Ordinary Differential Equations – Lecture #5

Mass-Spring-Dashpot systems

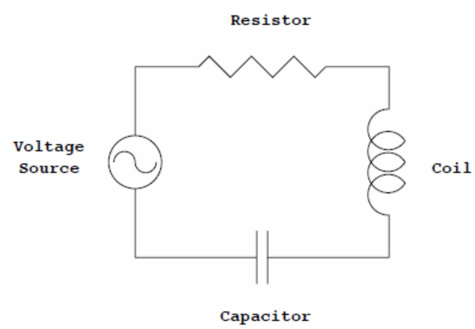
Of particular interest to us (for a variety of reasons) are mass-spring-dashpot systems in which a spring is governed by Hooke's Law but also subject to friction that is proportional to the velocity. [A picture was drawn in class illustrating a spring with an attached mass and the friction supplied by a piston (dashpot).] The simplest case is where this system is confined with the spring attached to one fixed wall, the dashpot on the other side of the mass attached to another fixed wall, and the mass moving relative to its equilibrium position. In this case, we would express the force acting on the mass as $F = -kx - cv$ where $v = \dot{x}$ and $F = ma = m\ddot{x}$. This gives the system $m\ddot{x} + c\dot{x} + kx = 0$ or $\boxed{\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0}$.

We could also imagine a system that is "driven" by moving either the fixed end of the spring or by moving the fixed end of the dashpot. If we incorporate this additional acceleration, the resulting system would be governed by an inhomogeneous ODE of the form $\boxed{\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = q(t)}$.

<u>**Note**</u>: We get similar equations in the case of an electric circuit with inductance (L), resistance (R), and capacitance (C), i.e. and ***LRC* circuit**.

Analogy between Mass-Spring-Dashpot systems and LRC circuits

A spring with an attached mass, friction supplied by a dashpot, and external force $F(t)$ is described by the differential equation $\boxed{m\ddot{x} + c\dot{x} + kx = F(t)}$. This purely mechanical system has an electrical analogue known as an LRC circuit where L represents the **inductance** associated with a coil, R represents the **resistance**, and C represents the **capacitance**. Given a voltage source with variable voltage $V(t)$ (measured in volts), the circuit will have at any time a current $I(t)$ (measured in amperes), and the capacitor will be carrying a charge $Q(t)$ (measured in Coulombs).



Series RLC Circuit

In physics, we learn that there are voltage drops associated with each of the elements of the circuit. Specifically, $V_L = L\dot{I} = L\frac{dI}{dt}$ due to the inductance, $V_R = IR$ due to the resistance, and $V_C = Q/C$ due to the capacitance. The sum of the voltage drops must match the voltage source, i.e. $V = V_L + V_R + V_C$. We also know that the current satisfies $\dot{Q} = \frac{dQ}{dt} = I$, so $\ddot{Q} = \frac{dI}{dt} = \dot{I}$ and $\dot{V}_C = \dot{Q}/C = I/C$. If we differentiate to get $\dot{V} = \dot{V}_L + \dot{V}_R + \dot{V}_C$ and substitute the above relations, we get that $\boxed{L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}}$ for the rate of change of the applied voltage.

In this mechanical/electrical analogy, the inductance becomes analogous to mass, the resistance is analogous to friction, and the (reciprocal of) capacitance is analogous to the stiffness of the spring. Also the rate of change of voltage is analogous to the external force (which is the rate of change of momentum).

Spring only case

The simplest case is a pure spring with no friction and no external driving force. In this case, the differential equation governing the motion would be simply $\boxed{\ddot{x} + \frac{k}{m}x = 0}$. In anticipation of what will follow, it's useful to let $\omega^2 = \frac{k}{m}$ or $\omega = \sqrt{\frac{k}{m}}$. This gives the differential equation $\ddot{x} + \omega^2 x = 0$. Its characteristic polynomial is $p(r) = r^2 + \omega^2 = 0 \quad \Rightarrow \quad r = \pm i\omega$. So all solutions to this homogeneous equation can be expressed as the span of $\{e^{i\omega t}, e^{-i\omega t}\}$, i.e. in the form $x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$ where c_1, c_2 are <u>complex</u> constants. We would, of course, prefer to express solutions as real-valued functions. Using Euler's Formula, we could rewrite the solutions as $x(t) = c_1(\cos\omega t + i\sin\omega t) + c_2(\cos\omega t - i\sin\omega t) = (c_1 + c_2)\cos\omega t + i(c_1 - c_2)\sin\omega t$ and then hope that any given initial condition will produce real coefficients (they will). Another way to look at this is to note that since $e^{i\omega t} = \cos\omega t + i\sin\omega t$ and $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ and we can also solve for $\cos\omega t = \frac{e^{i\omega t}+e^{-i\omega t}}{2}$ and $\sin\omega t = \frac{e^{i\omega t}-e^{-i\omega t}}{2i}$, it must be the case that $\operatorname{Span}\left\{e^{i\omega t}, e^{-i\omega t}\right\} = \operatorname{Span}\left\{\cos\omega t, \sin\omega t\right\}$. That is, all solutions must be of the form $\boxed{x(t) = a\cos\omega t + b\sin\omega t}$. We also have the option of expressing this as $\boxed{x(t) = A\cos(\omega t - \phi)}$ where $A = \sqrt{a^2 + b^2}$ and $\tan\phi = \frac{b}{a}$.

Note: If we felt the urge to inquire whether the set $\left\{e^{i\omega t}, e^{-i\omega t}\right\}$ or the set $\left\{\cos\omega t, \sin\omega t\right\}$ were linearly independent solutions, the corresponding Wronskians would give either $\begin{vmatrix} e^{i\omega t} & e^{-i\omega t} \\ i\omega e^{i\omega t} & -i\omega e^{-i\omega t} \end{vmatrix} = -2i\omega \neq 0$ or $\begin{vmatrix} \cos\omega t & \sin\omega t \\ -\omega\sin\omega t & \omega\cos\omega t \end{vmatrix} = \omega(\cos^2\omega t + \sin^2\omega t) = \omega \neq 0$. They both provide a linearly independent spanning set for the solutions, i.e. a **basis** for the solutions (in linear algebra terms).

Introducing friction (but no drive)

In the more general homogeneous case governed by $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$, we can again seek exponential-type solutions. The characteristic roots will be determined by $p(s) = s^2 + \frac{c}{m}s + \frac{k}{m} = 0$. We can solve this by the quadratic formula to get $s = \dfrac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}}{2} = \dfrac{-c \pm \sqrt{c^2 - 4km}}{2m}$. At this point, we have to look at cases.

__**Underdamped case**__

If $c^2 - 4km < 0$ or $c^2 < 4km$, we'll get complex roots that we can write as $s = \dfrac{-c \pm i\sqrt{4km - c^2}}{2m} = a \pm i\omega$ where $a = -\dfrac{c}{2m} < 0$ and $\omega = \dfrac{\sqrt{4km - c^2}}{2m}$. This yields solutions $\left\{e^{(a+i\omega)t}, e^{(a-i\omega)t}\right\} = \left\{e^{at}e^{i\omega t}, e^{at}e^{-i\omega t}\right\}$. As in the case of the spring without friction, we can use Euler's Formula to get the equivalent set $\left\{e^{at}\cos\omega t, e^{at}\sin\omega t\right\}$. Thus all homogeneous solutions can be expressed as $x(t) = e^{at}(c_1\cos\omega t + c_2\sin\omega t)$ with $a < 0$, i.e. decaying oscillatory solutions. We can equivalently express this in the form $x(t) = Ae^{at}\cos(\omega t - \phi)$ for appropriate values of A, ϕ. This is the **underdamped case** where the friction is small relative to the stiffness of the spring.

Example: Find the general solution of the homogeneous ODE: $\ddot{x} + 2\dot{x} + 3x = 0$

Solution: The characteristic polynomial is $p(s) = s^2 + 2s + 3 = 0$. This gives roots $s = -1 \pm i\sqrt{2}$. As described above, all solutions may be expressed in the form $x(t) = e^{-t}(c_1\cos(\sqrt{2}\,t) + c_2\sin(\sqrt{2}\,t))$ or as $x(t) = Ae^{-t}\cos(\sqrt{2}\,t - \phi)$ for appropriate A, ϕ.

__**Overdamped case**__

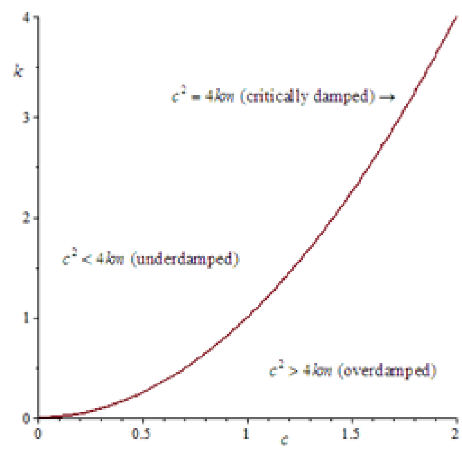
If $c^2 - 4km > 0$ or $c^2 > 4km$, we'll get two real roots $s = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km}$. Because $c^2 - 4km < c^2$ it's also easy to see that both of these roots will be negative and equidistant from $-\frac{c}{2m}$. If we call these two roots $s_1 < s_2 < 0$, these will yield two independent (decaying) solutions $e^{s_1 t}$ and $e^{s_2 t}$ (modes) and all homogeneous solutions can be expressed as $x(t)=c_1e^{s_1t}+c_2e^{s_2t}$ with $s_1<s_2<0$. This is the **overdamped case** where the friction is large relative to the stiffness of the spring.

Example: Find the general solution of the homogeneous ODE: $\ddot{x}+3\dot{x}+2x=0$
Solution: The characteristic polynomial is $p(s)=s^2+3s+2=(s+2)(s+1)=0$. This gives roots $s_1=-2$ and $s_2=-1$. As described above, all solutions may be expressed in the form $x(t)=c_1e^{-2t}+c_2e^{-t}$.

<u>**Critically damped case**</u>
Perhaps the most interesting case is when $c^2-4km=0$ or $c^2=4km$. In this case we get a repeated root $s=-\frac{c}{2m}=a$ with multiplicity 2. We know that e^{at} must be a solution, but <u>this will not span all solutions</u>. In fact, all solutions are spanned by $\left\{e^{at},te^{at}\right\}$, i.e. all solutions are of the form $x(t)=c_1e^{at}+c_2te^{at}$. Why? The idea is easily seen via an example.

Example: Find the general solution of the ODE: $\ddot{x}+4\dot{x}+4x=0$
Solution: The characteristic polynomial is $p(s)=s^2+4s+4=(s+2)^2=0$. This yields the solution e^{-2t}. Now consider this differential equation in terms of a composition of linear operators, namely $(D+2I)\circ(D+2I)x(t)=0$. If we let $(D+2I)x(t)=y(t)$, then $(D+2I)\circ y(t)=0$ or $\frac{dy}{dt}+2y=0$ or $\frac{dy}{dt}=-2y$. This is easily solved to get $y(t)=c_1e^{-2t}$. If we then substitute this into $(D+2I)x(t)=y(t)$, we have $\frac{dx}{dt}+2x=c_1e^{-2t}$. This can be solved using the integrating factor e^{2t}. This gives $e^{2t}\frac{dx}{dt}+2e^{2t}x=\frac{d}{dt}\left(e^{2t}x\right)=c_1$, and this is easily integrated to give $e^{2t}x=c_1t+c_2$. Multiplication by e^{-2t} then gives $x(t)=c_1te^{-2t}+c_2e^{-2t}$. That is, the solutions are given by $\mathrm{Span}\left\{e^{-2t},te^{-2t}\right\}$.

This method outlined in the example clearly works for any repeated root $s=a$ with multiplicity 2. That these solutions are linearly independent should be clear, but formally the Wronskian determinant gives that $\begin{vmatrix} e^{at} & te^{at} \\ ae^{at} & (1+at)e^{at} \end{vmatrix}=(1+at)e^{2at}-ate^{2at}=e^{2at}\neq 0$. In the case where the multiplicity is 3, we would write the differential equation as a composition of three identical operators and iterate the method to yield solutions $\left\{e^{at},te^{at},t^2e^{at}\right\}$. In general, we would produce as many independent solutions as the multiplicity of the root.



We can categorize the possibilities (underdamped, overdamped, and critically damped in terms of the relationship between the coefficient of friction c and the spring constant k. This can be viewed graphically in terms of the "**bifurcation locus**" shown at the right.

It's also helpful in each case to draw a "root diagram" showing where in the complex plane the characteristic roots lie. The underdamped case gives a complex conjugate pair with negative real part; the overdamped case gives two negative real roots; and the critically damped case gives a repeated (negative) real root.

Some terminology: zero input response (ZIR) and zero state response (ZSR)
Given a "driven" system governed by an ODE such as $\ddot{x}+p(t)\dot{x}+q(t)x=f(t)$ with initial conditions $x(t_0)=x_0$ and $\dot{x}(t_0)=\dot{x}_0$, we generally identify the left-hand expression $\ddot{x}+p(t)\dot{x}+q(t)x$ as "the system" and the inhomogeneity on the right-hand side $f(t)$ as the "input signal" or "impulse". We use the same terminology for higher order ODE's. There is some useful terminology relevant to these types of ODE's.

If $x(t_0)=0$ and $\dot{x}(t_0)=0$, we refer to this as the **zero state**.

If we solve $\ddot{x}+p(t)\dot{x}+q(t)x=f(t)$ for the zero state, we refer to this solution $x_f(t)$ as the **zero state response (ZSR)**.

If we seek homogeneous solutions to the ODE $\ddot{x}+p(t)\dot{x}+q(t)x=0$ for any state with $x(t_0)=x_0$ and $\dot{x}(t_0)=\dot{x}_0$, this will have a unique solution $x_h(t)$ called the **zero input response (ZIR)**.

In general, the solution to the ODE $\ddot{x}+p(t)\dot{x}+q(t)x=f(t)$ will be $x(t)=x_h(t)+x_p(t)$ for some particular solution $x_p(t)$, but note that the zero state response (ZSR) is such a particular solution, so $x(t)=x_h(t)+x_f(t)$. That is, $x(t)=\textbf{ZIR}+\textbf{ZSR}$. Note that $\left\{\begin{matrix} x(t_0)=x_h(t_0)+x_f(t_0)=x_h(t_0)+0=x_h(t_0)=x_0 \\ \dot{x}(t_0)=\dot{x}_h(t_0)+\dot{x}_f(t_0)=\dot{x}_h(t_0)+0=\dot{x}_h(t_0)=\dot{x}_0 \end{matrix}\right\}$, so $x(t)=x_h(t)+x_p(t)$ satisfies the initial value problem (IVP) without the need to introduce any additional constants.

We previously derived the following useful result for finding particular solutions for Linear Time Invariant (LTI) ODEs in the case of exponential or sinusoidal inputs:

Exponential Response Formula (ERF): Suppose the Linear Time-Invariant ODE $\left[p(D)\right]x(t)=ae^{rt}$ has characteristic polynomial $p(s)$ and that r is <u>not a characteristic root</u>, then a particular solution will be:

$$\boxed{x_p(t)=\frac{ae^{rt}}{p(r)}}.$$

This result can make easy work of solving constant coefficient linear ODE's in this form. However, this formula will fail in the case where r is a characteristic root (since the denominator will vanish). This formula is especially useful for dealing with sinusoidal inputs – either pure sinusoidal inputs or with exponential growth or decay. The key step is to use complex replacement in order to express the input in exponential form.

Example: Solve the ODE $\ddot{x}+3\dot{x}+2x=5e^{3t}$ with $x(0)=2$, $\dot{x}(0)=3$.
Solution: The characteristic polynomial is $p(s)=s^2+3s+2=(s+2)(s+1)$. This gives roots $s_1=-2$, $s_2=-1$, and the homogeneous solutions are of the form $x_h(t)=c_1e^{-2t}+c_2e^{-t}$. If we use the Exponential Response Formula, we calculate $p(3)=9+9+2=20$, so a particular solution is $x_p(t)=\frac{5e^{3t}}{p(3)}=\frac{5e^{3t}}{20}=\frac{1}{4}e^{3t}$. The general solution is therefore $x(t)=c_1e^{-2t}+c_2e^{-t}+\frac{1}{4}e^{3t}$. Differentiation gives $\dot{x}(t)=-2c_1e^{-2t}-c_2e^{-t}+\frac{3}{4}e^{3t}$. Evaluating these at $t=0$ gives $\left\{\begin{matrix} x(0)=c_1+c_2+\frac{1}{4}=2 \\ \dot{x}(0)=-2c_1-c_2+\frac{3}{4}=3 \end{matrix}\right\} \Rightarrow c_1=-4, c_2=\frac{23}{4}$, so $\boxed{x(t)=-4e^{-2t}+\frac{23}{4}e^{-t}+\frac{1}{4}e^{3t}}$.

Example: Find the general solution of the ODE $\ddot{x}+3\dot{x}+2x=2e^t\cos 3t$.
Solution: The characteristic polynomial is $p(s)=s^2+3s+2=(s+2)(s+1)$. This gives roots $s_1=-2$, $s_2=-1$, and the homogeneous solutions are of the form $x_h(t)=c_1e^{-2t}+c_2e^{-t}$. To produce a particular solution, we use <u>complex replacement</u> (and then recover the real part). Letting $z(t)=x(t)+i\,y(t)$, we'll simultaneously solve the ODEs $\ddot{x}+3\dot{x}+2x=2e^t\cos 3t$ and $\ddot{y}+3\dot{y}+2y=2e^t\sin 3t$. Using Euler's formula, we'll solve the ODE $\ddot{z}+3\dot{z}+2z=2e^t(\cos 3t+i\sin 3t)=2e^te^{3it}=2e^{(1+3i)t}$. Using the Exponential Response Formula, we calculate $p(1+3i)=(1+3i)^2+3(1+3i)+2=1+6i-9+3+9i+2=-3+15i$, so a particular solution is $z_p(t)=\frac{2e^{(1+3i)t}}{-3+15i}$.

We could do one of two things at this point. First, we could multiply the numerator and denominator by the complex conjugate $-3-15i$ and also use Euler's formula to express everything in terms of sines and cosines. This would give:

$$z_p(t)=\frac{2e^{(1+3i)t}}{-3+15i}=\tfrac{1}{117}e^t(-3-15i)(\cos 3t+i\sin 3t)=\tfrac{1}{117}e^t\left[(-3\cos 3t+15\sin 3t)+i(-15\cos 3t-3\sin 3t)\right].$$

We would then recover the real part as $x_p(t)=\tfrac{1}{117}e^t(-3\cos 3t+15\sin 3t)$.

The second option is particularly well suited to the Exponential Response Formula. If we express the denominator as a complex number, i.e. $-3+15i=\sqrt{234}\,e^{i\phi}$ where $\phi=\tan^{-1}\left(\tfrac{15}{-3}\right)=\tan^{-1}(-5)\cong 1.768$ radians (in the 3rd quadrant), we can then write $z_p(t)=\dfrac{2e^{(1+3i)t}}{\sqrt{234}\,e^{i\phi}}=\tfrac{2}{\sqrt{234}}e^t e^{i(3t-\phi)}=\tfrac{2}{\sqrt{234}}e^t\left[\cos(3t-\phi)+i\sin(3t-\phi)\right]$ and recover the real part to give $x_p(t)=\tfrac{2}{\sqrt{234}}e^t\cos(3t-\phi)$. We can then easily see that the <u>gain</u> is $\tfrac{1}{\sqrt{234}}$, the <u>lag</u> is $\phi=\tan^{-1}(-5)\cong 1.768$ and, by writing $x_p(t)=\tfrac{2}{\sqrt{234}}e^t\cos 3(t-\tfrac13\phi)$, the <u>time lag</u> is $\tfrac13\phi\cong 0.589$.

The general solution may then be expressed as $x(t)=c_1e^{-2t}+c_2e^{-t}+\tfrac{2}{\sqrt{234}}e^t\cos 3(t-\tfrac13\phi)$.

Resonance

The case where the Exponential Response Formula fails is when r in the exponential input ae^{rt} is a root of the characteristic polynomial. Though the term "resonance" is perhaps most appropriate when considering sinusoidal inputs with frequency matching the natural frequency of a harmonic oscillator (like a spring), we use the term more generally. Let's understand this situation by considering an example.

Example: Find a particular solution of the ODE $\ddot{x}+3\dot{x}+2x=5e^{-2t}$.

Solution: We cannot use the Exponential Response Formula here because $r=-2$ is a root of the characteristic polynomial $p(s)=s^2+3s+2$. So what do we do? If we think in terms of differential operators, we can express this ODE in the form $(D+2)\circ(D+1)\left[x(t)\right]=5e^{-2t}$ and we know that $(D+2)\left[e^{-2t}\right]=-2e^{-2t}+2e^{-2t}=0$. So, if we apply this differential operator to both sides of the former equation we get:

$$(D+2)\circ(D+2)\circ(D+1)\left[x(t)\right]=(D+2)\left[5e^{-2t}\right]=5(D+2)\left[e^{-2t}\right]=0.$$

So we should seek solutions of the 3rd order homogeneous ODE $(D+2)^2\circ(D+1)\left[x(t)\right]=0$. The characteristic polynomial in this case is $(s+2)^2(s+1)$ which gives the same characteristic roots as before only now the root $s=-2$ has multiplicity 2. This means that the homogeneous solutions are given by $\operatorname{Span}\left\{e^{-2t},te^{-2t},e^{-t}\right\}$. The original inhomogeneous equation already had homogeneous solutions $\operatorname{Span}\left\{e^{-2t},e^{-t}\right\}$, so we seek a particular solution of the form $x_p(t)=Ate^{-2t}$ and use undetermined coefficients. This gives $\dot{x}_p(t)=A(-2t+1)e^{-2t}$ and $\ddot{x}_p(t)=A(4t-4)e^{-2t}$, so $\ddot{x}+3\dot{x}+2x=A(4t-4)e^{-2t}+3A(-2t+1)e^{-2t}+2Ate^{-2t}=-Ae^{-2t}=5e^{-2t}$. Therefore $A=-5$ and the particular solution is $\boxed{x_p(t)=-5te^{-2t}}$.

It's possible to do this in general. Suppose we have an nth order linear ODE in the form $\left[p(D)\right]x(t)=ae^{rt}$ where r is a root with multiplicity k of the characteristic polynomial $p(s)$. This means that we can express the characteristic polynomial as $p(s)=q(s)(s-r)^k$ where $q(s)$ is a polynomial of degree $n-k$. The corresponding differential operator can then be expressed as $T=p(D)=q(D)\circ(D-rI)^k$. If we seek a particular solution of the form $x_p(t)=At^ke^{rt}$, we can calculate

$(D-rI)(At^ke^{rt})=D(At^ke^{rt})-rAt^ke^{rt}=A(rt^ke^{rt}+kt^{k-1}e^{rt}-rt^ke^{rt})=Akt^{k-1}e^{rt}$. If $k\geq 2$, we can apply this

operator again to get $(D-rI)^2(At^ke^{rt}) = Ak(k-1)t^{k-2}e^{rt}$. Continuing, we eventually get to $(D-rI)^k(At^ke^{rt}) = Ak(k-1)\cdots(2)(1)e^{rt} = Ak!e^{rt}$. Substituting this into the ODE we get:
$\left[p(D)\right](At^ke^{rt}) = q(D)\circ(D-rI)^k(At^ke^{rt}) = Ak![q(D)(e^{rt})] = Ak!q(r)e^{rt} = ae^{rt}$

So $Ak!q(r) = a$, and $A = \dfrac{a}{k!q(r)}$, and therefore $\boxed{x_p(t) = \dfrac{at^ke^{rt}}{k!q(r)}}$.

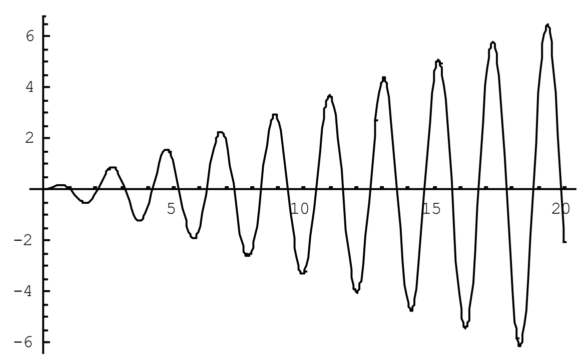
Though we could just use this as our "Resonant Response Formula", we can differentiate $p(s) = q(s)(s-r)^k$ repeatedly to get $p'(s) = q(s)k(s-r)^{k-1} + q'(s)(s-r)^k$,
$p''(s) = q(s)k(k-1)(s-r)^{k-2} + 2q'(s)(s-r)^{k-1} + q''(s)(s-r)^k$ and eventually
$p^{(k)}(s) = q(s)k! + (s-r)(\text{polynomial in } s)$, so $\boxed{p^{(k)}(r) = q(r)k!}$.

We can therefore in general express the **Resonant Response Formula (RRF)** as $\boxed{x_p(t) = \dfrac{at^ke^{rt}}{p^{(k)}(r)}}$ where $p^{(k)}(r)$ is the value of the kth derivative of the characteristic polynomial evaluated at r. Rarely will we need to use this for $k>1$, so the usual form is simply $\boxed{x_p(t) = \dfrac{ate^{rt}}{p'(r)}}$. The **Exponential Response Formula (ERF)** is just the $k=0$ case, i.e. $\boxed{x_p(t) = \dfrac{ae^{rt}}{p(r)}}$.

If we had applied the RRF to the previous example, we would have $p(s) = s^2+3s+2$ and we would calculate $p'(s) = 2s+3$, so $p'(-2) = -1$ and the particular solution would be $x_p(t) = \dfrac{5te^{-2t}}{-1} = -5te^{-2t}$.

Resonance with input frequency matching the natural frequency of a harmonic oscillator
Intuitively one should expect that if a spring system had a natural frequency ω with no external influence and if this system was driven with an applied sinusoidal force exactly matching this natural frequency, the amplitude might grow if the oscillations were in synch. A simple example illustrates this.

Example: Find the general solution of the ODE $x''+9x = 2\cos 3t$.
Solution: The homogeneous system $x''+9x=0$ has characteristic polynomial $p(s) = s^2+9$ which yields characteristic roots $s = \pm 3i$, and we can choose $\{\cos 3t, \sin 3t\}$ as a basis for the homogeneous solutions. Note that the input has this same frequency. If we choose to solve this using complex replacement, we would solve $z''+9z = 2e^{3it}$ and then recover the real part. Because $r = 3i$ is a simple characteristic root, we use the RRF. We have $p'(s) = 2s$ and $p'(3i) = 6i$, so a particular solution is:



$z_p(t) = \dfrac{2te^{3it}}{p'(3i)} = \dfrac{2te^{3it}}{6i} = -\tfrac{1}{3}it(\cos 3t + i\sin 3t) = \tfrac{1}{3}t\sin 3t - \tfrac{1}{3}it(\cos 3t)$. Therefore $\boxed{x_p(t) = \tfrac{1}{3}t\sin 3t}$. Note that this solution oscillates between the lines $y = \tfrac{1}{3}t$ and $y = -\tfrac{1}{3}t$ (hence the amplitude grows linearly in time).

The general solution will be $\boxed{x(t) = c_1\cos 3t + c_2\sin 3t + \tfrac{1}{3}t\sin 3t}$.

Superposition of (particular) solutions

In the case where a linear differential equation has an input expressed as the sum of two or more functions, linearity allows us to find solutions for each input individually and then sum these solutions to produce a solution for the original ODE. That is, if we have a linear ODE of the form $T(f) = g_1 + g_2$ and if can individually find functions f_1 and f_2 such that $T(f_1) = g_1$ and $T(f_2) = g_2$, then since $T(f_1 + f_2) = T(f_1) + T(f_2) = g_1 + g_2$, it follows that $f_1 + f_2$ is a solution to $T(f) = g_1 + g_2$. In fact, the same reason shows that if $T(f_1) = g_1$ and $T(f_2) = g_2$, then $c_1 f_1 + c_2 f_2$ will be a solution of $T(f) = c_1 g_1 + c_2 g_2$.

Example: Find a particular solution to the ODE $\ddot{x} + 3\dot{x} + 2x = 5e^{-2t} + t^2$

Solution: We have already solved $\ddot{x} + 3\dot{x} + 2x = 5e^{-2t}$ to get a solution $x_1(t) = -5te^{-2t}$. We can solve $\ddot{x} + 3\dot{x} + 2x = t^2$ using undetermined coefficients and a solution of the form $x(t) = at^2 + bt + c$. This gives $2a + 3(2at + b) + 2(at^2 + bt + c) = 2at^2 + (6a + 2b)t + (2a + 3b + 2c) = t^2$, so $\left\{\begin{aligned} 2a &= 1 \\ 6a + 2b &= 0 \\ 2a + 3b + 2c &= 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} a &= \tfrac{1}{2} \\ b &= -\tfrac{3}{2} \\ c &= \tfrac{7}{4} \end{aligned}\right\}$. So $x_2(t) = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4}$. Therefore the desired solution is $x_1(t) + x_2(t) = \boxed{-5te^{-2t} + \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4}}$.

Linear Time Invariant (LTI) ODEs

Recall the case of an autonomous 1st order ODE of the form $\frac{dx}{dt} = F(x)$, i.e. where the prescribed slope does not vary with the time *t*. The slope field for such a differential equation is horizontally invariant. For any such differential equation, if $x(t)$ is a solution, then $x(t-c)$ must also be a solution for any translation *c*. As a simple illustration, if $\frac{dx}{dt} = kx$ yields a solution $x(t) = ae^{kt}$ (natural exponential growth or decay), then $x(t-c) = ae^{k(t-c)} = (ae^{-kc})e^{kt}$ is also a solution of this same form.

There is a similar property in the case of constant coefficient linear ODEs of any order. In the case of a general linear ODE of the form $T[x(t)] = \frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t) = q(t)$, the "system" *T* may explicitly depend on the time *t* as evidenced by the coefficient functions $p_k(t)$ being functions of *t*. However, in the case of constant coefficients, i.e. $T[x(t)] = \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1\frac{dx}{dt} + a_0 x(t) = q(t)$, the system *T* does not explicitly depend on the time *t*. If a function $x(t)$ solves $T[x(t)] = q(t)$, then $T[x(t-t)] = q(t-c)$, so the translated function $y(t) = x(t-c)$ satisfies $\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y(t) = q(t-c)$. That is, if a given input signal is delayed for an amount of time but is otherwise unchanged, the response will be the same but delayed by that same period of time. Here's a simple example to illustrate this:

Example: Find a particular solution to the ODE $\ddot{x} + 3\dot{x} + 2x = 4\sin(t-3)$.

Solution: It would be simple enough to solve this using undetermined coefficients, but we can also just solve the equation $\ddot{x} + 3\dot{x} + 2x = 4\sin t$ and then use time invariance to get a solution to the given equation by translating our solution be replacing *t* with $t-3$. If we try a solution of the form $x = a\cos t + b\sin t$, we quickly conclude that to satisfy the equation we must have $a = -\frac{6}{5}$ and $b = \frac{2}{5}$, so $x_p(t) = -\frac{6}{5}\cos t + \frac{2}{5}\sin t$ satisfies $\ddot{x} + 3\dot{x} + 2x = 4\sin t$. Therefore $x_p(t) = -\frac{6}{5}\cos(t-3) + \frac{2}{5}\sin(t-3)$ satisfies $\ddot{x} + 3\dot{x} + 2x = 4\sin(t-3)$.
Had we instead used complex replacement to solve $\ddot{x} + 3\dot{x} + 2x = 4\sin t$. We would have obtained $x_p(t) = \frac{4}{\sqrt{10}}\sin(t-\phi)$ where $\phi = \tan^{-1}(3)$. Therefore $x_p(t) = \frac{4}{\sqrt{10}}\sin(t-3-\phi) = \frac{4}{\sqrt{10}}\sin(t-(3+\phi))$ is a particular solution to $\ddot{x} + 3\dot{x} + 2x = 4\sin(t-3)$. You can explicitly see how the new solution is just shifted further in time.

Problematic inputs

It won't always be the case that the input function for a linear ODE will be in a familiar form for which there is a simple formula or an obvious choice for undetermined coefficients. In such problematic cases, we can either be more inventive with our guesswork and choice of undetermined coefficients or perhaps come up with another method for handling problematic cases. For example:

Example: Find a particular solution to the ODE $y''+9y = xe^x \cos x$.

Solution: Finding the homogeneous solutions for this ODE is straightforward. Though it may not be obvious what form of solution might be best to try for a particular solution, perhaps one sufficiently general form might be $y=(Ax+B)e^x\cos x+(Cx+D)e^x\sin x$. As you can see, calculating the derivatives, substituting into the ODE, and solving for the undetermined coefficients is no picnic. You might want to try *Mathematica*.
That said, it is possible to carry out the calculations:

$$\left\{\begin{array}{l} y=(Ax+B)e^x\cos x+(Cx+D)e^x\sin x \\ y'=[(A+C)x+(A+B+D)]e^x\cos x+[(-A+C)x+(-B+C+D)]e^x\sin x \\ y''=[2Cx+(2A+2C+2D)]e^x\cos x+[-2Ax+(2A-2B+2C)]e^x\sin x \end{array}\right\}$$

So:

$$y''+9y=[(9A+2C)x+(2A+9B+2C+2D)]e^x\cos x+[(-2A+9C)x+(2A-2B+2C+9D)]e^x\sin x = xe^x\cos x$$

Matching coefficients gives: $\left\{\begin{array}{r} 9A+2C=1 \\ 2A+9B+2C+2D=0 \\ -2A+9C=0 \\ 2A-2B+2C+9D=0 \end{array}\right\} \Rightarrow \begin{bmatrix} 9 & 0 & 2 & 0 \\ 2 & 9 & 2 & 2 \\ -2 & 0 & 9 & 0 \\ 2 & -2 & 2 & 9 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}=\begin{bmatrix} 9/85 \\ -154/7225 \\ 2/85 \\ -242/7225 \end{bmatrix}$.

So $y_p=(\frac{9}{85}x-\frac{154}{7225})e^x\cos x+(\frac{2}{85}x-\frac{242}{7225})e^x\sin x$. As we said, it's doable but it's no picnic.

In the previous example there was at least some basis for guessing the form of the particular solution. Sometimes this is not a realistic expectation. **What then?**

Variation of Parameters

Recall the method in the case of a 1st order linear ODE of the form $\frac{dy}{dx}+p(x)y=q(x)$. First we found a homogeneous solution $y_h(x)$, so all homogeneous solutions would be of the form $A\,y_h(x)$. To find a particular solution to the inhomogeneous equation we "vary the parameter" A and seek a solution of the form $y(x)=v(x)y_h(x)$. We then derived that $v'(x)=\frac{q(x)}{y_h(x)}$ and, in principle, we can then integrate this to determine $v(x)$ and hence $y(x)$.

The situation with higher order linear ODEs is similar but more complicated. For example, if we have a 2nd order linear ODE of the form $y''+P(x)y'+Q(x)y=R(x)$, we would again first find homogeneous solutions $y_1(x)$ and $y_2(x)$ with all homogeneous solutions of the form $c_1y_1(x)+c_2y_2(x)$. We would again "vary the parameters" by seeking functions $v_1(x)$ and $v_2(x)$ such that $y(x)=v_1(x)y_1(x)+v_2(x)y_2(x)$ satisfies the inhomogeneous equation. For simplicity, let's write this as $y=v_1y_1+v_2y_2$. Differentiation (using the product and sum rules) gives $y'=u_1y_1'+v_1'y_1+v_2y_2'+v_2'y_2=\left(v_1y_1'+v_2y_2'\right)+\left(v_1'y_1+v_2'y_2\right)$. We have some flexibility here, so let's <u>assume</u> that we can find $v_1(x)$ and $v_2(x)$ such that $\boxed{v_1'y_1+v_2'y_2=0}$. Then $y'=v_1y_1'+v_2y_2'$. If we differentiate again, we get $y''=v_1y_1''+v_1'y_1'+v_2y_2''+v_2'y_2'$. If we substitute these expressions into the ODE we get:

$$y''+P(x)y'+Q(x)y=\left(v_1y_1''+v_1'y_1'+v_2y_2''+v_2'y_2'\right)+P(x)\left(v_1y_1'+v_2y_2'\right)+Q(x)\left(v_1y_1+v_2y_2\right)=R(x)$$

If we rearrange terms we get:

$$y''+P(x)y'+Q(x)y=v_1\left(y_1''+P(x)y_1'+Q(x)y_1\right)+v_2\left(y_2''+P(x)y_2'+Q(x)y_2\right)+\left(v_1'y_1'+v_2'y_2'\right)=R(x)$$

However, since $y_1(x)$ and $y_2(x)$ are homogeneous solutions, the two of these terms vanish and we're left with $\boxed{v_1'y_1'+v_2'y_2'=R(x)}$. If we join this with the previous assumption, we are seeking solutions to the system $\left\{\begin{matrix} y_1v_1'+y_2v_2'=0 \\ y_1'v_1'+y_2'v_2'=R\end{matrix}\right\}\Rightarrow\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2'\end{bmatrix}\begin{bmatrix} v_1' \\ v_2'\end{bmatrix}=\begin{bmatrix} 0 \\ R\end{bmatrix}$. This will yield solutions precisely when the Wronskian determinant $W(x)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2'\end{vmatrix}=y_1y_2'-y_2y_1'\neq 0$, and we then solve for $\begin{bmatrix} v_1' \\ v_2'\end{bmatrix}=\frac{1}{W(x)}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1\end{bmatrix}\begin{bmatrix} 0 \\ R\end{bmatrix}=\frac{1}{W(x)}\begin{bmatrix} -Ry_2 \\ Ry_1\end{bmatrix}$. That is, $\boxed{v_1'=-\dfrac{R(x)y_2(x)}{W(x)},v_2'=\dfrac{R(x)y_1(x)}{W(x)}}$ where $\boxed{W(x)=y_1(x)y_2'(x)-y_2(x)y_1'(x)\neq 0}$. Integration then gives $v_1(x)$ and $v_2(x)$ and hence $y_1(x)$ and $y_2(x)$. This method can be generalized to higher order linear ODEs. It depends on the ability to first find a complete basis of homogeneous solutions as well as the ability to find antiderivatives or, alternatively, to express these antiderivatives as integrals using the 2nd Fundamental Theorem of Calculus.

Example: Find a particular solution to the ODE $y''+3y'+2y=e^{-x}$.
Solution: The simplest way to solve this (and hence the best way) is to use either the ERF or, if necessary, the RRF. In this case the characteristic polynomial is $p(s)=s^2+3s+2=(s+2)(s+1)$, and $r=-1$ in the exponent of the input function is a simple characteristic root. The ERF is therefore not applicable, but the RRF is applicable. We have $p'(s)=2s+3$, so $p'(-1)=1$ and a particular solution is $y_p(x)=\dfrac{xe^{-x}}{p'(-1)}=\dfrac{xe^{-x}}{1}=\boxed{xe^{-x}}$.

If we had used variation of parameters, we would have the homogeneous solutions $y_1=e^{-2x}$, $y_2=e^{-x}$, their derivatives would be $y_1'=-2e^{-2x}$, $y_2'=-e^{-x}$, and the Wronskian determinant would be $W(x)=e^{-3x}$. So $v_1'=-\dfrac{e^{-x}e^{-x}}{e^{-3x}}=-e^{x}$, $v_2'=\dfrac{e^{-x}e^{-2x}}{e^{-3x}}=1$. These give $v_1=-e^x$ and $v_2=x$, so a particular solution would be $y_p(x)=-e^xe^{-2x}+xe^{-x}=-e^{-x}+xe^{-x}$. The first term is actually a homogeneous solution, so we can discard it to give $y_p(x)=\boxed{xe^{-x}}$.

Example: Find a particular solution to the ODE $y''+9y=xe^x\cos x$.
Solution: The homogeneous equation $y''+9y=0$ has characteristic polynomial $p(s)=s^2+9$ and characteristic roots $s=\pm 3i$. These yield the homogeneous solutions $\left\{e^{3ix},e^{-3ix}\right\}$ or, if you prefer $\left\{\cos 3x,\sin 3x\right\}$.

If we choose the latter basis for the homogeneous solutions, we would take $y_1=\cos 3x$, $y_2=\sin 3x$. We calculate $y_1'=-3\sin 3x$, $y_2'=3\cos 3x$, and the Wronskian determinant is $W(x)=3(\cos^2 3x+\sin^2 3x)=3$. So $v_1'=-\frac{1}{3}xe^x\cos x\sin 3x$, $v_2'=\frac{1}{3}xe^x\cos x\cos 3x$. It is unlikely that either we humans or Mathematica can produce nice antiderivatives for these, but we can express $v_1=-\frac{1}{3}\int_0^x te^t\cos t\sin 3t\,dt$, $v_2=\frac{1}{3}\int_0^x te^t\cos t\cos 3t\,dt$, so formally a particular solution is $\boxed{y_p(x)=-\frac{1}{3}\cos 3x\int_0^x te^t\cos t\sin 3t\,dt+\frac{1}{3}\sin 3x\int_0^x te^t\cos t\cos 3t\,dt}$.

Had we instead chosen the complex exponentials as a basis we would have similar challenges.

Notes by Robert Winters