Math E-21c – Lecture #6

In today's lecture we'll finish up a few miscellaneous topics on *n*th order ODEs – superposition of solutions; time invariance, the Exponential Shift Rule, and Variation of Parameters. Also included in these notes are an example of how to deal with discontinuous inputs for a first order ODE and an introduction to the idea of Fourier series as a way to manage periodic inputs, including discontinuous periodic inputs.

Superposition of (particular) solutions

In the case where a linear differential equation has an input expressed as the sum of two or more functions, linearity allows us to find solutions for each input individually and then sum these solutions to produce a solution for the original ODE. That is, if we have a linear ODE of the form $T(f) = g_1 + g_2$ and if can individually find functions f_1 and f_2 such that $T(f_1) = g_1$ and $T(f_2) = g_2$, then since $T(f_1 + f_2) = T(f_1) + T(f_2) = g_1 + g_2$, it follows that $f_1 + f_2$ is a solution to $T(f) = g_1 + g_2$. In fact, the same reason shows that if $T(f_1) = g_1$ and $T(f_2) = g_2$, then $c_1 f_1 + c_2 f_2$ will be a solution of $T(f) = c_1 g_1 + c_2 g_2$.

Example: Find a particular solution to the ODE $\ddot{x} + 3\dot{x} + 2x = 5e^{-2t} + t^2$

Solution: We have already solved $\ddot{x} + 3\dot{x} + 2x = 5e^{-2t}$ to get a solution $x_1(t) = -5te^{-2t}$. We can solve $\ddot{x} + 3\dot{x} + 2x = t^2$ using undetermined coefficients and a solution of the form $x(t) = at^2 + bt + c$. This gives

$$2a + 3(2at + b) + 2(at^2 + bt + c) = 2at^2 + (6a + 2b)t + (2a + 3b + 2c) = t^2, \text{ so } \left\{ \begin{aligned} 2a &= 1 \\ 6a + 2b &= 0 \\ 2a + 3b + 2c &= 0 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} a &= \tfrac{1}{2} \\ b &= -\tfrac{3}{2} \\ c &= \tfrac{7}{4} \end{aligned} \right\}. \text{ So}$$

$x_2(t) = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4}$. Therefore the desired solution is $x_1(t) + x_2(t) = \boxed{-5te^{-2t} + \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4}}$.

Linear Time Invariant (LTI) ODEs

Recall the case of an autonomous 1st order ODE of the form $\frac{dx}{dt} = F(x)$, i.e. where the prescribed slope does not vary with the time *t*. The slope field for such a differential equation is horizontally invariant. For any such differential equation, if $x(t)$ is a solution, then $x(t-c)$ must also be a solution for any translation *c*. As a simple illustration, if $\frac{dx}{dt} = kx$ yields a solution $x(t) = ae^{kt}$ (natural exponential growth or decay), then $x(t-c) = ae^{k(t-c)} = (ae^{-kc})e^{kt}$ is also a solution of this same form.

There is a similar property in the case of constant coefficient linear ODEs of any order. In the case of a general linear ODE of the form $T[x(t)] = \frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t) = q(t)$, the "system" *T* may explicitly depend on the time *t* as evidenced by the coefficient functions $p_k(t)$ being functions of *t*. However, in the case of constant coefficients, i.e. $T[x(t)] = \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1\frac{dx}{dt} + a_0 x(t) = q(t)$, the system *T* does not explicitly depend on the time *t*. If a function $x(t)$ solves $T[x(t)] = q(t)$, then $T[x(t-t)] = q(t-c)$, so the translated function $y(t) = x(t-c)$ satisfies $\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y(t) = q(t-c)$. That is, if a given input signal is delayed for an amount of time but is otherwise unchanged, the response will be the same but delayed by that same period of time. Here's a simple example to illustrate this:

Example: Find a particular solution to the ODE $\ddot{x} + 3\dot{x} + 2x = 4\sin(t-3)$.

Solution: It would be simple enough to solve this using undetermined coefficients, but we can also just solve the equation $\ddot{x} + 3\dot{x} + 2x = 4\sin t$ and then use time invariance to get a solution to the given equation by translating our solution be replacing *t* with $t-3$. If we try a solution of the form $x = a\cos t + b\sin t$, we quickly conclude that to satisfy the equation we must have $a=-\frac{6}{5}$ and $b=\frac{2}{5}$, so $x_p(t)=-\frac{6}{5}\cos t+\frac{2}{5}\sin t$ satisfies $\ddot{x}+3\dot{x}+2x=4\sin t$. Therefore $x_p(t)=-\frac{6}{5}\cos(t-3)+\frac{2}{5}\sin(t-3)$ satisfies $\ddot{x}+3\dot{x}+2x=4\sin(t-3)$. Had we instead used complex replacement to solve $\ddot{x}+3\dot{x}+2x=4\sin t$. We would have obtained $x_p(t)=\frac{4}{\sqrt{10}}\sin(t-\phi)$ where $\phi=\tan^{-1}(3)$. Therefore $x_p(t)=\frac{4}{\sqrt{10}}\sin(t-3-\phi)=\frac{4}{\sqrt{10}}\sin(t-(3+\phi))$ is a particular solution to $\ddot{x}+3\dot{x}+2x=4\sin(t-3)$. You can explicitly see how the new solution is just shifted further in time.

Exponential Shift Rule

We have spent quite a bit of time solving ODEs of the form $[p(D)]x(t)=q(t)$, especially in the case where the input function $q(t)$ is sinusoidal, exponential, or a product of these. One special case is where the ODE is of the form $[p(D)]x(t)=e^{rt}q(t)$. For example, we might wish to solve the equation $\ddot{x}+3\dot{x}+2x=t^2e^{3t}$. For problems like this, there's a handy rule that can help avoid some of the more annoying guesswork and algebra that can occur with either undetermined coefficients or variation of parameters.

Exponential Shift Rule (ESR): Suppose we wish to solve an ODE of the form $[p(D)]x(t)=e^{rt}q(t)$ where $p(D)=D^n+a_{n-1}D^{n-1}+\cdots+a_1D+a_0I$ is a linear differential operator with constant coefficients. If $u(t)$ is a solution of the ODE $[p(D+rI)]u(t)=q(t)$, then $x(t)=e^{rt}u(t)$ will solve $[p(D)]x(t)=e^{rt}q(t)$.

Proof: The key step in understanding this rule is the ordinary product rule from Calculus. In terms of differential operators, if $u=u(t)$ and $v=v(t)$ and $D=\frac{d}{dt}$, then the product rule is simply $D(uv)=uDv+vDu$. In particular, $D(e^{rt}u)=e^{rt}Du+re^{rt}u=e^{rt}(Du+ru)=e^{rt}(D+rI)u$. Similarly, we have:

$$D^2(e^{rt}u)=D\left[e^{rt}(D+rI)u\right]=e^{rt}(D^2+rD)u+re^{rt}(D+rI)u=e^{rt}(D^2+2rD+r^2I)u=e^{rt}(D+rI)^2u$$

$$D^3(e^{rt}u)=D\left[e^{rt}(D+rI)^2u\right]=e^{rt}D(D+rI)^2u+re^{rt}(D+rI)^2u=e^{rt}(D+rI)^3u, \text{ and so on.}$$

Therefore:

$$\begin{aligned}[p(D)](e^{rt}u)&=D^n(e^{rt}u)+a_{n-1}D^{n-1}(e^{rt}u)+\cdots+a_1D(e^{rt}u)+a_0e^{rt}u\\ &=e^{rt}(D+rI)^nu+a_{n-1}e^{rt}(D+rI)^{n-1}u+\cdots+a_1e^{rt}(D+rI)u+a_0e^{rt}u\\ &=e^{rt}[(D+rI)^n+a_{n-1}(D+rI)^{n-1}+\cdots+a_1(D+rI)+a_0I]u=e^{rt}[p(D+rI)](u)\end{aligned}$$

So, if we write $x(t)=e^{rt}u(t)$, we see that if we can solve $[p(D+rI)]u(t)=q(t)$, then $[p(D)]x(t)=e^{rt}[p(D+rI)]u(t)=e^{rt}q(t)$, so $x(t)$ will solve the original ODE.

Example: Find a particular solution to $\ddot{x}+3\dot{x}+2x=t^2e^{3t}$.

Solution: First let's do some guesswork and undetermined coefficients to solve this problem before trying our new method. A reasonable guess might be a solution of the form $x(t)=(at^2+bt+c)e^{3t}$. We calculate:

$$\dot{x}(t)=3(at^2+bt+c)e^{3t}+(2at+b)e^{3t}=[3at^2+(3b+2a)t+(3c+b)]e^{3t}$$

$$\ddot{x}(t)=[3at^2+(3b+2a)t+(3c+b)]3e^{3t}+[6at+(3b+2a)]e^{3t}=[9at^2+(9b+12a)t+(9c+6b+2a)]e^{3t}$$

$$\begin{aligned}\ddot{x}+3\dot{x}+2x&=[9at^2+(9b+12a)t+(9c+6b+2a)]+3[3at^2+(3b+2a)t+(3c+b)]+2(at^2+bt+c)\\ &=[20at^2+(18a+20b)t+(2a+9b+20c)]e^{3t}=t^2e^{3t}\end{aligned}$$

Therefore $20a=1 \Rightarrow \boxed{a=\frac{1}{20}}$, $18a+20b=0 \Rightarrow \boxed{b=-\frac{9}{200}}$, and $2a+9b+20c=0 \Rightarrow \boxed{c=\frac{61}{4000}}$.

These calculations will also arise if we use the Exponential Shift Rule, but we get there more efficiently and with less likelihood of error. Specifically, because $p(D)=D^2+3D+2I$, we would first calculate

$p(D+3I)=(D+3I)^2+3(D+3I)+2I=D^2+6D+9I+3D+9I+2I=D^2+9D+20I$. You may find it easier to work with the characteristic polynomials rather that with the differential operator. If the characteristic polynomial of the original ODE is $p(s)$, the characteristic polynomial after the shift will be $p(s+r)$ and we can determine $p(D+rI)$ from that. In this example we have $p(s)=s^2+3s+2=(s+2)(s+1)$, so $p(s+3)=(s+3+2)(s+3+1)=(s+5)(s+4)=s^2+9s+20$. Therefore $p(D+3I)=D^2+9D+20I$.

We then solve the ODE $\ddot{u}+9\dot{u}+20u=t^2$ using undetermined coefficients. The clear choice is a solution of the form $u(t)=at^2+bt+c$ and we calculate

$\ddot{u}+9\dot{u}+20u=[2a+9(2at+b)+20(at^2+bt+c)]=20at^2+(18a+20b)t+(2a+9b+20c)=t^2$, so again

$20a=1 \Rightarrow \boxed{a=\frac{1}{20}}$, $18a+20b=0 \Rightarrow \boxed{b=-\frac{9}{200}}$, and $2a+9b+20c=0 \Rightarrow \boxed{c=\frac{61}{4000}}$.

The particular solution is therefore $x(t)=e^{3t}u(t)=e^{3t}\left(\frac{1}{20}t^2-\frac{9}{200}t+\frac{61}{4000}\right)$.

Example: Find a particular solution to the ODE $\ddot{x}+3\dot{x}+2x=e^{-t}$.

Solution: In this case, the characteristic polynomial is $p(s)=s^2+3s+2=(s+2)(s+1)$ and $r=-1$ is a characteristic root. We cannot use the Exponential Response Formula, but the Resonant Response Formula provides a simple solution. We calculate $p'(s)=2s+3$ and $p'(-1)=1$, so $x(t)=\dfrac{te^{-t}}{p'(-1)}=te^{-t}$.

The Exponential Shift Rule doesn't provide a faster method here, but it does provide a different way of seeing how this solution comes about. We take $q(t)=1$ and $r=-1$ and solve $[p(D-I)]u=1$. We have $p(s)=(s+2)(s+1)$, so $p(s-1)=(s-1+2)(s-1+1)=(s+1)s=s^2+s$, and therefore $p(D-I)=D^2+D$. We then solve $\ddot{u}+\dot{u}=1$. If we try $u(t)=At+B$ we get $\ddot{u}+\dot{u}=0+A=1$, so $A=1$ and we can choose any B we like. So take $B=0$ to get $u(t)=t$. Therefore $x(t)=e^{-t}t=te^{-t}$ is the desired solution.

Variation of Parameters

Recall the method in the case of a 1st order linear ODE of the form $\frac{dx}{dt}+p(t)x=q(t)$. First we found a homogeneous solution $x_1(t)$, so all homogeneous solutions would be of the form $Ax_1(t)$. To find a particular solution to the inhomogeneous equation we "vary the parameter" A and seek a solution of the form $x(t)=v(t)x_1(t)=v\,x_1$. We then derived that $\dot{v}(t)=\frac{q(t)}{x_1(t)}$ and, in principle, we can then integrate this to determine $v(t)$ and hence $x(t)=v(t)x_1(t)$.

The situation with higher order linear ODEs is similar but more complicated. For example, if we have a 2nd order linear ODE of the form $\ddot{x}+p_1(t)\dot{x}+p_0(t)x=R(t)$ and we needed to find a particular solution, we would again first find homogeneous solutions $x_1(t)$ and $x_2(t)$ with all homogeneous solutions of the form $x_h(t)=c_1x_1(t)+c_2x_2(t)$. We would again "vary the parameters" by seeking functions $v_1(t)$ and $v_2(t)$ such that $x(t)=v_1(t)x_1(t)+v_2(t)x_2(t)=v_1x_1+v_2x_2$ satisfies the inhomogeneous equation. Differentiation gives $\dot{x}=v_1\dot{x}_1+\dot{v}_1x_1+v_2\dot{x}_2+\dot{v}_2x_2=\left(v_1\dot{x}_1+v_2\dot{x}_2\right)+\left(x_1\dot{v}_1+x_2\dot{v}_2\right)$.

We have some flexibility here, so let's <u>assume</u> that we can find $v_1(t)$ and $v_2(t)$ such that $\boxed{x_1\dot{v}_1+x_2\dot{v}_2=0}$. Then $\dot{x}=v_1\dot{x}_1+v_2\dot{x}_2=\dot{x}_1v_1+\dot{x}_2v_2$. If we differentiate again, we get $\ddot{x}=\dot{x}_1\dot{v}_1+\dot{x}_2\dot{v}_2+\ddot{x}_1v_1+\ddot{x}_2v_2$. If we substitute these expressions into the ODE we get:

$$\ddot{x}+p_1(t)\dot{x}+p_0(t)x=\left(\dot{x}_1\dot{v}_1+\dot{x}_2\dot{v}_2+\ddot{x}_1v_1+\ddot{x}_2v_2\right)+p_1(t)\left(\dot{x}_1v_1+\dot{x}_2v_2\right)+p_0(t)\left(v_1x_1+v_2x_2\right)=R(t)$$

If we rearrange terms we get:

$$\ddot{x} + p_1(t)\dot{x} + p_0(t)x = \left(\ddot{x}_1 + p_1\dot{x}_1 + p_0x_1\right)v_1 + \left(\ddot{x}_2 + p_1\dot{x}_2 + p_0x_2\right)v_2 + \dot{x}_1\dot{v}_1 + \dot{x}_2\dot{v}_2 = R(t)$$

However, since $x_1(t)$ and $x_2(t)$ are homogeneous solutions, the first two terms vanish and we're left with $\boxed{\dot{x}_1\dot{v}_1 + \dot{x}_2\dot{v}_2 = R(t)}$. If we join this with the previous assumption, we will be seeking solutions to the system $\begin{Bmatrix} x_1\dot{v}_1 + x_2\dot{v}_2 = 0 \\ \dot{x}_1\dot{v}_1 + \dot{x}_2\dot{v}_2 = R \end{Bmatrix} \Rightarrow \begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix}\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix}$. This will yield solutions precisely when the Wronskian determinant $W(t) = \begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} = x_1\dot{x}_2 - x_2\dot{x}_1 \neq 0$, and we then solve for $\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \frac{1}{W(t)}\begin{bmatrix} \dot{x}_2 & -x_2 \\ -\dot{x}_1 & x_1 \end{bmatrix}\begin{bmatrix} 0 \\ R \end{bmatrix} = \frac{1}{W(t)}\begin{bmatrix} -x_2R \\ x_1R \end{bmatrix}$.

That is, $\boxed{\dot{v}_1 = -\frac{x_2(t)R(t)}{W(t)}, \dot{v}_2 = \frac{x_1(t)R(t)}{W(t)}}$ where $\boxed{W(t) = x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t) \neq 0}$.

You may find it easier to remember simply that $\begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix}\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix}$. Integration then gives $v_1(t)$ and $v_2(t)$ and hence $x_p(t) = v_1(t)x_1(t) + v_2(t)x_2(t)$.

This method can be generalized to higher order linear ODEs. It depends on the ability to first find a complete basis of homogeneous solutions as well as the ability to find antiderivatives or, alternatively, to express these antiderivatives as integrals using the 2nd Fundamental Theorem of Calculus.

For example, for a third order linear ODE we would find three independent homogeneous solutions $\{x_1(t), x_2(t), x_3(t)\}$ and all homogeneous solutions of the form $x_h(t) = c_1x_1(t) + c_1x_2(t) + c_1x_3(t)$, then vary the parameters to seek a particular solution of the form $x_p(t) = v_1(t)x_1(t) + v_2(t)x_2(t) + v_3(t)x_3(t)$. A similar derivation (where we set two expressions equal to zero to facilitate things) would result in the system of equations $\begin{bmatrix} x_1 & x_2 & x_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 \end{bmatrix}\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ R \end{bmatrix}$ from which we would determine $\dot{v}_1(t)$, $\dot{v}_2(t)$, and $\dot{v}_3(t)$. We would then integrate these to get $v_1(t)$, $v_2(t)$, and $v_3(t)$ and finally $x_p(t) = v_1(t)x_1(t) + v_2(t)x_2(t) + v_3(t)x_3(t)$.

Higher order linear ODEs would yield analogous results.

Example: Find a particular solution to the ODE $\ddot{x} + 3\dot{x} + 2x = e^{-t}$.

Solution: The simplest way to solve this (<u>and hence the best way</u>) is to use either the ERF or, if necessary, the RRF. In this case the characteristic polynomial is $p(s) = s^2 + 3s + 2 = (s+2)(s+1)$, and $r = -1$ in the exponent of the input function is a simple characteristic root. The ERF is therefore not applicable, but the RRF is applicable. We have $p'(s) = 2s + 3$, so $p'(-1) = 1$ and a particular solution is $x_p(t) = \frac{te^{-t}}{p'(-1)} = \frac{te^{-t}}{1} = \boxed{te^{-t}}$.

If we had used variation of parameters, we would have the homogeneous solutions $x_1 = e^{-2t}$, $x_2 = e^{-t}$, their derivatives would be $\dot{x}_1 = -2e^{-2t}$, $\dot{x}_2 = -e^{-t}$, and the Wronskian determinant would be $W(t) = e^{-3t}$.

So $\dot{v}_1 = -\frac{e^{-t}e^{-t}}{e^{-3t}} = -e^{t}$, $\dot{v}_2 = \frac{e^{-t}e^{-2t}}{e^{-3t}} = 1$.

These give $v_1 = -e^t$ and $v_2 = t$, so a particular solution would be $x_p(t) = -e^te^{-2t} + te^{-t} = -e^{-t} + te^{-t}$. The first term is actually a homogeneous solution, so we can discard it to give $x_p(t) = \boxed{te^{-t}}$.

Example: Find a particular solution to the ODE $\ddot{x}+9x=te^t\cos t$.

Solution: The homogeneous equation $\ddot{x}+9x=0$ has characteristic polynomial $p(s)=s^2+9$ and characteristic roots $s=\pm 3i$. These yield the homogeneous solutions $\{e^{3it},e^{-3it}\}$ or, if you prefer $\{\cos 3t,\sin 3t\}$.

If we choose the latter basis for the homogeneous solutions, we would take $x_1=\cos 3t,\ x_2=\sin 3t$. We calculate $\dot{x}_1=-3\sin 3t,\ \dot{x}_2=3\cos 3t$, and the Wronskian determinant is $W(t)=3(\cos^2 3t+\sin^2 3t)=3$. So $\dot{v}_1=-\frac{1}{3}te^t\cos t\sin 3t,\ \dot{v}_2=\frac{1}{3}te^t\cos t\cos 3t$. It is unlikely that either we humans or Mathematica can produce nice antiderivatives for these, but we can express $v_1=-\frac{1}{3}\int_0^t ue^u\cos u\sin 3u\,du,\ v_2=\frac{1}{3}\int_0^t ue^u\cos u\cos 3u\,du$, so formally a particular solution is $\boxed{x_p(t)=-\frac{1}{3}\cos 3t\int_0^t ue^u\cos u\sin 3u\,du+\frac{1}{3}\sin 3t\int_0^t ue^u\cos u\cos 3u\,du}$.

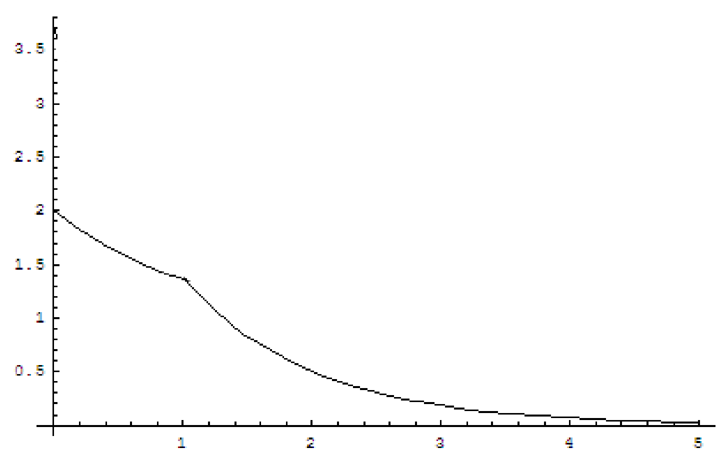
Had we instead chosen the complex exponentials as a basis we would have similar challenges.

A Discontinuous Input

We will soon be developing a more general method (Laplace transform) for dealing with discontinuous inputs, but we can solve 1st order linear ODEs now using methods already developed.

Example: Solve the ODE $\frac{dx}{dt}+kx=q(t)=\begin{Bmatrix}0 & t<0\\ 1 & 0\le t\le 1\\ 0 & t>1\end{Bmatrix}$. This is a "switch on, switch off" input.

Solution: The homogeneous solutions are easy, namely $x_h(t)=ce^{-kt}$. Because this is a 1st order ODE, we can use an integrating factor to solve. (This option is not available for higher order ODEs.) The integrating factor is e^{kt} and we have $e^{kt}\frac{dx}{dt}+ke^{kt}x=\frac{d}{dt}\left(e^{kt}x\right)=e^{kt}q(t)=\begin{Bmatrix}0 & t<0\\ e^{kt} & 0\le t\le 1\\ 0 & t>1\end{Bmatrix}$. If we use $t=0$ as a starting point with $x(0)=x_0$ as initial condition, we integrate to get $e^{kt}x(t)-x(0)=\int_{\tau=0}^{\tau=t}e^{k\tau}d\tau=\frac{1}{k}(e^{kt}-1)$ for $0\le t\le 1$; and for $t>1$ we have $e^{kt}x(t)-x(0)=\int_{\tau=0}^{\tau=1}e^{k\tau}d\tau=\frac{1}{k}(e^k-1)$. So $x(t)=\begin{Bmatrix}e^{-kt}[x_0+\frac{1}{k}(e^{kt}-1)] & 0\le t\le 1\\ e^{-kt}[x_0+\frac{1}{k}(e^k-1)] & t>1\end{Bmatrix}$. This response is actually continuous even though there was a discontinuous input.



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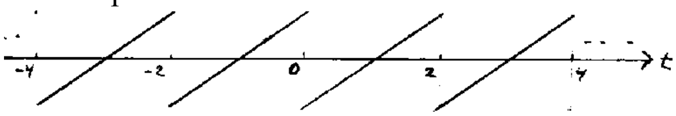
Periodic Inputs and Fourier Series

The solution of an ODE of the form $[p(D)]x(t) = a\cos kt$ or $[p(D)]x(t) = a\sin kt$ is now relatively straightforward through the use of complex replacement, the Exponential Response Formula, and, when needed, the Resonant Response Formula. How might we solve an ODE of the form $[p(D)]x(t) = f(t)$ where $f(t)$ is some other periodic function such:



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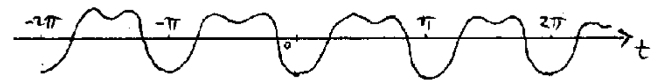


The way we'll handle this is to successively approximate any such periodic function as a sum of trigonometric functions, solve term-by-term, and then reassemble a solution using linearity (superposition). The approximation method involved **Fourier Series**.

Definition: A function $f(t)$ is called **periodic** with period *T* if $f(t+nT) = f(t)$ for all *t* and all integers *n*. We say that *T* is the **base period** if it is the least such $T > 0$.

Examples: The functions $\sin t$ and $\cos t$ are both periodic with base period 2π. The functions $\sin \omega t$ and $\cos \omega t$ are both periodic with base period $\frac{2\pi}{\omega}$.

Note: Any constant function is also periodic, but with no base period.

For the sake of simplicity, we'll begin by considering periodic functions with base period 2π. We will later rescale to adapt our methods to other base periods. Our methods will be based on the following theorem:

Theorem (Fourier): Suppose a function $f(t)$ is periodic with base period 2π and continuous except for a finite number of jump discontinuities. Then $f(t)$ may be represented by a (convergent) Fourier Series:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

where: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$.

The numbers $\{a_0, a_1, b_1, \cdots, a_n, b_n, \cdots\}$ are called the **Fourier coefficients** of the function $f(t)$.

This representation is an equality at all points of continuity of the function $f(t)$. At any point of discontinuity $t = a$, the series converges to the average of $f(a^-)$ and $f(a^+)$, i.e. the value $\frac{1}{2}[f(a^-) + f(a^+)]$.

Note: (a) If $f(t)$ is an even function [$f(-t) = f(t)$ for all *t*], then $b_n = 0$ for all *n* by basic facts from calculus.

(b) If $f(t)$ is an odd function [$f(-t) = -f(t)$ for all *t*], then $a_0 = 0$ and $a_n = 0$ for all *n* by basic facts from calculus.

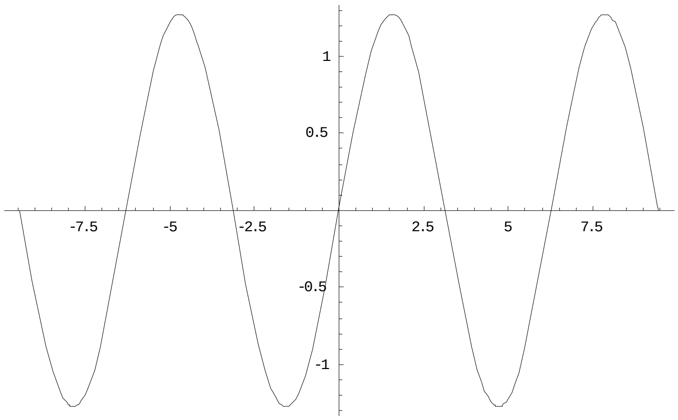
Example (Square wave function): $f(t) = sq(t) = \begin{Bmatrix} -1 & t \in [-\pi, 0) \\ +1 & t \in [0, \pi) \end{Bmatrix}$, extended periodically for all *t*.

This function is periodic (with period 2π) and antisymmetric, i.e. an odd function. Therefore $a_0 = 0$ and $a_n = 0$ for all *n*. We calculate $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin nt\,dt + \int_{0}^{\pi}\sin nt\,dt\right] = \frac{1}{\pi}\left[\left[\frac{\cos nt}{n}\right]_{-\pi}^{0} - \left[\frac{\cos nt}{n}\right]_{0}^{\pi}\right]$

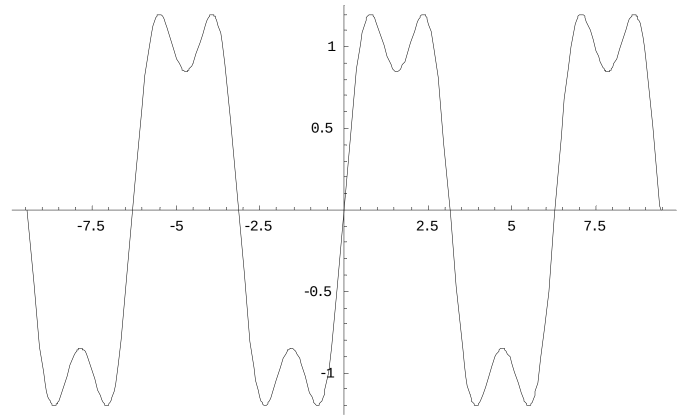
$$= \frac{1}{n\pi}\left[\left[1-(-1)^n\right] - \left[(-1)^n - 1\right]\right] = \begin{Bmatrix} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{Bmatrix}.$$

So $\boxed{sq(t) \sim \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n} = \frac{4}{\pi}[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots]}$.

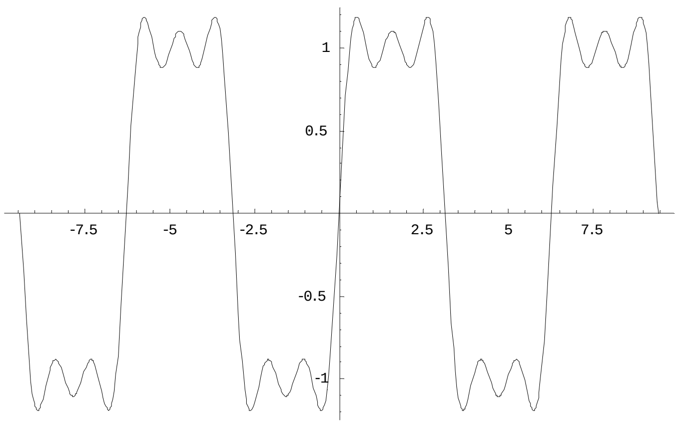
The nature of the convergence of this Fourier series toward the square wave function can be seen by graphing the partial sums:



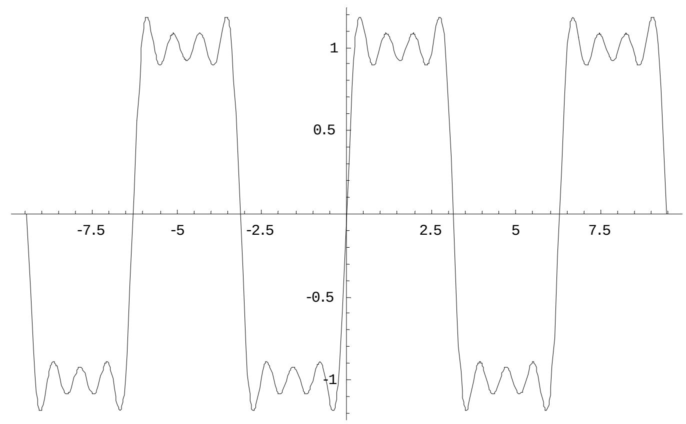
n = 1



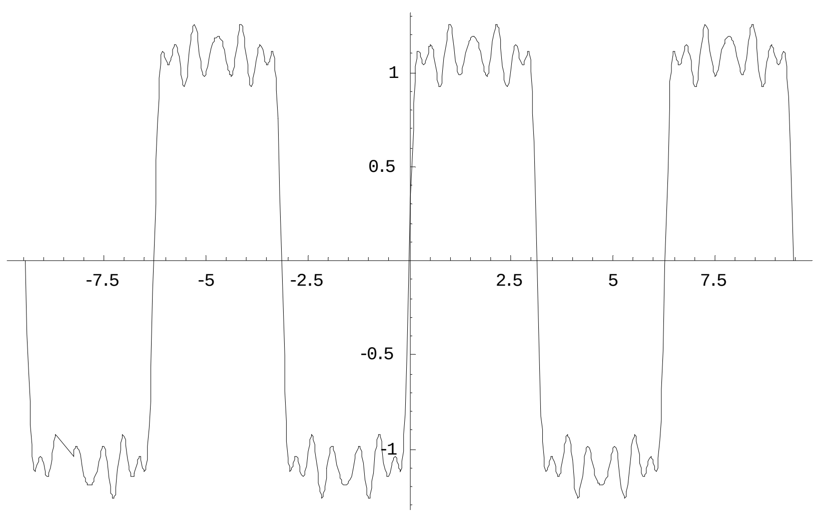
n = 3



n = 5



n = 7



n = 15

Notes by Robert Winters