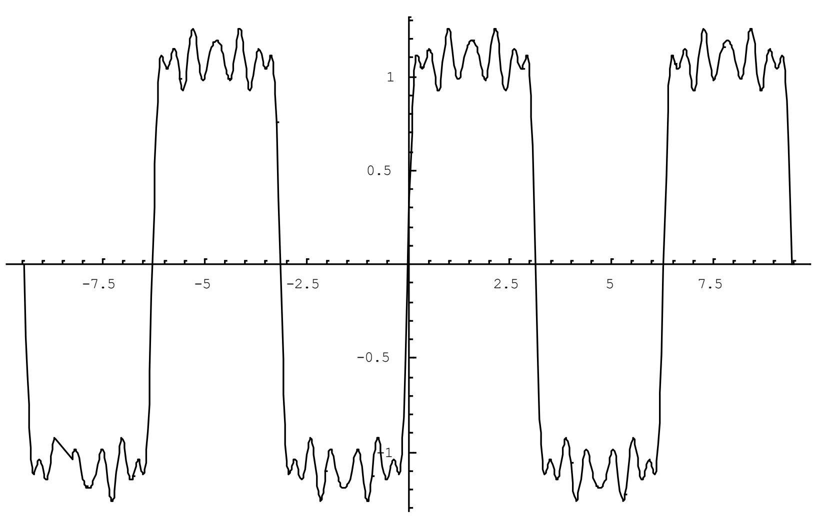
n = 3



n = 5



n = 7



n = 15

Inner Products and Orthogonality

We are all familiar with the mutually perpendicular (orthogonal) unit vectors $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ in $\mathbf{R}^3$ and how we can express any vector $\mathbf{v}=\langle x,y,z\rangle$ in $\mathbf{R}^3$ as $\mathbf{v}=\langle x,y,z\rangle = x\langle 1,0,0\rangle + y\langle 0,1,0\rangle + z\langle 0,0,1\rangle = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. The components of the vector are just the scalar projections of $\mathbf{v}$ in the directions of $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$, respectively. We find the scalar projection of a vector in any given direction by calculating its dot product with a unit vector $\mathbf{u}$ in the given direction, i.e. $\mathbf{v}\cdot\mathbf{u}$. So we can also express $\mathbf{v}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=(\mathbf{v}\cdot\mathbf{i})\mathbf{i}+(\mathbf{v}\cdot\mathbf{j})\mathbf{j}+(\mathbf{v}\cdot\mathbf{k})\mathbf{k}$.

We could do the same thing with any set of three mutually orthogonal unit vectors $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ in $\mathbf{R}^3$. That is, if $\mathbf{v}\in\mathbf{R}^3$ we could write $\mathbf{v}=(\mathbf{v}\cdot\mathbf{u}_1)\mathbf{u}_1+(\mathbf{v}\cdot\mathbf{u}_2)\mathbf{u}_2+(\mathbf{v}\cdot\mathbf{u}_3)\mathbf{u}_3$.

Just as we can add and scale vectors, we can also do this with functions. We add functions by adding their values and scale them by scaling their values. That is, $(f+g)(x)=f(x)+g(x)$ and $(cf)(x)=cf(x)$. We might speculate that if functions can be combined in a manner analogous to vectors (where we add respective values instead of respective components), perhaps there may be something analogous to the dot product that we could use to define notions such as orthogonality in spaces of functions.

Think about how the dot product of two vectors is calculated: We multiply the respective components of the two vectors and sum these products, i.e. in $\mathbf{R}^3$ we have $\mathbf{u}\cdot\mathbf{v}=\langle u_1,u_2,u_3\rangle\cdot\langle v_1,v_2,v_3\rangle = u_1v_1+u_2v_2+u_3v_3\in\mathbf{R}$ and, more generally, in $\mathbf{R}^n$ we have $\mathbf{u}\cdot\mathbf{v}=\langle u_1,\cdots,u_n\rangle\cdot\langle v_1,\cdots,v_n\rangle = u_1v_1+\cdots+u_nv_n\in\mathbf{R}$. We also derive using the Law of Cosines that $\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. It is from this fact that we conclude that nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u}\cdot\mathbf{v}=0$.

Vectors in $\mathbf{R}^3$ (or any $\mathbf{R}^n$) have just a finite list of components, whereas functions of a real variable have infinitely many values. If we think of the values of a function as analogous to the components of a vector, and if we use integration as analogous to a discrete sum, this suggests the following definition:

Definition: If f,g are functions defined on some interval $[a,b]$, we can define an **inner product** of these functions by $\langle f,g\rangle=\int_a^b f(t)g(t)\,dt$. Such an inner product will then satisfy the following properties (where defined for any functions f,g,h):

(1) $\langle g,f\rangle=\langle f,g\rangle$ (symmetric)

(2) $\langle f+g,h\rangle=\langle f,h\rangle+\langle g,h\rangle$ and $\langle f,g+h\rangle=\langle f,g\rangle+\langle f,h\rangle$ (left and right distributive laws)

(3) $\langle cf,g\rangle=c\langle f,g\rangle=\langle f,cg\rangle$ for any scalar c

These properties are analogous to the algebraic properties of the dot product. We can use this inner product to define orthogonality.

Definition: We say that two (nonzero) functions f,g are orthogonal if $\langle f,g\rangle=0$.

There is a 4th property of the dot product that doesn't quite work as simply in the context of functions. That is, for any vector $\mathbf{v}$, we have $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2\ge 0$ and $\mathbf{v}\cdot\mathbf{v}=0$ only if $\mathbf{v}=\mathbf{0}$. The corresponding statement for functions is not generally true. It will always be the case that $\langle f,f\rangle=\int_a^b [f(t)]^2\,dt\ge 0$, but this inner product could be equal to zero for a function that is not identically 0 with isolated discontinuities where the function takes on nonzero values. If we only consider continuous functions, then $\langle f,f\rangle=\int_a^b [f(t)]^2\,dt=0$ would imply that $f(t)$ is identically zero on the interval $[a,b]$. In any case, we can still define the **norm of a function *f*** by $\|f\|^2 = \langle f, f\rangle = \int_a^b [f(t)]^2\, dt$ or $\|f\| = \sqrt{\langle f, f\rangle}$.

It's certainly possible that these integrals might not be defined, so we generally restrict the set of functions to those for which $\|f\|^2 = \langle f, f\rangle = \int_a^b [f(t)]^2\, dt$ is finite. These are called "square summable" functions, and the set of all such functions for the interval $[a,b]$ is denoted by $L^2([a,b])$. It can be shown that in this space of functions, $|\langle f, g\rangle| \le \|f\|\|g\|$ (Cauchy-Schwarz Inequality) and $\|f+g\| \le \|f\| + \|g\|$ (Triangle Inequality).

Note: All of the above properties work just as well if we define the inner product as $\langle f, g\rangle = K\int_a^b f(t)g(t)dt$ for some fixed constant *K*. This will alter the way in which the norm of a given function is defined, but using such a normalizing constant is often desirable when working with a particular interval. We'll start by considering functions defined on the interval $[-\pi, \pi]$ and choose our normalizing constant so that $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt$.

Proposition: In $L^2([-\pi, +\pi])$ with inner product $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt$, the finite collection $\mathscr{B}_n = \left\{\frac{1}{\sqrt{2}}, \cos t, \sin t, \cos 2t, \sin 2t, \cdots, \cos nt, \sin nt\right\}$ is an **orthonormal** set. That is, each function in $\mathscr{B}_n$ has norm 1 and any distinct pair has inner product equal to 0. We think of this set as consisting of mutually orthogonal unit elements. In linear algebra terminology, we would say that these $2n+1$ functions **span** a **subspace** (referred to as T_n) and that they form a **basis** for this subspace.

Proof: This is just a list of integral calculations. We'll calculate a few of them and just quote the rest (though you may want to try some integration techniques or consult an integral table to see why they are true). We have:

$$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{2}}\right)^2 dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}\,dt = \frac{1}{\pi}(\tfrac{1}{2})2\pi = 1$$

$$\left\langle \frac{1}{\sqrt{2}}, \cos kt \right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\sqrt{2}}\cos kt\, dt = \frac{1}{\pi\sqrt{2}}\left.\frac{\sin kt}{k}\right|_{-\pi}^{\pi} = 0$$

$$\left\langle \frac{1}{\sqrt{2}}, \sin kt \right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\sqrt{2}}\sin kt\, dt = 0 \qquad \text{(integral of an odd function over a symmetric interval)}$$

$$\langle \cos kt, \cos kt\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 kt\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}[1+\cos(2kt)]\,dt = \frac{1}{2\pi}\left[t + \frac{\sin(2kt)}{2k}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}(2\pi) = 1$$

$$\langle \sin kt, \sin kt\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 kt\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}[1-\cos(2kt)]\,dt = \frac{1}{2\pi}\left[t - \frac{\sin(2kt)}{2k}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}(2\pi) = 1$$

$$\langle \cos jt, \cos kt\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos jt\cos kt\, dt = 0 \text{ for integers } j \ne k \qquad \text{(consult integral table)}$$

$$\langle \sin jt, \sin kt\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin jt\sin kt\, dt = 0 \text{ for integers } j \ne k \qquad \text{(consult integral table)}$$

$$\langle \cos jt, \sin kt\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos jt\sin kt\, dt = 0 \text{ for integers } j, k \qquad \text{(consult integral table)}$$

We can define the orthogonal projection of a function *f* onto the subspace T_n analogous to $\mathbf{R}^n$, namely:

$$f_n = \mathrm{Proj}_n(f) = \left\langle f, \frac{1}{\sqrt{2}}\right\rangle\frac{1}{\sqrt{2}} + \langle f, \cos t\rangle\cos t + \langle f, \sin t\rangle\sin t + \cdots + \langle f, \cos nt\rangle\cos nt + \langle f, \sin nt\rangle\sin nt$$

If we express these in terms of integrals, we get:

$$f_n = \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt\right] + \left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos t\,dt\right]\cos t + \left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin t\,dt\right]\sin t + \cdots$$
$$\cdots + \left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt\right]\cos nt + \left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt\right]\sin nt$$

This function f_n is known as the ***n*th order Fourier approximation** of the function *f*.

This can be expressed more succinctly by defining the **Fourier coefficients** by:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt$$

Then $f_n = \frac{a_0}{2} + a_1\cos t + b_1\sin t + \cdots + a_n\cos nt + b_n\sin nt$ is the *n*th order Fourier approximation.

If, for any given *n*, we express $f = (f - f_n) + f_n$, we can think of $f_n \in T_n$ and $(f - f_n) \in T_n^{\perp}$ (known as the orthogonal complement of T_n). There is the **analogue of the Pythagorean Theorem** in this context that gives that $\|f\|^2 = \|f - f_n\|^2 + \|f_n\|^2$. With some careful analysis it can be shown that as *n* gets larger, the Fourier approximation converges in the sense that $\lim_{n\to\infty}\|f - f_n\|^2 = 0$, and this implies that $\boxed{\lim_{n\to\infty}\|f_n\|^2 = \|f\|^2}$ which can provide some very interesting results.

We also by letting $n \to \infty$ produce the **Fourier Series** of *f* as $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt)$, and there's this accompanying theorem (proven elsewhere):

Theorem (Fourier): Suppose a function $f(t)$ is periodic with base period 2π and continuous except for a finite number of jump discontinuities. Then $f(t)$ may be represented by a (convergent) Fourier Series:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt)$$

where: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$.

The numbers $\{a_0, a_1, b_1, \cdots, a_n, b_n, \cdots\}$ are called the **Fourier coefficients** of the function $f(t)$.

This representation is an equality at all points of continuity of the function $f(t)$. At any point of discontinuity $t = a$, the series converges to the average of $f(a^-)$ and $f(a^+)$, i.e. the value $\frac{1}{2}[f(a^-) + f(a^+)]$.

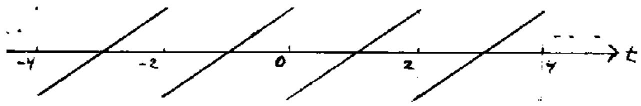
We performed these calculations earlier for the **square-wave function** $f(t) = sq(t) = \begin{Bmatrix} -1 & t \in [-\pi, 0) \\ +1 & t \in [0, \pi) \end{Bmatrix}$, extended periodically for all *t*, and derived that $\boxed{sq(t) \sim \frac{4}{\pi}\sum_{n\text{ odd}}\frac{\sin nt}{n} = \frac{4}{\pi}[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots]}$.

If we translate $\lim_{n\to\infty}\|f_n\|^2 = \|f\|^2$ for this function we get that $\|f\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} 1dt = 2$, and

$\lim_{n\to\infty}\|f_n\|^2 = \frac{16}{\pi^2}\left[1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots\right] = \frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = 2$. Therefore $\boxed{\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}}$, a curious fact.

We can also apply the last statement in Fourier's Theorem by evaluating the square-wave function at $\pi/2$, a point of continuity, to get that $sq(\pi/2)=1=\frac{4}{\pi}[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots]$, so $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$, but the convergence is so abysmally slow as to be of no practical consequence – another curiosity.

Much more relevant to us is how Fourier series can be applied to the solution of linear time-invariant ODEs of the form $[p(D)]x(t)=f(t)$ where $f(t)$ is some other periodic function such as:



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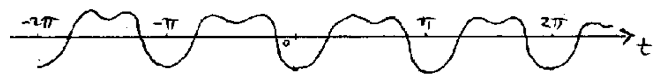

Our plan will be to (a) express $f(t)$ as a Fourier series, (b) solve the ODE with each term of the Fourier series as the (sinusoidal) input, and then (c) reassemble the solution for the input $f(t)$ using linearity principles.

Harmonic Response to Periodic Inputs

If we couple the Fourier series representation of a periodic input with linearity, we can produce series representations to linear time-independent (LTI) differential equations.

Example: Find the general solution to the differential equation $\ddot{x}+4x=sq(t)$, where $sq(t)$ is the square-wave function.

Solution: The system corresponds to a harmonic oscillator. The characteristic polynomial is $p(s)=s^2+4$ with characteristic roots $s=\pm 2i$ and the homogeneous solutions are of the form $x_h(t)=c_1\cos 2t+c_2\sin 2t$.

For a particular solution, we use linearity. Using the Fourier series representation $sq(t)\sim\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{2n+1}$, we individually solve $\ddot{x}+4x=\sin(2n+1)t$ for each *n*. To do this we use complex replacement and solve $\ddot{z}+4z=e^{i(2n+1)t}$ using the Exponential Response Formula (ERF). We have $p(i(2n+1))=4-(2n+1)^2$, so $\frac{e^{i(2n+1)t}}{4-(2n+1)^2}=\frac{\cos(2n+1)t+i\sin(2n+1)t}{4-(2n+1)^2}$ is a solution, and we extract its imaginary part to get $\frac{\sin(2n+1)t}{4-(2n+1)^2}$.

Using linearity for the ODE $\ddot{x}+4x=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{2n+1}$, we appropriately scale the individual terms and sum to get the particular solution $x_p(t)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{(2n+1)[4-(2n+1)^2]}$. If we expand this to show the first few terms, we have $x_p(t)=\frac{4}{\pi}\left[\frac{1}{3}\sin t-\frac{1}{15}\sin 3t-\frac{1}{105}\sin 5t-\frac{1}{315}\sin 7t-\frac{1}{693}\sin 9t-\cdots\right]$. Note how the amplitudes of the higher frequencies decrease rapidly. As always, the general solution is $x(t)=x_h(t)+x_p(t)$.

More generally, we could solve $\ddot{x}+\omega^2x=sq(t)$ to get $x_p(t)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{(2n+1)[\omega^2-(2n+1)^2]}$. This will usually yield a convergent series, but we have a problem in the case where ω is an odd integer since one term of the series will "blow up" in that case. This is a case of resonance and we'll look at that case shortly.

Harmonic response with resonance

One of the more interesting aspects of using Fourier Series is analyzing how a linear time-independent ODE with a periodic signal yields a response that exhibits **resonance**. The basic idea is that if we expand a periodic signal in a Fourier Series, it's sometimes that a single term in the series may be responsible for resonance. The signal may be composed of a whole range of frequencies, but one of them may produce resonance that may be the dominant feature of the response.

Suppose we wish to solve the ODE $\ddot{x}+\omega^2 x = sq(t) \sim \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{2n+1}$, where $sq(t)$ is the square-wave function. We previously observed that this would yield the series solution:

$$x_p(t)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{(2n+1)[\omega^2-(2n+1)^2]}$$

There is a catch, however. All of the terms in the series make sense unless ω is an odd integer. If this is the case, then all but one of the terms in the series will continue to make sense, but we'll have to treat the one term where $\omega = 2n+1$ differently. Let's consider a specific example.

Example: Find a particular solution to the ODE $\ddot{x}+9x = sq(t)$.

In this case, all of the terms in the above series are as stated, but we have to deal with the $n=1$ term separately since $\omega = 3$. For this one term we solve the ODE $\ddot{x}+9x=\frac{4}{3\pi}\sin 3t$. If we use complex replacement and later extract the imaginary part, we be solving the ODE $\ddot{z}+9z=\frac{4}{3\pi}e^{3it}$. Since the characteristic polynomial is $p(s)=s^2+9$ and $s=3i$ is a characteristic root, we must use the Resonant Response Formula, i.e. $z=\dfrac{\frac{4}{3\pi}te^{3it}}{p'(3i)}$.
Since $p'(s)=2s$ and $p'(3i)=6i$, we have the (complex) solution
$z=\dfrac{\frac{4}{3\pi}te^{3it}}{6i}=\frac{2}{9\pi}t(-i)[\cos 3t+i\sin 3t]=\frac{2}{9\pi}t[\sin 3t-i\cos 3t]$. Extracting the imaginary part gives
$x_3(t)=-\frac{2}{9\pi}t\cos 3t$. This term can then be added into the previous sum to replace the $n=1$ term. Note, however, that this term is oscillatory but its amplitude grows linearly in time. This is exactly the sort of thing we would expect when the system has resonance – even if it is caused by just one resonant frequency embedded among others.

Another Fourier Series calculation

Problem: Find the Fourier series for the function $f(t)=\begin{cases}0 & t\in[-\pi,0) \\ t & t\in[0,\pi)\end{cases}$, extended periodically for all *t*.

Solution: This function is neither symmetric nor antisymmetric, so we have to compute all the Fourier coefficients.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)dt=\frac{1}{\pi}\int_0^{\pi}t\,dt=\frac{1}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi}=\frac{\pi}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos nt\,dt=\frac{1}{\pi}\int_0^{\pi}t\cos nt\,dt=\begin{cases}0 & n \text{ even}\\ -\frac{2}{\pi n^2} & n\text{ odd}\end{cases}$$ after a little integration by parts.

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt\,dt=\frac{1}{\pi}\int_0^{\pi}t\sin nt\,dt=\frac{(-1)^{n+1}}{n}$$ after a little integration by parts.

So $\boxed{f(t)\sim\frac{\pi}{4}-\frac{2}{\pi}\sum_{n\text{ odd}}\left(\frac{\cos nt}{n^2}\right)+\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}\sin nt\right)}$

Curiosity: Note that for this function $t=0$ is a point of continuity and $f(0)=0$, so $\frac{\pi}{4}-\frac{2}{\pi}\sum_{n\text{ odd}}\left(\frac{1}{n^2}\right)=0$.

Therefore $\sum_{n\text{ odd}}\left(\frac{1}{n^2}\right)=1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\cdots=\frac{\pi^2}{8}$ (which we have previously shown).

Tips & Tricks – Manipulation of Fourier series

<u>Different period</u>: We developed our Fourier series representation for functions with a standard period 2π and fundamental interval $[-\pi,\pi]$. If we instead have a function $f(t)$ with period $2L$ and fundamental interval $[-L,L]$, we can simply change variables to produce the corresponding Fourier series in this case. We let $u=\frac{\pi t}{L}$ (so $t=\frac{Lu}{\pi}$) and define $g(u)=f\left(\frac{Lu}{\pi}\right)$ with period 2π and fundamental interval $[-\pi,\pi]$. The Fourier series for $g(u)$ is then

$g(u)\sim\dfrac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nu+b_n\sin nu\right)$ and if we use the substitution $u=\frac{\pi t}{L}$ (and $du=\frac{\pi}{L}dt$), we'll have

$$a_0=\tfrac{1}{\pi}\int_{-\pi}^{\pi}g(u)du=\tfrac{1}{L}\int_{-L}^{L}g(\tfrac{\pi t}{L})\,dt=\tfrac{1}{L}\int_{-L}^{L}f(u)\,du=\tfrac{1}{L}\int_{-L}^{L}f(t)\,dt,$$

$$a_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi}g(u)\cos nu\,du=\tfrac{1}{L}\int_{-L}^{L}g(\tfrac{\pi t}{L})\cos(\tfrac{n\pi t}{L})\,dt=\tfrac{1}{L}\int_{-L}^{L}f(t)\cos(\tfrac{n\pi t}{L})\,dt,$$

$b_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi}g(u)\sin nu\,du=\tfrac{1}{L}\int_{-L}^{L}g(\tfrac{\pi t}{L})\sin(\tfrac{n\pi t}{L})\,dt=\tfrac{1}{L}\int_{-L}^{L}f(t)\sin(\tfrac{n\pi t}{L})\,dt$, and we can write:

$$\boxed{f(t)=g(\tfrac{n\pi t}{L})\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos(\tfrac{n\pi t}{L})+b_n\sin(\tfrac{n\pi t}{L})\right)}$$

Fourier series can be differentiated or integrated term-by-term to produce other Fourier series:

Example: If we start with $sq(t)=\begin{Bmatrix}-1 & t\in[-\pi,0)\\ +1 & t\in[0,\pi)\end{Bmatrix}\sim\frac{4}{\pi}\sum_{n\text{ odd}}\frac{\sin nt}{n}$ and integrate term-by-term, we get $F(t)\sim\frac{4}{\pi}\sum_{n\text{ odd}}-\frac{\cos nt}{n^2}+C$. If we also insist that $F(0)=0$ and that $F(t)$ be continuous, we get that $-\frac{4}{\pi}\left(\sum_{n\text{ odd}}\frac{1}{n^2}\right)+C=-\frac{4}{\pi}\left(\frac{\pi^2}{8}\right)+C=-\frac{\pi}{2}+C=0$, so $C=\frac{\pi}{2}$. This gives

$F(t)=|t|=\begin{Bmatrix}-t & t\in[-\pi,0)\\ +t & t\in[0,\pi)\end{Bmatrix}\sim\frac{\pi}{2}-\frac{4}{\pi}\sum_{n\text{ odd}}\frac{\cos nt}{n^2}$, extended periodically for all *t*, a "sawtooth function".

This series could also have been calculated directly using the formulas for the Fourier coefficients and some integration by parts.

Fourier series can be scaled, shifted, etc. to produce other Fourier series

Example #1: Start with $sq(t)=\begin{Bmatrix}-1 & t\in[-\pi,0)\\ +1 & t\in[0,\pi)\end{Bmatrix}\sim\frac{4}{\pi}\sum_{n\text{ odd}}\frac{\sin nt}{n}$.

Then $1+sq(t)=\begin{Bmatrix}0 & t\in[-\pi,0)\\ 2 & t\in[0,\pi)\end{Bmatrix}\sim 1+\frac{4}{\pi}\sum_{n\text{ odd}}\frac{\sin nt}{n}$.

So $\frac{1}{2}\left[1+sq(t)\right]=\begin{Bmatrix}0 & t\in[-\pi,0)\\ 1 & t\in[0,\pi)\end{Bmatrix}\sim\frac{1}{2}+\frac{2}{\pi}\sum_{n\text{ odd}}\frac{\sin nt}{n}$, extended periodically for all *t*, a different sort of square-wave function.

Example #2: Find the Fourier series for the function $f(t)=\cos(t-\pi/3)$.

Solution: This function is periodic with period 2π. There's no need to consider the formulas for the Fourier coefficients. Simply note that $f(t)=\cos(t-\pi/3)=\cos t\cos(\pi/3)+\sin t\sin(\pi/3)=\frac{1}{2}\cos t+\frac{\sqrt{3}}{2}\sin t$.

Notes by Robert Winters