Ordinary Differential Equations – Lecture #4

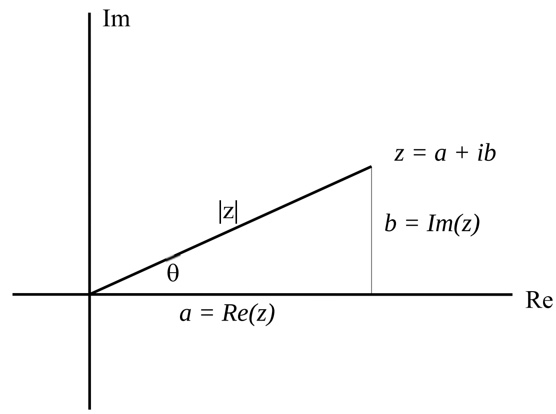
Today's topics include a) a recap of the methods so far for finding particular solutions for linear ODEs; b) the introduction of complex numbers and related facts to reformulate some of the methods involving exponential and sinusoidal inputs – specifically the Exponential Response Formula (ERF) and "complex replacement," and c) finding homogeneous and particular solutions for 2nd order (and higher) linear ODEs.

Methods we've seen so far:
Separation of variables
Integrating factors for solving 1st order linear ODEs
Linearity – homogeneous and particular solutions for linear ODEs
    Method of undetermined coefficients for finding particular solutions to linear ODEs
    Variation of Parameters for finding particular solutions

Complex variable methods for working with sinusoidal and exponential inputs (and other applications)
We often have to solve linear ODEs of the form $T(f)=g$ where the input is either sinusoidal or exponential or both. That is, we might encounter inputs of the form $g(t)=ke^{at}$ or $g(t)=k\cos\omega t$ or $g(t)=k\sin\omega t$ or $g(t)=ke^{at}\cos\omega t$ or $g(t)=ke^{at}\sin\omega t$ or a sum of such inputs for various choices of the constants k,a,ω.

Somewhere in your mathematical history you most likely learned a few things about **complex numbers**. We initially express complex numbers in the rectangular form $z=a+ib$ where $i^2=-1$. Complex numbers can be viewed in vector-like terms in the complex plane as shown in the diagram. We define:

$$\text{modulus}(z)=\text{mod}(z)=|z|=\sqrt{a^2+b^2}$$
$$\text{argument}(z)=\arg(z)=\theta=\tan^{-1}\left(\tfrac{b}{a}\right)$$



We add complex numbers by adding their respective real and imaginary parts, in much the same way as vector addition is defined. We multiply complex numbers via the distributive law and the fact that $i^2=-1$. For example:

$$(3+2i)(-1-4i)=-3-2i-12i-8i^2=-3-14i+8=5-14i.$$

If we note that $a=|z|\cos\theta$ and $b=|z|\sin\theta$ (see picture), then we can write $z=a+bi=|z|(\cos\theta+i\sin\theta)$. There's a simpler way to express this using Euler's formula.

The Maclaurin series for e^t, $\cos t$, and $\sin t$ are: $\left\{\begin{array}{l} e^t=1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots \\ \cos t=1-\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots \\ \sin t=t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots \end{array}\right\}$.

If we formally replace t by it and use the usual algebra rules, we get that:

$$e^{it}=1+it+\frac{(it)^2}{2!}+\frac{(it)^3}{3!}+\frac{(it)^4}{4!}+\cdots=\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\cdots\right)+i\left(t-\frac{t^3}{3!}+\cdots\right)=\cos t+i\sin t$$

That is, $\boxed{e^{it}=\cos t+i\sin t}$ [**Euler's Formula**]

A curious corollary of this is Euler's Identity: $e^{i\pi}=-1$.

Using Euler's Formula, we can express any complex number as $z=a+bi=|z|(\cos\theta+i\sin\theta)=|z|e^{i\theta}$ where $|z|$ is the modulus and θ is the argument of the complex number. This **polar form** allows us to understand the multiplication of complex numbers in very geometric terms. That is, if $z_1=|z_1|e^{i\theta_1}$ and $z_2=|z_2|e^{i\theta_2}$ are two complex numbers, their product is $z_1 z_2 = |z_1||z_2| e^{i\theta_1} e^{i\theta_2} = |z_1||z_2| e^{i(\theta_1+\theta_2)}$. That is, the modulus of the product is given by $|z_1 z_2| = |z_1||z_2|$ and the argument of the product is given by $Arg(z_1 z_2) = \theta_1 + \theta_2 = Arg(z_1) + Arg(z_2)$. When we multiply complex numbers, we multiply the moduli and we add the arguments.

As a special case, note that the complex number i has modulus 1 and argument $\pi/2 = 90°$. So i^2 should have modulus 1 and argument $\pi = 180°$, and this does indeed correspond to -1.

Perhaps more interesting is what this tells us about the "**roots of unity**". If we seek solutions to the equation $z^n = 1$ or, equivalently, $z^n - 1 = 0$, we know that $z = 1$ is a solution, but what are the other solutions? One way to approach this might be via factoring, i.e. $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1) = 0$ and we'd be seeking a factorization of $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$. If, instead, we think of this geometrically, it should be pretty clear that any such root would have to have modulus 1 (so it would lie on the unit circle in the complex plane) and it's argument θ would have to be such that $n\theta = 2\pi k$ for some integer *k*. Any such number must be of the form $z = e^{i(2\pi k/n)}$, and these consist of *n* points evenly distributed on the unit circle including $z = 1$. For example , the solutions to $z^3 = 1$ would be $\left\{1, e^{i(2\pi/3)}, e^{i(4\pi/3)}\right\}$, i.e. $\left\{1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\right\}$.

Definition: The **complex conjugate** of $z = a + ib$ is defined to be $\bar{z} = a - ib$. In the complex plane, *z* and $\bar{z}$ are reflections of each other across the real axis. It's not hard to show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

When factoring polynomials with real coefficients, the Fundamental Theorem of Algebra and the Quadratic Formula guarantee that any complex roots must come in complex conjugate pairs.

A little more trigonometry

We can use Euler's formula to produce a quick derivation of the sum of angle formulas for both the sine and cosine functions. We have:

$$e^{i(\theta+\phi)} = e^{i\theta} e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi).$$

So, since $e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$, comparing the real parts and the imaginary parts give that:

$$\boxed{\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi} \text{ and } \boxed{\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi}.$$

Application to integration

We can actually find the integrals $\int e^{at}\cos bt\,dt$ and $\int e^{at}\sin bt\,dt$ simultaneously using complex numbers.

If we write $e^{ibt} = \cos bt + i\sin bt$, then $e^{(a+ib)t} = e^{at}e^{ibt} = e^{at}\cos bt + ie^{at}\sin bt$.

Integration acts linearly, and if we extend this to complex-valued functions, we have that:

$$\int e^{(a+ib)t}dt = \int e^{at}\cos bt\,dt + i\int e^{at}\sin bt\,dt.$$

Exponential functions are easy to integrate (even when we extend to complex-valued exponential functions), and we calculate that $\int e^{(a+ib)t}dt = \frac{1}{a+ib}e^{(a+ib)t}$. We can proceed several ways here, but for the purpose of calculating these integrals, let's get rid of the complex denominator by multiplying both numerator and denominator by its complex conjugate (and use the fact that $z\bar{z} = (a+ib)(a-ib) = a^2 + b^2 = |z|^2$). We get:

$$\begin{aligned}\int e^{(a+ib)t}dt &= \frac{1}{a+ib}e^{(a+ib)t} = \frac{a-ib}{a^2+b^2}e^{at}e^{ibt} = \frac{1}{a^2+b^2}e^{at}(a-ib)(\cos bt + i\sin bt) \\ &= \frac{1}{a^2+b^2}e^{at}\left[(a\cos bt + b\sin bt) + i(-b\cos bt + a\sin bt)\right]\end{aligned}$$

If we compare this with $\int e^{(a+ib)t}dt = \int e^{at}\cos bt\,dt + i\int e^{at}\sin bt\,dt$, we see that:

$$\boxed{\int e^{at}\cos bt\,dt = \tfrac{1}{a^2+b^2}e^{at}(a\cos bt + b\sin bt)} \text{ and } \boxed{\int e^{at}\sin bt\,dt = \tfrac{1}{a^2+b^2}e^{at}(-b\cos bt + a\sin bt)}.$$

Previously we might have calculated such integrals using repeated application of integration by parts where we would have to use algebra to solve for the desired integral. Alternatively, we can use the above results. For example:

$$\boxed{\int e^{2t}\cos 3t\,dt = \tfrac{1}{13}e^{2t}\left(2\cos 3t + 3\sin 3t\right) + C}$$

(First order) Linear response to exponential, sinusoidal inputs

Motivating example: Heating/cooling can be modeled by the ODE $\frac{dx}{dt} = k(y-x)$ where $x(t)$ measures the temperature inside some box, room, or other space, and where the outside temperature varies according to some prescribed function $y(t)$, with $k>0$ the coupling constant. This can also be written as $\boxed{\frac{dx}{dt} + kx = ky}$, so this can be thought of as a 1st order linear inhomogeneous differential equation with input $g(t) = k\,y(t)$.

Imagine a situation where the initial inside temperature is $x(0) = x_0$ and where the outside temperature varies sinusoidally according to $y(t) = A\cos\omega t$. Here ω is the frequency and the period is $T = \frac{2\pi}{\omega}$.

What do we expect will happen?

(a) The temperature variation (amplitude) inside will likely not be as great as the variation outside.

(b) Any initial temperature inside will be transient – as the system eventually takes over.

(c) The change in temperature inside will likely lag or be out of phase with the outside temperature (wine cellar effect)

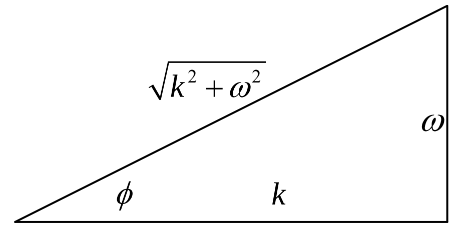
(d) If the frequency ω is very small (slow change), we might expect the inside temperature to "keep up" with the outside temperature.

(e) If ω is very large (rapid oscillation of temperature), we expect that the inside temperature will have very small variation around the average temperature (which is 0 in this case).

To solve the given linear differential equation, we start by finding the homogeneous solutions. We rewrite $\frac{dx}{dt} + kx = 0$ as $\frac{dx}{dt} = -kx$ and get $x_h(t) = ce^{-kt}$. For a particular solution, we could use undetermined coefficients and a solution of the form $x_p(t) = a\cos\omega t + b\sin\omega t$, but based on our expectations we might alternatively seek a solution of the form $x_p(t) = gA\cos(\omega t - \phi)$, where g is the ratio of response amplitude to input amplitude A. [This is equivalent to a solution of the form $x_p(t) = a\cos\omega t + b\sin\omega t$.] This ratio g is called the **gain**. We then substitute $x_p(t) = gA\cos(\omega t - \phi)$ into the original inhomogeneous ODE to determine g and ϕ.

We calculate $\frac{dx}{dt} + kx = -g\omega A\sin(\omega t - \phi) + kgA\cos(\omega t - \phi) = kA\cos\omega t$. To facilitate the determination of the unknowns g and ϕ, we rewrite $kA\cos\omega t = kA\cos(\omega t - \phi + \phi) = kA\cos\phi\cos(\omega t - \phi) - kA\sin\phi\sin(\omega t - \phi)$. So $\frac{dx}{dt} + kx = -g\omega A\sin(\omega t - \phi) + kgA\cos(\omega t - \phi) = kA\cos\phi\cos(\omega t - \phi) - kA\sin\phi\sin(\omega t - \phi)$. Equating coefficients gives $-g\omega A = -kA\sin\phi \;\Rightarrow\; \boxed{\sin\phi = g\omega/k}$, and $kgA = kA\cos\phi \;\Rightarrow\; \boxed{\cos\phi = g}$. So $\boxed{\tan\phi = \omega/k}$

This is most easily pictured with a right triangle as shown.
From this we see that $\boxed{\tan\phi = \frac{\omega}{k}}$ and $\boxed{g = \frac{k}{\sqrt{k^2+\omega^2}}}$.
So, the particular solution is $x_p(t) = gA\cos(\omega t - \phi)$ with these values for the gain g and the phase angle ϕ, and the general solution is therefore $x(t) = ce^{-kt} + gA\cos(\omega t - \phi)$. <u>Does this match with our expectations</u>?



<u>Notes</u>: (1) When the frequency ω is small (slow change), the gain g will be close to $1 = 100\%$, i.e. the inside temperature will vary almost as much as the outside temperature, and the lag will be close to 0 (temperature inside will "keep up" with the outside temperature change).

(2) When the frequency ω is large (rapid change), the gain g will be close to 0, so the inside temperature will have very small variation around the average temperature of 0. It will also be the case that the lag will approach 90°, but this will likely go unnoticed due to the minimal temperature variation.

(3) The initial temperature inside will determine the constant c in the exponentially decaying (transient) term, and this term will become negligible over time.

Another approach to finding a solution is to introduce complex-valued functions. For this we'll actually be solving two differential equations simultaneously. In addition to the ODE $\frac{dx}{dt} + kx = kA\cos\omega t$, let's also consider the ODE $\frac{dy}{dt} + ky = kA\sin\omega t$. If we let $z(t) = x(t) + i\,y(t)$, then $\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$, so we'll have
$\frac{dz}{dt} + kz = \left(\frac{dx}{dt} + i\frac{dy}{dt}\right) + k\left(x + iy\right) = \left(\frac{dx}{dt} + kx\right) + i\left(\frac{dy}{dt} + ky\right) = kA(\cos\omega t + i\sin\omega t) = kAe^{i\omega t}$, using Euler's formula.

This gives the complex ODE $\frac{dz}{dt} + kz = kAe^{i\omega t}$ where now the right-hand-side is now an exponential function. This approach is known as **complex replacement**.

We will soon develop a handy tool called the **Exponential Response Formula** (ERF) for handling similar linear ODE's of any order, but for now we can solve this directly using undetermined coefficients. The homogeneous solutions will again be of the form $z_h(t) = ce^{-kt}$, but we must understand the constant c to be an arbitrary <u>complex</u> constant, i.e. $c = c_1 + ic_2$. The homogeneous solutions may this be written as $z_h(t) = c_1e^{-kt} + ic_2e^{-kt}$. Since $k > 0$ this will decay, so we think of this as a **transient**.

For a particular solution (which will actually represent the **steady-state** solution), we try $z_p(t) = GAe^{i\omega t}$ where G is a complex constant called the **complex gain**. Differentiation and substitution into the ODE gives $\frac{dz}{dt} + kz = GAi\omega e^{i\omega t} + kGAe^{i\omega t} = GA(k + i\omega)e^{i\omega t} = kAe^{i\omega t}$, so we must have $G(k + i\omega) = k$ or $G = \frac{k}{k+i\omega}$. If we refer to the triangle from before and write in polar form $k + i\omega = \sqrt{k^2+\omega^2}\,e^{i\phi}$, we'll have $G = \frac{k}{\sqrt{k^2+\omega^2}\,e^{i\phi}}$, and the particular solution will be $z_p(t) = \frac{kA}{\sqrt{k^2+\omega^2}\,e^{i\phi}}e^{i\omega t} = \frac{kA}{\sqrt{k^2+\omega^2}}e^{i(\omega t-\phi)}$. Note that $|G| = \left|\frac{k}{\sqrt{k^2+\omega^2}\,e^{i\phi}}\right| = \frac{k}{\sqrt{k^2+\omega^2}} = g$ is the gain, and we can write $z_p(t) = \frac{kA}{\sqrt{k^2+\omega^2}}e^{i(\omega t-\phi)} = \left[\frac{kA}{\sqrt{k^2+\omega^2}}\cos(\omega t - \phi)\right] + i\left[\frac{kA}{\sqrt{k^2+\omega^2}}\sin(\omega t - \phi)\right]$.

So we have $z(t) = \left[c_1e^{-kt} + \frac{kA}{\sqrt{k^2+\omega^2}}\cos(\omega t - \phi)\right] + i\left[c_2e^{-kt} + \frac{kA}{\sqrt{k^2+\omega^2}}\sin(\omega t - \phi)\right] = x(t) + i\,y(t)$ as the general solution. This individually gives solutions $\boxed{x(t) = c_1e^{-kt} + \frac{kA}{\sqrt{k^2+\omega^2}}\cos(\omega t - \phi)}$ to the first ODE and $\boxed{y(t) = c_2e^{-kt} + \frac{kA}{\sqrt{k^2+\omega^2}}\sin(\omega t - \phi)}$ to the second ODE, and the solution to the first ODE is consistent with what we derived previously. Here $g = \frac{k}{\sqrt{k^2+\omega^2}}$ is the gain and $\tan\phi = \frac{\omega}{k}$ determines the lag. This method involving complex solutions is especially appropriate when considering gain and lag in the solution of higher order linear ODEs when the inhomogeneity is of the form $q(t) = ke^{at}\cos\omega t$ or $q(t) = ke^{at}\sin\omega t$.

Engineers often plot the gain and lag as functions of the input frequency ω. The plots of $\log\left[g(\omega)\right]$ vs. $\log(\omega)$ and $-\phi(\omega)$ vs. $\log(\omega)$ are known as **Bode plots**. They measure the response to a given signal.

Higher order linear ordinary differential equations with constant coefficients
In general, an *n*th order linear ordinary differential equation is a differential equation of the form $\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t) = q(t)$, where $p_{n-1}(t),\ldots,p_1(t),p_0(t),q(t)$ are functions of the independent variable *t*. We solve this by (1) finding an expression for all homogeneous solutions $x_h(t)$, (2) using some productive method to find one particular solution $x_p(t)$ to the inhomogeneous equation, and then (3) adding these to get the general solution $x(t) = x_h(t) + x_p(t)$. If we are solving an initial value problem, we would then use the initial conditions to determine any unknown constants in the expression for $x(t)$.

One case of special interest is the case where all of the coefficient functions $p_i(t) = a_i$ are constant. In this case the differential equation simplifies to $\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1\frac{dx}{dt} + a_0 x(t) = q(t)$. If we write $D = \frac{d}{dt}$, $D^2 = D\circ D = \frac{d^2}{dt^2}$, etc. and $I = Identity$, we can express this ODE as $\left[D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 I\right]x(t) = q(t)$. Note that this **linear operator** $T = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 I$ has a very polynomial-like quality. It has a corresponding **characteristic polynomial** $p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ that permits us to formally express $T = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 I = p(D)$. We will often write such an ODE in the form $\left[p(D)\right]x(t) = q(t)$.

If we seek exponential solutions of the form e^{rt} for the homogeneous equation $\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1\frac{dx}{dt} + a_0 x(t) = 0$, we calculate $\frac{dx}{dt} = re^{rt}$, $\frac{d^2x}{dt^2} = r^2 e^{rt}$, …, $\frac{d^n x}{dt^n} = r^n e^{rt}$, and substitution gives $r^n e^{rt} + a_{n-1}r^{n-1}e^{rt} + \cdots + a_2 r^2 e^{rt} + a_1 re^{rt} + a_0 e^{rt} = (r^n + a_{n-1}r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0)e^{rt} = p(r)e^{rt} = 0$.
This yields a solution only when the characteristic polynomial $p(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0$.

So for any root r_i of the characteristic polynomial, $e^{r_i t}$ will be a homogeneous solution. The Fundamental Theorem of Algebra guarantees (in principle) that we can factor $p(r)$ into a product of linear factors and irreducible quadratic factors. **As long as there are no repeated roots**, and since we can use the quadratic formula to produce a complex conjugate pair of roots for each irreducible quadratic factor, we will be able to produce *n* distinct roots and a corresponding set of exponential solutions $\left\{e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}\right\}$. In the case of repeated roots, this will yield fewer solutions of this form.

By linearity, any function of the form $x_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \ldots + c_n e^{r_n t}$ will solve the homogeneous equation.

Question: Does this yield <u>all</u> solutions?

A second order example should explain why the answer is <u>YES</u>. Suppose we wish to solve the ODE $\ddot{x} + 3\dot{x} + 2x = 0$. Any exponential solution e^{rt} would give $p(r) = r^2 + 3r + 2 = (r+2)(r+1) = 0$. Its characteristic roots are $r_1 = -2$ and $r_2 = -1$, and these yield solutions e^{-2t} and e^{-t}. Why are ALL homogeneous solutions of the form $x(t) = c_1 e^{-2t} + c_2 e^{-t}$?

If we write the differential equation in terms of linear differential operators, we might write this as $\left[D+2I\right]\circ\left[D+I\right]x(t)=0$, i.e. as a composition of two 1st order linear differential operators. [Feel free to check this.] If we let $\left[D+I\right]x(t)=y(t)$, this gives two 1st order equations: $\frac{dx}{dt}+x=y(t)$ and $\frac{dy}{dt}+2y=0$. The latter equation is easily solved to give all solutions $y(t)=c_1e^{-2t}$ where c_1 is a constant. We then substitute this into the former equation to get $\frac{dx}{dt}+x=c_1e^{-2t}$. This is an inhomogeneous equation with integrating factor e^t. Multiplication by this gives $e^t\frac{dx}{dt}+e^tx=\frac{d}{dt}(e^tx)=c_1e^{-t}$, so $e^tx(t)=-c_1e^{-t}+c_2$. Finally, multiplying both sides by e^{-t} gives $x(t)=-c_1e^{-2t}+c_2e^{-t}$. Except for the sign switch on the first arbitrary constant, this demonstrates that ***all*** homogeneous solutions are of the form $x(t)=c_1e^{-2t}+c_2e^{-t}$ for some choices of the constants c_1 and c_2, i.e. all linear combinations of the two basic exponential solutions that we found.

It should be relatively clear that this approach can be generalized to the *n*th order case as long as the characteristic polynomial can be factored into distinct linear factors. (We write the differential equation as a composition of *n* 1st order linear operators and iterate the above process.) **This even works in the case of complex roots as long as they are not repeated.** The more difficult case is when there are repeated roots of the characteristic polynomial, but, as we'll soon see, this case also yields a relatively simple solution.

In Linear Algebra terms, we say that $\left\{e^{r_1t},e^{r_2t},\ldots,e^{r_nt}\right\}$ **span all solutions** in the above case. It is a valid question to ask whether all of these solutions are necessary, i.e. if we could span all solutions with a subset of these exponential solutions. In Linear Algebra terms, we would ask: **Are these solutions are linearly independent**? In other words, is it possible to express any of these solutions as a linear combination of the other solutions?

Definition: A set of functions $\{f_1, f_2,\ldots, f_n\}$ is called **linearly independent** if the equation $c_1f_1(t)+c_2f_2(t)+\ldots+c_nf_n(t)=0$ (for all t) implies that $c_1=c_2=\ldots=c_n=0$.

When seeking solutions to an *n*th order linear differential equation of the form $\left[p(D)\right]x(t)=q(t)$, **we actually want more than this**. We want to guarantee a unique solution to any well-posed initial value problem with initial conditions given for the function and its derivatives up to order $(n-1)$, i.e. $x(t_0)=x_0$, $\dot{x}(t_0)=\dot{x}_0$, … $x^{(n-1)}(t_0)=x_0^{(n-1)}$. If $x_p(t)$ is one particular solution and if we can express all homogeneous solutions as $x_h(t)=c_1f_1(t)+c_2f_2(t)+\cdots+c_nf_n(t)$,then we would have the general solution $x(t)=x_h(t)+x_p(t)=c_1f_1(t)+c_2f_2(t)+\cdots+c_nf_n(t)+x_p(t)$ and we would then also want that:

$$\left\{\begin{array}{c} x_h(t_0)+x_p(t_0)=x(t_0) \\ \dot{x}_h(t_0)+\dot{x}_p(t_0)=\dot{x}(t_0) \\ \vdots \\ x_h^{(n-1)}(t_0)+x_p^{(n-1)}(t_0)=x^{(n-1)}(t_0) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} c_1f_1(t_0)+c_2f_2(t_0)+\cdots+c_nf_n(t_0)+x_p(t_0)=x(t_0) \\ c_1f_1'(t_0)+c_2f_2'(t_0)+\cdots+c_nf_n'(t_0)+\dot{x}_p(t_0)=\dot{x}(t_0) \\ \vdots \\ c_1f_1^{(n-1)}(t_0)+c_2f_2^{(n-1)}(t_0)+\cdots+c_nf_n^{(n-1)}(t_0)+x_p^{(n-1)}(t_0)=x^{(n-1)}(t_0) \end{array}\right\}$$

To guarantee a unique solution to the initial value problem, we would have to produce unique values for $\{c_1,c_2,\ldots,c_n\}$. We can rewrite the above system of linear equations in the form:

$$\left\{\begin{array}{c} f_1(t_0)c_1+f_2(t_0)c_2+\cdots+f_n(t_0)c_n=x(t_0)-x_p(t_0) \\ f_1'(t_0)c_1+f_2'(t_0)c_2+\cdots+f_n'(t_0)c_n=\dot{x}(t_0)-\dot{x}_p(t_0) \\ \vdots \\ f_1^{(n-1)}(t_0)c_1+f_2^{(n-1)}(t_0)c_2+\cdots+f_n^{(n-1)}(t_0)c_n=x^{(n-1)}(t_0)-x_p^{(n-1)}(t_0) \end{array}\right\}$$

In terms of matrices, we can express these as:

$$\begin{bmatrix} f_1(t_0) & f_2(t_0) & \cdots & f_n(t_0) \\ f_1'(t_0) & f_2'(t_0) & \cdots & f_n'(t_0) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(t_0) & f_2^{(n-1)}(t_0) & \cdots & f_n^{(n-1)}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} x(t_0) - x_p(t_0) \\ \dot{x}(t_0) - \dot{x}_p(t_0) \\ \vdots \\ x^{(n-1)}(t_0) - x_p^{(n-1)}(t_0) \end{bmatrix} \Rightarrow \text{ unique } \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Two fundamental results in linear algebra say that this will only be the case when the above matrix is invertible, and this will only be the case when its determinant is never equal to 0.

Definition: $\det \begin{bmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{bmatrix} = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix} = W(t)$

is called the **Wronskian determinant**.

Corollary: If the Wronskian determinant is never 0, the given ODE will yield unique solutions in the form $x(t) = x_h(t) + x_p(t) = c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) + x_p(t)$ for any given initial conditions given for the function and its derivatives up to order $(n-1)$.

Though not routinely used to ensure a linearly independent set of solutions, (there are arguments with less tedious calculations that can be made), the Wronskian is one tool for ensuring that a set of homogeneous solutions to a linear ODE is valid for ***uniquely*** expressing ***all*** solutions to a given initial value problem.

Example: Solve the initial value problem $\ddot{x} + 5\dot{x} + 4x = 3\sin 2t$ with initial conditions $x(0) = 3$, $\dot{x}(0) = 2$.

Solution: We first solve the homogeneous equation $\ddot{x} + 5\dot{x} + 4x = 0$. Its characteristic polynomial is $p(r) = r^2 + 5r + 4 = (r+4)(r+1)$ and this yields two distinct roots $r = -4$ and $r = -1$. The corresponding exponential solutions are e^{-4t} and e^{-t}. We can check that these are, in fact, linearly independent by calculating the Wronskian determinant: $\begin{vmatrix} e^{-4t} & e^{-t} \\ -4e^{-4t} & -e^{-t} \end{vmatrix} = -e^{-5t} + 4e^{-5t} = 3e^{-5t} \neq 0$. From our previous arguments, we know that all homogeneous solutions are of the form $\boxed{x_h(t) = c_1 e^{-4t} + c_2 e^{-t}}$.

Next, we seek a particular solution. There are at least two good ways to do this. We could do "complex replacement" and simultaneously solve $\ddot{x} + 5\dot{x} + 4x = 3\cos 2t$ and $\ddot{y} + 5\dot{y} + 4y = 3\sin 2t$ by solving the inhomogeneous equation $\ddot{z} + 5\dot{z} + 4z = 3e^{2it}$ and then taking the "imaginary" part. It is perhaps easier to solve using undetermined coefficients.

If we let $x = a\cos 2t + b\sin 2t$, we get $\left\{ \begin{aligned} x &= a\cos 2t + b\sin 2t \\ \dot{x} &= 2b\cos 2t - 2a\sin 2t \\ \ddot{x} &= -4a\cos 2t - 4b\sin 2t \end{aligned} \right\} \Rightarrow \ddot{x} + 5\dot{x} + 4x = (10b)\cos 2t + (-10a)\sin 2t$

We must therefore have $10b = 0$ and $-10a = 3$, so $a = -\frac{3}{10}$ and $b = 0$. So $\boxed{x_p(t) = -\frac{3}{10}\cos 2t}$.

The general solution is therefore $\boxed{x(t) = c_1 e^{-4t} + c_2 e^{-t} - \frac{3}{10}\cos 2t}$, and we have $\dot{x}(t) = -4c_1 e^{-4t} - c_2 e^{-t} + \frac{3}{5}\cos 2t$.

If we substitute the initial conditions $x(0) = 3$, $\dot{x}(0) = 2$, we have:

$$\begin{Bmatrix} x(0) = c_1 + c_2 - \frac{3}{10} = 3 \\ \dot{x}(0) = -4c_1 - c_2 = 2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} c_1 + c_2 = \frac{33}{10} \\ -4c_1 - c_2 = 2 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{33}{10} \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{53}{30} \\ \frac{76}{15} \end{bmatrix}.$$

We don't have to use matrices to solve these two equations, but it's worth noting that the non-vanishing of the Wronskian determinant is precisely why there is a unique solution for these constants. The unique solution to this initial value problem is therefore $\boxed{x(t) = -\frac{53}{30}e^{-4t} + \frac{76}{15}e^{-t} - \frac{3}{10}\cos 2t}$.

Note: In this example, the exponential terms are transients (they decay quickly) and the "steady state" solution is the particular solution that we calculated.

Characteristic polynomial and the Exponential Response Formula

Given the operator $D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 I$ and the fact that $D[e^{rt}] = re^{rt}$, $D^2[e^{rt}] = r^2 e^{rt}$, etc., it follows that $\left[D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 I\right]e^{rt} = r^n e^{rt} + a_{n-1}r^{n-1}e^{rt} + \cdots + a_1 re^{rt} + a_0 e^{rt} = (r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0)e^{rt}$.

We have already defined $p(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0$ as the **characteristic polynomial**. We can also formally write $p(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 I$, and write simply $\left[p(D)\right]e^{rt} = p(r)e^{rt}$. In the case of a homogeneous system, this means we would have $\left[p(D)\right]e^{rt} = p(r)e^{rt} = 0$ for all *t*, and this is only possible when $p(r) = 0$, i.e. when *r* is a root of the characteristic polynomial (called a characteristic root). According to the Fundamental Theorem of Algebra, we should, in principle, be able to fact $p(r)$ into a product of linear and irreducible quadratic factors and produce *n* roots, possible with multiplicity, and possibly including complex conjugate pairs.

If the ODE is not homogeneous but is in the simple form $\left[p(D)\right]x(t) = ae^{rt}$ for some (possibly complex) numbers *a* and *r*, we can use the method of undetermined coefficients to produce a particular solution. That is, if we let $x(t) = Ae^{rt}$, this will be a particular solution if:

$$\left[p(D)\right]Ae^{rt} = \left[D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 I\right]Ae^{rt} = A(r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0)e^{rt} = Ap(r)e^{rt} = ae^{rt}$$

Cancellation of the exponential factors and division by $p(r)$ gives $A = \frac{a}{p(r)}$, so $x_p(t) = \dfrac{ae^{rt}}{p(r)}$.

Exponential Response Formula (ERF): Suppose the ODE $\left[p(D)\right]x(t) = ae^{rt}$ has characteristic polynomial $p(s)$ and that *r* is not a characteristic root. Then a particular solution will be $\boxed{x_p(t) = \dfrac{ae^{rt}}{p(r)}}$.

This result can make easy work of solving constant coefficient linear ODE's in this form.

Example: Solve the ODE $\ddot{x} + 3\dot{x} + 2x = 5e^{3t}$ with $x(0) = 2$, $\dot{x}(0) = 3$.

Solution: The characteristic polynomial is $p(s) = s^2 + 3s + 2 = (s+2)(s+1)$. This gives roots $s_1 = -2$, $s_2 = -1$, and the homogeneous solutions are of the form $x_h(t) = c_1 e^{-2t} + c_2 e^{-t}$. If we use the Exponential Response Formula, we calculate $p(3) = 9 + 9 + 2 = 20$, so a particular solution is $x_p(t) = \dfrac{5e^{3t}}{p(3)} = \dfrac{5e^{3t}}{20} = \frac{1}{4}e^{3t}$. The general solution is therefore $x(t) = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{4}e^{3t}$. Differentiation gives $\dot{x}(t) = -2c_1 e^{-2t} - c_2 e^{-t} + \frac{3}{4}e^{3t}$. Evaluating these at $t = 0$ gives $\begin{Bmatrix} x(0) = c_1 + c_2 + \frac{1}{4} = 2 \\ \dot{x}(0) = -2c_1 - c_2 + \frac{3}{4} = 3 \end{Bmatrix} \Rightarrow c_1 = -4, c_2 = \frac{23}{4}$, so $\boxed{x(t) = -4e^{-2t} + \frac{23}{4}e^{-t} + \frac{1}{4}e^{3t}}$.

Example: Find the general solution of the ODE $\ddot{x}+3\dot{x}+2x=2e^t\cos 3t$.
Solution: The characteristic polynomial is $p(s)=s^2+3s+2=(s+2)(s+1)$. This gives roots $s_1=-2$, $s_2=-1$, and the homogeneous solutions are of the form $x_h(t)=c_1e^{-2t}+c_2e^{-t}$. To produce a particular solution, we use <u>complex replacement</u> (and then recover the real part). Letting $z(t)=x(t)+i\,y(t)$, we'll simultaneously solve the ODEs $\ddot{x}+3\dot{x}+2x=2e^t\cos 3t$ and $\ddot{y}+3\dot{y}+2y=2e^t\sin 3t$. Using Euler's formula, we'll solve the ODE $\ddot{z}+3\dot{z}+2z=2e^t(\cos 3t+i\sin 3t)=2e^te^{3it}=2e^{(1+3i)t}$. Using the Exponential Response Formula, we calculate $p(1+3i)=(1+3i)^2+3(1+3i)+2=1+6i-9+3+9i+2=-3+15i$, so a particular solution is $z_p(t)=\dfrac{2e^{(1+3i)t}}{-3+15i}$.

We could do one of two things at this point. First, we could multiply the numerator and denominator by the complex conjugate $-3-15i$ and also use Euler's formula to express everything in terms of sines and cosines. This would give:

$$z_p(t)=\frac{2e^{(1+3i)t}}{-3+15i}=\tfrac{1}{117}e^t(-3-15i)(\cos 3t+i\sin 3t)=\tfrac{1}{117}e^t\left[(-3\cos 3t+15\sin 3t)+i(-15\cos 3t-3\sin 3t)\right].$$

We would then recover the real part as $x_p(t)=\tfrac{1}{117}e^t(-3\cos 3t+15\sin 3t)$.

The second option is particularly well suited to the Exponential Response Formula. If we express the denominator as a complex number, i.e. $-3+15i=\sqrt{234}\,e^{i\phi}$ where $\phi=\tan^{-1}\left(\frac{15}{-3}\right)=\tan^{-1}(-5)\cong 1.768$ radians (in the 3rd quadrant), we can then write $z_p(t)=\dfrac{2e^{(1+3i)t}}{\sqrt{234}\,e^{i\phi}}=\frac{2}{\sqrt{234}}e^te^{i(3t-\phi)}=\frac{2}{\sqrt{234}}e^t\left[\cos(3t-\phi)+i\sin(3t-\phi)\right]$ and recover the real part to give $x_p(t)=\frac{2}{\sqrt{234}}e^t\cos(3t-\phi)$. We can then easily see that the <u>gain</u> is $\frac{1}{\sqrt{234}}$, the <u>lag</u> is $\phi=\tan^{-1}(-5)\cong 1.768$ and, by writing $x_p(t)=\frac{2}{\sqrt{234}}e^t\cos 3(t-\frac{1}{3}\phi)$, the <u>time lag</u> is $\frac{1}{3}\phi\cong 0.589$.
The general solution may then be expressed as $x(t)=c_1e^{-2t}+c_2e^{-t}+\frac{2}{\sqrt{234}}e^t\cos 3(t-\frac{1}{3}\phi)$.

Superposition of (particular) solutions
In the case where a linear differential equation has an input expressed as the sum of two or more functions, linearity allows us to find solutions for each input individually and then sum these solutions to produce a solution for the original ODE. That is, if we have a linear ODE of the form $T(f)=g_1+g_2$ and if can individually find functions f_1 and f_2 such that $T(f_1)=g_1$ and $T(f_2)=g_2$, then since $T(f_1+f_2)=T(f_1)+T(f_2)=g_1+g_2$, it follows that f_1+f_2 is a solution to $T(f)=g_1+g_2$. In fact, the same reason shows that if $T(f_1)=g_1$ and $T(f_2)=g_2$, then $c_1f_1+c_2f_2$ will be a solution of $T(f)=c_1g_1+c_2g_2$.

Example: Find a particular solution to the ODE $\ddot{x}+5\dot{x}+4x=3\sin 2t+t^2$
Solution: We have already solved $\ddot{x}+5\dot{x}+4x=3\sin 2t$ to get a solution $x_1(t)=-\frac{3}{10}\cos 2t$. We can solve $\ddot{x}+5\dot{x}+4x=t^2$ using undetermined coefficients and a solution of the form $x(t)=at^2+bt+c$. This gives $2a+5(2at+b)+4(at^2+bt+c)=4at^2+(10a+4b)t+(2a+5b+4c)=t^2$, so $\left\{\begin{aligned}4a&=1\\10a+4b&=0\\2a+5b+4c&=0\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}a&=\tfrac{1}{4}\\b&=-\tfrac{5}{8}\\c&=\tfrac{21}{32}\end{aligned}\right\}$. So $x_2(t)=\frac{1}{4}t^2-\frac{5}{8}t+\frac{21}{32}$. Therefore the desired solution is $x(t)=x_1(t)+x_2(t)=\boxed{-\frac{3}{10}\cos 2t+\frac{1}{4}t^2-\frac{5}{8}t+\frac{21}{32}}$.

Notes by Robert Winters