Ordinary Differential Equations – Lecture #3

Definition: A differential equation of the form $\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t) = q(t)$, where $p_{n-1}(t), \ldots, p_1(t), p_0(t), q(t)$ are functions of the independent variable *t*, is called an ***n*th order linear ordinary differential equation**. In the case where $q(t) = 0$ for all *t*, we call the equation **homogeneous**. Otherwise we call it **inhomogeneous**.

Input-Response formulation for linear ODEs

A linear ODE of the form $x^{(n)}(t) + p_{n-1}(t)x^{(n-1)}(t) + \cdots + p_1(t)x'(t) + p_0(t)x(t) = q(t)$ where $p_{n-1}(t), \ldots, p_1(t), p_0(t), q(t)$ are functions of the independent variable *t* can be expressed in the form $T(x(t)) = g(t)$ where *T* is a **linear operator** of the form $T = \frac{d^n}{dt^n} + p_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + p_1(t)\frac{d}{dt} + p_0(t)\cdot$. The last term refers to multiplication by $p_0(t)$. A useful way of formulating such an ODE is to thing of the left-hand-side as corresponding to "the system" and the inhomogeneity $q(t)$ on the right-hand-side as corresponding to the "input signal" or, more simply, the "input." The general solution of the ODE is then referred to as the "output signal" or "response." Some motivating examples are in order.

Banking: If we let $x = x(t)$ represent how much money (in dollars) we have in a bank account after *t* years with a fixed interest rate I, the simple model for this is $\frac{dx}{dt} = Ix$ and we have solved this to get $x(t) = Ae^{It}$. If we have only the initial deposit $x(0) = x_0$, then we'll have $x(t) = x_0 e^{It}$. Note, however, that we can write the ODE as $\frac{dx}{dt} - Ix = 0$, a homogeneous 1st order linear ODE. This corresponds to the situation where you make your deposit and then go home and let the system grow your money without further intervention.

Now let's suppose that you make deposits and withdrawals according to some function $q(t)$ (in dollars/year). If we add this rate into our model, we have $\frac{dx}{dt} = Ix + q(t)$, or $\boxed{\frac{dx}{dt} - Ix = q(t)}$. Note how this intervention (or input) corresponds to the inhomogeneity of this linear ODE. The "system" will carry on as before but will be subject to the input associated with the deposits and withdrawals. The "response" to all this internal and external activity will be the output $x(t)$, i.e. how much money you'll have in the bank at any given time.

Newton's Law of Cooling (diffusion): Suppose we have an enclosed space such as a building or a cooler chest and that the temperature at any given time *t* in the interior space is measured as some function $x(t)$ and that the initial temperature is $x(0) = x_0$. If the outside temperature is given by some function $y(t)$ (possibly constant or possibly variable), then we might expect the interior temperature to change depending on the quality of the insulation and on the difference between outside and inside temperatures. That is, the rate of change of temperature might be modeled as $\frac{dx}{dt} = F(y - x)$ for some function *F*. We would expect that when $y > x$ the temperature would increase, i.e. that $\frac{dx}{dt} > 0$; when $y < x$ the temperature would decrease, i.e. that $\frac{dx}{dt} < 0$; and that when the outside temperatures are the same there would be no change in temperature, i.e. $\frac{dx}{dt} = 0$. The simplest model for this would be $\boxed{\frac{dx}{dt} = k(y - x)}$ for some positive constant *k* (called the coupling constant) that depends on the level of insulation. We can rewrite this as $\boxed{\frac{dx}{dt} + kx = ky}$. In this form, we can think of the homogeneous equation $\frac{dx}{dt} + kx = 0$ as representing how this system would be governed if the outside temperature remained constant (at 0) and the interior temperature gradually rose or fell to that level. The inhomogeneous equation $\frac{dx}{dt} + k\,x(t) = k\,y(t)$ would govern how the interior temperature would respond to the input $y(t)$ (or $ky(t)$) in the case where the outside temperature varied according to some known pattern, e.g. the sinusoidal temperature change that might be associated with either a day/night cycle or seasonal cycle, or perhaps some other temperature variation.

Hooke's Law: A simple model for a frictionless mass-spring system is given by Hooke's Law $F=-kx$ where F represents an applied force, x represents the displacement of the mass from the equilibrium position, and where k is the spring constant that corresponds to the stiffness of the spring. If we combine this with Newton's 2nd Law that $F=ma$ where m is the mass, $v=\frac{dx}{dt}$ is the velocity, and $a=\frac{dv}{dt}=\frac{d^2x}{dt^2}$ is the acceleration of the mass, we have $ma=-kx$ or $m\frac{d^2x}{dt^2}=-kx$. We can rewrite this as $\frac{d^2x}{dt^2}+\frac{k}{m}x=0$. If these is some friction in the system, a simple model suggests that this friction would grow proportionally to the velocity, i.e. there would be an additional force $F_f=-cv$ opposing the motion. The revised equation becomes $m\frac{d^2x}{dt^2}=-kx-cv$ or $\frac{d^2x}{dt^2}+\frac{c}{m}\frac{dx}{dt}+\frac{k}{m}x=0$. Physicists often favor the "dot notation" for time derivatives with $\dot{x}=\frac{dx}{dt}$ and $\ddot{x}=\frac{d^2x}{dt^2}$, so the equation may also be expressed as $\ddot{x}+\frac{c}{m}\dot{x}+\frac{k}{m}x=0$.

Now imagine that you mess with this spring system by "driving" the system with an additional external force $E(t)$. The model might then look like $m\frac{d^2x}{dt^2}=-kx-cv+E(t)$ and if we write $E(t)=mq(t)$ for simplicity, the ODE becomes $\frac{d^2x}{dt^2}+\frac{c}{m}\frac{dx}{dt}+\frac{k}{m}x=\frac{E(t)}{m}=q(t)$ or $\boxed{\ddot{x}+\frac{c}{m}\dot{x}+\frac{k}{m}x=q(t)}$. Once again, the inhomogeneity corresponds to the external input imposed on the system, and the homogeneous ODE would govern how the system would evolve without this intervention.

As we've seen previously, a good approach to solving all of these linear ODEs is to use linear methods that involve finding all homogeneous solutions (the system), finding one particular solution, and combining these to determine the overall response $x(t)$.

Example (diffusion): Suppose a closed container has an initial interior temperature of 32°F at 10:00am and that the outside temperature (also in °F) rises steadily according to $y(t)=60+6t$ where time t is measured in hours. Further suppose that Newton's Law of Cooling applies where the coupling constant is $k=\frac{1}{3}$. (a) How will the interior temperature vary in time, and (b) at what time will the interior temperature reach 60°F?

Solution: The temperature will be governed by $\frac{dx}{dt}=\frac{1}{3}(y-x)$ or $\frac{dx}{dt}+\frac{1}{3}x=\frac{1}{3}y(t)=\frac{1}{3}(60+6t)=20+2t$, so the inhomogeneous ODE is $\frac{dx}{dt}+\frac{1}{3}x=20+2t$. This can be solved using an integrating factor, but let's use linearity.

(1) The homogeneous equation $\frac{dx}{dt}+\frac{1}{3}x=0$ easily yields the solutions of the form $\boxed{x_h(t)=ce^{-\frac{1}{3}t}}$. It's worth noting that over time any such homogeneous solution will tend toward 0 and become negligible. For this reason we often refer to this as a <u>transient</u>. In the short term it may be relevant, but in the long term it is not.

(2) We can use undetermined coefficients to find a particular solution. The nature of the inhomogeneity $q(t)=20+2t$ suggests that we seek a solution of the form $x_p(t)=A+Bt$. We have $\frac{dx_p}{dt}(t)=B$, so we must have $B+\frac{1}{3}(A+Bt)=(B+\frac{1}{3}A)+\frac{1}{3}Bt=20+2t$, so $B+\frac{1}{3}A=20$ and $\frac{1}{3}B=2$. This gives $B=6$ and $A=42$, so $x_p(t)=42+6t$. Once the transients have become negligible, this is all that will remain. For this reason we might refer to this as the "steady state" solution.

(3) The general solution is $x(t)=x_h(t)+x_p(t)=ce^{-\frac{1}{3}t}+42+6t$. If we substitute the initial condition $x(0)=32$, we have $x(0)=c+42=32$, so $c=-10$ and $\boxed{x(t)=42+6t-10e^{-\frac{1}{3}t}}$. Note that eventually the interior temperature will be rising at the same rate as the outside temperature but always 18°F cooler.

The interior temperature will reach 60°F at a time T when $42+6T-10e^{-\frac{1}{3}T}=60$ or $6T-10e^{-\frac{1}{3}T}=18$. This cannot be solved algebraically, but it's easy to get a numerical solution using a graphing calculator and the *trace* function. It gives a time $T \approx 3.33 \approx 3\text{ hrs, }20\text{ min}$, i.e. about 1:20pm.

Example #2 (exponential input): Solve the initial value problem $\frac{d^2x}{dt^2}+3\frac{dx}{dt}+2x=e^t$, $x(0)=4$, $x'(0)=2$.

Solution: This ODE is of the type we might expect from a mass-spring system, though the external driving force is not especially realistic (relentlessly exponential in a single direction). It is nonetheless good for illustrating the methods, and the exponential input will be very relevant in the days and weeks to come. For simplicity, let's write the ODE as $x''+3x'+2x=e^t$.

(1) For the homogeneous solutions, look for exponential solutions $x=e^{rt}$ to the equation $x''+3x'+2x=0$. This gives $r^2e^{rt}+3re^{rt}+2e^{rt}=(r^2+3r+2)e^{rt}=0$, so $r^2+3r+2=(r+1)(r+2)=0 \Rightarrow r=-1, r=-2$. Individual homogeneous solutions are $x_1(t)=e^{-t}$ and $x_2(t)=e^{-2t}$, and by linearity any solution of the form $x_h(t)=c_1e^{-t}+c_2e^{-2t}$ will satisfy the homogeneous ODE. It's not hard to prove that these give all homogeneous solutions if we think of the 2nd order homogenous linear ODE as a composition of two 1st order linear ODEs and use the fact that we can always solve such equations. [See if you can complete the argument.]

Note that, in this case, the homogeneous solutions are transient.

(2) Once again, undetermined coefficients provide the simplest way to find a particular solution in this case. The obvious choice is to try a solution of the form $x=Ae^t$. This gives $x'=Ae^t, x''=Ae^t$, and we get that $Ae^t+3Ae^t+2Ae^t=6Ae^t=e^t \Rightarrow A=\frac{1}{6}$, so our particular solution is $x_p(t)=\frac{1}{6}e^t$.

(3) Our general solution is then $\boxed{x(t)=c_1e^{-t}+c_2e^{-2t}+\frac{1}{6}e^t}$. We compute $x'(t)=-c_1e^{-t}-2c_2e^{-2t}+\frac{1}{6}e^t$, and the initial conditions give $\left\{\begin{array}{l} x(0)=c_1+c_2+\frac{1}{6}=4 \\ x'(0)=-c_1-2c_2+\frac{1}{6}=2 \end{array}\right\} \Rightarrow \left\{\begin{array}{r} c_1+c_2=\frac{23}{6} \\ -c_1-2c_2=\frac{11}{6} \end{array}\right\} \Rightarrow c_1=\frac{19}{2}, c_2=-\frac{17}{3}$. So the unique solution to the initial value problem is $\boxed{x(t)=\underbrace{\frac{19}{2}e^{-t}-\frac{17}{3}e^{-2t}}_{transient}+\underbrace{\frac{1}{6}e^t}_{steady-state}}$.

Example #3 (sinusoidal input): Find the solution to the ODE $\frac{dx}{dt}+2x=10\cos 3t$, $x(0)=4$.

What do we expect will happen?

(a) The temperature variation (amplitude) inside will likely not be as great as the variation outside.

(b) Any initial temperature inside will be transient – as the system eventually takes over.

(c) The change in temperature inside will likely lag or be out of phase with the outside temperature (wine cellar effect)

(d) If the frequency ω were very small (slow change), we might expect the inside temperature to "keep up" with the outside temperature.

(e) If ω was very large (rapid oscillation of temperature), we expect that the inside temperature will have very small variation around the average temperature (which is 0 in this case).

Solution: As with all 1st order linear equations, solving using an integrating factor is always an option, though it could lead to some difficult integration. In this example, the integrating factor is e^{2t} which gives $e^{2t}\frac{dx}{dt}+2e^{2t}x=\frac{d}{dt}(e^{2t}x)=10e^{2t}\cos 3t$. Integration gives $e^{2t}x(t)=10\int e^{2t}\cos 3t+C$ and $x(t)=e^{-2t}\left[10\int e^{2t}\cos 3t+C\right]$. The integration can be done using integration by parts (twice) and some additional algebra.

If we solve this using linearity:

(1) $\frac{dx}{dt}+2x=0$ gives the homogeneous solutions $x_h(t)=ce^{-2t}$

(2) For a particular solution, try $x=a\cos 3t+b\sin 3t$. We calculate $x'=3b\cos 3t-3a\sin 3t$, and substitution gives $\dot{x}+2x=(2a+3b)\cos 3t+(-3a+2b)\sin 3t=10\cos 3t$, so

$$\left\{\begin{array}{l} 2a+3b=10 \\ -3a+2b=0 \end{array}\right\} \Rightarrow \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} 10 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix}=\frac{1}{13}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 10 \\ 0 \end{bmatrix}=\begin{bmatrix} \frac{20}{13} \\ \frac{30}{13} \end{bmatrix}, \text{ so } x_p(t)=\frac{20}{13}\cos 3t+\frac{30}{13}\sin 3t \text{ or}$$

$x_p(t)=\frac{10}{13}(2\cos 3t+3\sin 3t)$.

(3) The general solution is therefore $\boxed{x(t)=\underbrace{ce^{-2t}}_{transient}+\underbrace{\frac{10}{13}(2\cos 3t+3\sin 3t)}_{steady-state}}$.

(4) Substituting the initial condition gives $x(0)=c+\frac{20}{13}=4$, so $c=\frac{32}{13}$ and

$\boxed{x(t)=\frac{32}{13}e^{-2t}+\frac{10}{13}(2\cos 3t+3\sin 3t)}$.

A little trigonometry

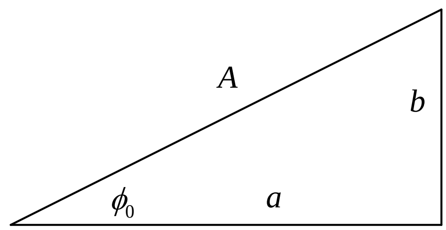
Any expression of the form $a\cos\omega t+b\sin\omega t$ actually represents a single sinusoidal curve with frequency ω and an appropriate translation (phase shift), i.e. a function of the form $A\cos(\omega t-\phi_0)$. We can see this quickly using the sum of angle formula for cosine:

$$A\cos(\omega t-\phi_0)=A\cos\omega t\cos\phi_0+A\sin\omega t\sin\phi_0=a\cos\omega t+b\sin\omega t$$

We must therefore have $\left\{\begin{array}{l} A\cos\phi_0=a \\ A\sin\phi_0=b \end{array}\right\}$.

This is most easily understood with a right triangle as shown.

From this we see that $A=\sqrt{a^2+b^2}$ and $\tan\phi_0=\frac{b}{a}$.



In our example with $x_p(t)=\frac{10}{13}(2\cos 3t+3\sin 3t)$ we would get $A=\frac{10}{13}\sqrt{2^2+3^2}=\frac{10\sqrt{13}}{13}=\frac{10}{\sqrt{13}}$ and $\tan\phi_0=\frac{3}{2}$.

This gives $\phi_0\cong 56.31°$ or $\phi_0\cong 0.9828$ radians. The period of the oscillation would be $\frac{2\pi}{\omega}=\frac{2\pi}{3}$.

The "gain" would be $\dfrac{\text{Response Amplitude}}{\text{Input Amplitude}}=\dfrac{1}{\sqrt{13}}$.

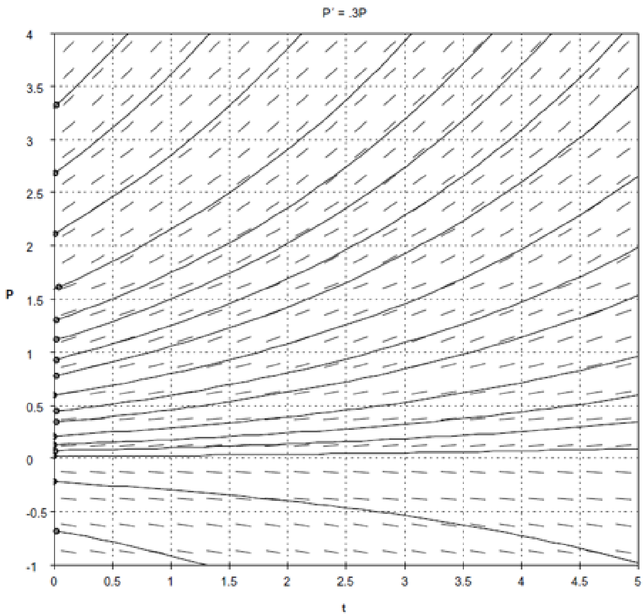
Autonomous 1st Order Differential Equations

Definition: An **first order autonomous differential equation** is an ODE of the form $\frac{dx}{dt}=F(x)$, i.e. an ODE where the rate $\frac{dx}{dt}$ depends only on the value of *x*. If *t* represents time, this means that the rate of change is time-independent.

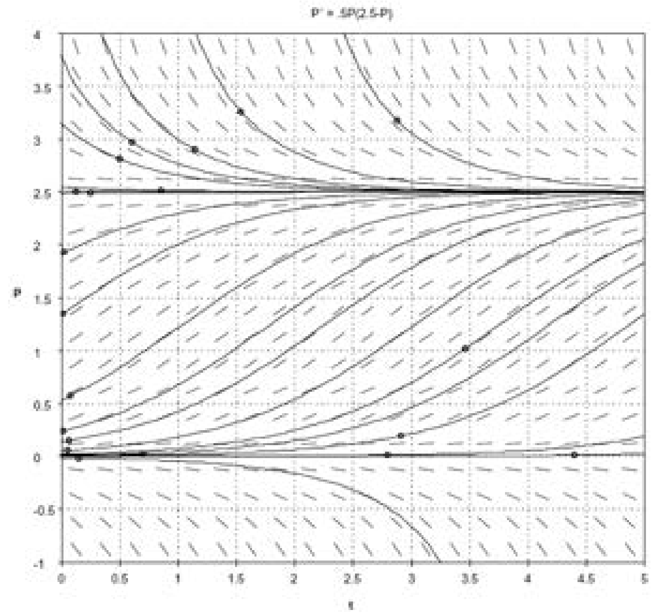
If we draw the slope field corresponding to an autonomous equation, the slopes will be constant horizontally but may vary vertically. Two familiar autonomous ODE's are:

(a) Natural (unrestricted) growth: $\boxed{\frac{dx}{dt}=kx}$ (exponential growth for $k>0$, exponential decay for $k<0$)

(b) Logistic growth: $\boxed{\frac{dx}{dt}=kx(1-\frac{x}{L})}$ (*L* is the "carrying capacity", the relative growth rate $\frac{1}{x}\frac{dx}{dt}=k(1-\frac{x}{L})$ decays linearly with increasing population with rate 0 when $x=L$ and negative growth for $x>L$)



Natural (exponential) growth: $\frac{dx}{dt} = kx$



Logistic growth: $\frac{dx}{dt} = kx(1 - \frac{x}{L})$

Even if we can solve an autonomous differential equation analytically to get a formula for the solutions, it is often more important to understand the solutions <u>qualitatively</u>.

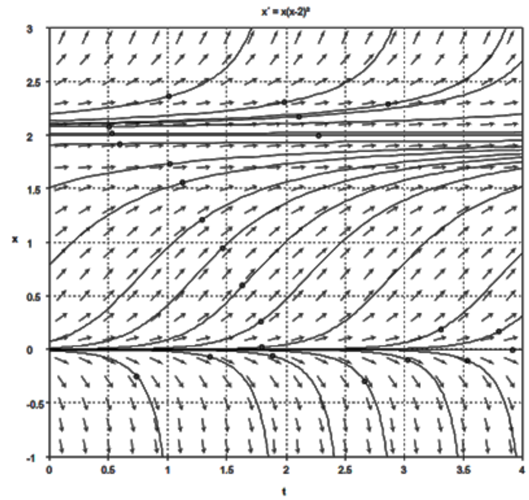
Definition: Given an autonomous differential equation $\frac{dx}{dt} = F(x)$, we call a point x_0 an **equilibrium** if $F(x_0) = 0$. The constant solution $x(t) = x_0$ will be a solution to the differential equation with initial condition $x(0) = x_0$.
As we can see in the illustrations above, some equilibria are such that nearby solutions converge toward the equilibrium and other equilibria are such that nearby solutions diverge away from the equilibrium.

Definition: If x_0 is an equilibrium of $\frac{dx}{dt} = F(x)$ and if for all initial conditions in some interval around x_0 the solutions $x(t)$ are such that $\lim_{t \to \infty}\left[x(t)\right] = x_0$, then we call x_0 a **stable equilibrium**. Otherwise we call it an **unstable equilibrium**. However, we usually consider an unstable equilibrium to be such that nearby solutions diverge away from the equilibrium. If we draw only the *x*-axis and indicate equilibria as points with arrows indicating the direction of nearby solutions, we refer to this as the **phase line**.

There's a simple **derivative test** for distinguishing stable and unstable equilibria. Suppose x_0 is an equilibrium for the differential equation $\frac{dx}{dt} = F(x)$ and that $F(x)$ is differentiable at x_0. We learned in Calculus about linear approximation, and in the vicinity of x_0 we'll have $F(x) \cong F(x_0) + F'(x_0)(x - x_0) = F'(x_0)(x - x_0)$ because $F(x_0) = 0$. We also know that if we let $u = (x - x_0)$, then $\frac{du}{dt} = \frac{d}{dt}(x - x_0) = \frac{dx}{dt}$, so we'll have $\frac{du}{dt} \cong F'(x_0)(x - x_0) = F'(x_0)u$. The differential equation $\frac{du}{dt} = F'(x_0)u$ yields growth (away from $u = 0$ or $x = x_0$) if $F'(x_0) > 0$, and decay (toward $u = 0$ or $x = x_0$) if $F'(x_0) < 0$. This enables us to distinguish unstable and stable equilibria. In the case where $F'(x_0) = 0$, we'll have to look at the slope field or use similar analysis.

<u>Note</u>: It may happen that on one side of an equilibrium nearby solutions converge toward the equilibrium but on the other side they diverge away from the equilibrium. In this case we would call the equilibrium **semistable**.

Example: Determine the equilibria of the differential equation $\frac{dx}{dt} = x(x-2)^2$ and classify their stability.



Solution: The equilibria will be where $F(x) = x(x-2)^2 = 0$, i.e. at $x=0$ and at $x=2$. The derivative gives $F'(x) = (x-2)(3x-2)$. We have $F'(0) = 4 > 0$ so this equilibrium will be unstable. On the other hand, $F'(2) = 0$ so we must use other means to determine the stability of this equilibrium. Note that $F'(x) > 0$ for $x > 2$ (repelling) and $F'(x) < 0$ for $x < 2$ (attracting), so this equilibrium will be semistable.

Analytic solution of the logistic equation

The logistic equation is $\frac{dx}{dt} = kx(1-\frac{x}{L})$ where $k > 0$ is constant. It has an unstable equilibrium at $x=0$ and a stable equilibrium at $x = L$ (the carrying capacity). Suppose $x(0) = x_0$ is the initial condition. We can write $\frac{dx}{x(1-\frac{x}{L})} = kdt$ and $\int \frac{dx}{x(1-\frac{x}{L})} = \int kdt = kt + C$. The integral on the left is done using partial fractions. Specifically, $\frac{1}{x(1-\frac{x}{L})} = \frac{L}{x(L-x)} = \frac{A}{x} + \frac{B}{L-x} \quad \Rightarrow \quad L = A(L-x) + Bx$. Choosing $x=0$ gives $AL = L$ or $A=1$. Choosing $x = L$ gives $BL = L$ or $B = 1$.

So $\frac{1}{x(1-\frac{x}{L})} = \frac{1}{x} + \frac{1}{L-x} \;\Rightarrow\; \int \frac{dx}{x(1-\frac{x}{L})} = \int \left(\frac{1}{x} + \frac{1}{L-x}\right) dx = \ln|x| - \ln|L-x| = \ln\left|\frac{x}{L-x}\right|$.

So $\ln\left|\frac{x}{L-x}\right| = kt + C \quad \Rightarrow \quad \frac{x}{L-x} = Ae^{kt} \quad \Rightarrow \quad x = LAe^{kt} - Axe^{kt} \quad \Rightarrow \quad x(1+Ae^{kt}) = LAe^{kt} \quad \Rightarrow \quad x(t) = \frac{LAe^{kt}}{1+Ae^{kt}}$.

The initial condition gives $x(0) = \frac{LA}{1+A} = x_0 \quad \Rightarrow \quad LA = x_0 + Ax_0 \quad \Rightarrow \quad A(L-x_0) = x_0 \quad \Rightarrow \quad A = \frac{x_0}{L-x_0}$.

So $x(t) = \dfrac{L\left(\dfrac{x_0}{L-x_0}\right)e^{kt}}{1+\left(\dfrac{x_0}{L-x_0}\right)e^{kt}} = \dfrac{Lx_0e^{kt}}{(L-x_0)+x_0e^{kt}} = \dfrac{Lx_0}{x_0+(L-x_0)e^{-kt}}$. So the solution is $\boxed{x(t) = \dfrac{Lx_0}{x_0+(L-x_0)e^{-kt}}}$.

Note, in particular, that $\lim_{t\to\infty}\left[x(t)\right] = \lim_{t\to\infty}\left[\dfrac{Lx_0}{x_0+(L-x_0)e^{-kt}}\right] = L$, as expected.

Example: Suppose population growth is governed by the logistic differential equation $\frac{dx}{dt} = kx(1-\frac{x}{L})$ with $k=1$ and carrying capacity $L = 1000$. Further suppose that the initial population is $x(0) = 100$. The analytic solution will then be $x(t) = \dfrac{100000}{100+900e^{-t}} = \dfrac{1000}{1+9e^{-t}}$. If we would like to know when the population will reach 500, we have $x(t) = \dfrac{1000}{1+9e^{-t}} = 500 \;\Rightarrow\; 1000 = 500 + 4500e^{-t} \;\Rightarrow\; e^t = 9 \;\Rightarrow\; t = \ln 9 \cong 2.197$.

If we ask when the population will reach 990, we have

$x(t) = \dfrac{1000}{1+9e^{-t}} = 990 \;\Rightarrow\; 1000 = 990 + 8910e^{-t} \;\Rightarrow\; e^t = 891 \;\Rightarrow\; t = \ln 891 \cong 6.792$.

Notes by Robert Winters