Ordinary Differential Equations – Lecture #2

Definition: A differential equation of the form $\frac{d^n x}{dt^n}+p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}}+\cdots+p_1(t)\frac{dx}{dt}+p_0(t)x(t)=q(t)$, where $p_{n-1}(t),\ldots,p_1(t),p_0(t),q(t)$ are functions of the independent variable *t*, is called an ***n*****th order linear ordinary differential equation**. In the case where $q(t)=0$ for all *t*, we call the equation **homogeneous**. Otherwise we call it **inhomogeneous**.

We are initially concerned with **1st order linear ODEs** of the form $\boxed{\frac{dx}{dt}+p(t)x=q(t)}$ (or $\boxed{\frac{dy}{dx}+p(x)y=q(x)}$).

Integrating factors

Definition: An **integrating factor** for a given first order ODE is a function $v(t)$ such that when both sides of the ODE are multiplied by $v(t)$ the resulting differential equation consists of known derivatives on both sides of the equation. The ODE can then be solved by integrating both sides and then solving for the dependent variable in terms of the independent variable.

It's always possible to formally solve $\frac{dx}{dt}+p(t)x=q(t)$ via an integrating factor. We seek $v(t)$ such that we can integrate both sides of the equation $v(t)\left[\frac{dx}{dt}+p(t)x\right]=v(t)q(t)$. The left-hand-side is $v\frac{dx}{dt}+pvx$, and if we note that $\frac{d}{dt}(vx)=v\frac{dx}{dt}+\frac{dv}{dt}x$, we can then look for $v(t)$ such that $\cancel{v\frac{dx}{dt}}+pvx=\cancel{v\frac{dx}{dt}}+\frac{dv}{dt}x$ or simply $pvx=\frac{dv}{dt}x \quad\Rightarrow\quad pv=\frac{dv}{dt}$. This is separable and can be rewritten as $\frac{dv}{v}=pdt \quad\Rightarrow\quad \int\frac{dv}{v}=\int p(t)dt$. This gives $\ln|v(t)|=\int p(t)dt+C$. Since we're just looking for one integrating factor, we can arbitrarily take $C=0$ and integrate to get $\boxed{v(t)=e^{\int p(t)dt}}$ as an integrating factor. This approach, of course, works best if you can find an antiderivative of the function $p(t)$.

We then have the new ODE $\frac{d}{dt}\big(v(t)x(t)\big)=v(t)q(t)$, so integration gives $v(t)x(t)=\int v(t)q(t)dt+C$. We can then solve for $x(t)=\frac{1}{v(t)}\left[\int v(t)q(t)dt+C\right]$. If we insert the integrating factor $v(t)=e^{\int pdt}$, we can write this solution as $\boxed{x(t)=e^{-\int pdt}\left[\int q(t)e^{\int pdt}dt+C\right]}$. It may not be pretty, but it works if you can actually do the integrals. You may find it simpler to just multiply both sides of the original ODE by the integrating factor $v(t)=e^{\int pdt}$ and proceed with the integrations.

Example: Solve the initial value problem $\frac{dx}{dt}=5x+3,\quad x(0)=4$.

Solution via separation of variables: Some quick algebra enables us to rearrange this ODE as $\frac{dx}{5x+3}=dt$ and multiplying both sides by 5 and integrating gives $\int\frac{5dx}{5x+3}=\int 5dt \quad\Rightarrow\quad \ln|5x+3|=5t+C \quad\Rightarrow\quad 5x+3=Ae^{5t}$. Substituting the initial condition gives $23=A$, so $5x+3=23e^{5t} \quad\Rightarrow\quad \boxed{x(t)=\frac{1}{5}(23e^{5t}-3)}$.

Solution via integrating factor: We start by first putting the ODE in the correct form, i.e. $\frac{dx}{dt}-5x=3$. We then recognize that $p(t)=-5$, so the integrating factor is $v(t)=e^{\int -5dt}=e^{-5t}$. Multiplying both sides of the ODE by this gives: $e^{-5t}\frac{dx}{dt}-5e^{-5t}x=3e^{-5t}$ or $\frac{d}{dt}\left(e^{-5t}x\right)=3e^{-5t}$, so $e^{-5t}x=3\int e^{-5t}dt=-\frac{3}{5}e^{-5t}+C$. We then solve for

$x(t) = -\frac{3}{5} + Ce^{5t}$. Inserting the initial condition gives $x(0) = 4 = -\frac{3}{5} + C$, so $C = \frac{23}{5}$ and $x(t) = -\frac{3}{5} + \frac{23}{5}e^{5t}$. So $\boxed{x(t) = \frac{1}{5}(23e^{5t} - 3)}$.

Considering the relatively simple expression for this solution, you might think that there could be a simpler approach. There is, but it requires us to start our way down an important path that will lead to some of the most important methods and perspectives in this entire course. This is the Linearity path. This method will prove to be the method that most easily generalizes to higher order linear ODEs. The method borrows some ideas from linear algebra and will require some introduction, especially the concept of a **linear operator**.

Linearity

In the context of functions of one variable, linearity is an often abused word. In fact, a function of the form $f(x) = mx + b$ is NOT a linear function. It is more appropriately called a 1st order *affine* function. Linearity is a property most simply characterized by the fact that linear functions preserve scaling and adding. The linear functions of one variable consist only of those of the form $f(x) = mx$. Note that $f(ax) = m(ax) = a(mx) = af(x)$, i.e. it preserves scaling, and $f(x+y) = m(x+y) = mx + my = f(x) + f(y)$, i.e. it preserves addition.

Definition: Formally we say that a function is **linear** if for all inputs x_1, x_2 and constants c_1, c_2 we must have $f(c_1x_1 + c_2x_2) = c_1 f(x_1) + c_2 f(x_2)$.

In the case of functions $T: \mathbf{R}^n \to \mathbf{R}^m$, linearity means that the scaling of vectors and the addition of vectors is preserved via a linear transformation. All such transformations are of the form $T(\mathbf{x}) = \mathbf{Ax}$ where $\mathbf{A}$ is an $m \times n$ matrix with constant entries. Linearity then translates into the matrix algebra facts that $\mathbf{A}(k\mathbf{x}) = k(\mathbf{Ax})$ and $\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{Ax} + \mathbf{Ay}$, or (combined) $\mathbf{A}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{Ax} + \beta\mathbf{Ay}$ for all scalars α, β and all vectors $\mathbf{x}, \mathbf{y}$.

Our current situation involves working with functions in the same way that we looked at vectors in $\mathbf{R}^n$. Just as we can scale and add vectors, we can also scale and add functions. A transformation that acts on functions in a manner analogous to the way matrices act on vectors is known as a **linear (differential) operator**. The basic examples are differentiation and multiplication by a fixed function. We can then compose these basic operators and add them to form more complicated operators.

There are many spaces of functions in which we can seek solutions to differential equations. Perhaps the most common such space is the space of functions that are differentiable to all orders.

Multiplication by a fixed function is a linear operator.

Suppose we have a fixed function $p(t)$ and we define a transformation of functions by $[T(f)](t) = p(t)f(t)$. We can easily see that for any constant *c*, $[T(cf)](t) = p(t)cf(t) = cp(t)f(t) = c[T(f)](t)$, so $T(cf) = cT(f)$, i.e. *T* preserves scaling. Similarly, if f_1 and f_2 are two functions, then $[T(f_1 + f_2)](t) = p(t)(f_1 + f_2)(t) = p(t)(f_1(t) + f_2(t)) = p(t)f_1(t) + p(t)f_2(t) = [T(f_1)](t) + [T(f_2)](t)$.
This is really just the distributive law, but the result is that formally $T(f_1 + f_2) = T(f_1) + T(f_2)$, i.e. *T* preserves addition of functions. Together, this shows that *T* is a linear operator.

Differentiation of functions is a linear operator.

Let *D* be the transformation defined by $D(f) = f'$, i.e. $D = \frac{d}{dt}$. That is, $[D(f)](t) = \frac{df}{dt} = f'(t)$. The old refrains you learned in first semester calculus are precisely what makes this a linear operator: (a) The derivative of a constant times a function is the constant times the derivative of the function; and (b) The derivative of a sum is the sum of the derivatives. In symbolic terms, $D(cf) = cf'$ and $D(f+g) = f' + g'$. We can put these together as a single linearity rule: $D(c_1 f_1 + c_2 f_2) = c_1 D(f_1) + c_2 D(f_2)$.

The composition of linear operators (or any linear function), where defined, is also linear.
If S and T are both linear operators and if the composition $S \circ T$ is defined, then using the linearity properties of both we have that for all scalars c_1, c_2 and functions f_1, f_2,

$$\begin{aligned}\left(S \circ T\right)(c_1 f_1 + c_2 f_2) &= S(T(c_1 f_1 + c_2 f_2)) = S(c_1 T(f_1) + c_2 T(f_2)) \\ &= c_1 S(T(f_1) + c_2 S(T(f_2)) = c_1 \left(S \circ T\right)(f_1) + c_2 \left(S \circ T\right)(f_2)\end{aligned}$$

For example, since differentiation acts linearly, we can compose this with itself to get the 2nd derivative and this also acts linearly. The same holds for higher order derivatives.

The sum of two linear operators is also a linear operator.
The sum of two operators is defined in the same way we add any functions, i.e. $(S+T)(f) = S(f) + T(f)$.
If S and T are both linear operators, then we'll have that for all scalars c_1, c_2 and functions f_1, f_2,

$$\begin{aligned}\left[S+T\right](c_1 f_1 + c_2 f_2) &= S(c_1 f_1 + c_2 f_2) + T(c_1 f_1 + c_2 f_2) = c_1 S(f_1) + c_2 S(f_2) + c_1 T(f_1) + c_2 T(f_2) \\ &= c_1 S(f_1) + c_1 T(f_1) + c_2 S(f_2) + c_2 T(f_2) = c_1\left[S(f_1) + T(f_1)\right] + c_2\left[S(f_2) + T(f_2)\right] = c_1\left[S+T\right](f_1) + c_2\left[S+T\right](f_2)\end{aligned}$$

If we put together the facts that composition of linear operators and the addition of linear operators yields another linear operator, we see that the expression $\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t)$ for functions $p_{n-1}(t), \ldots, p_1(t), p_0(t)$ represents a linear operator acting on an undetermined function $x(t)$. If we write this operator as $T(x(t)) = \frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t)$, we then know by linearity that $T(x_1(t) + x_2(t)) = T(x_1(t)) + T(x_2(t))$ and $T(c\,x(t)) = c\,T(x(t))$ and, more generally, $T(c_1\,x_1(t) + c_2\,x_2(t)) = c_1\,T(x_1(t)) + c_2\,T(x_2(t))$.

Now that we have paved the road to Linearity, we can apply this idea to solving linear differential equations.

Linearity method using homogeneous solutions and particular solutions
Suppose we have an inhomogeneous linear ODE of the form $T(f) = g$ where T is an nth order linear differential operator. We can produce ALL solutions to $T(f) = g$ as follows:

(1) First solve the homogeneous equation $T(f) = 0$ to find a general expression for all such solutions. We call these the homogeneous solutions f_h. It will generally involve n arbitrary constants.

(2) Find a single particular solution to the inhomogeneous equation $T(f) = g$. Call this particular solution f_p.

(3) The general solution to $T(f) = g$ is then $\boxed{f = f_h + f_p}$.

(4) Use the initial condition(s) to determine the unique solution to the given initial value problem (IVP).

Proof of the method: We know that $T(f_p) = g$, so suppose f is any other solution to $T(f) = g$. Then, by linearity, $T(f - f_p) = T(f) - T(f_p) = g - g = 0$. So $f - f_p$ solves the homogeneous equation and must be included among all homogeneous solution, i.e. $f - f_p = f_h$. Therefore $f = f_h + f_p$.

This fact is really the same thing that we see when solving a consistent, inhomogeneous system of linear algebraic equations. In matrix form, if the system is represented as $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is an $m \times n$ matrix, and if $\mathbf{x}_h$ represents all solutions to the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{x}_p$ is a single solution to $\mathbf{Ax} = \mathbf{b}$, then all solutions to $\mathbf{Ax} = \mathbf{b}$ will be of the form $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$. Typically, these homogeneous solutions are lines, planes or higher-dimensional analogues (subspaces) passing through the origin. This just says that the inhomogeneous solutions are parallel translates of these subspaces.

So, let's solve the problem $\frac{dx}{dt} = 5x + 3, \quad x(0) = 4$ using linearity methods:

We start by wring the ODE in the form $\frac{dx}{dt} - 5x = 3$, a first order, linear, inhomogeneous ODE.

(1) The homogeneous equation is just $\frac{dx}{dt}-5x=0$ or $\frac{dx}{dt}=5x$. We've already solved problems like this (it's separable) to get all solutions in the form $x_h(t)=Ae^{5t}$.

(2) We can find an inhomogeneous solution by educated guessing (formally called the **method of undetermined coefficients**). Try a solution of the form $x=at+b$. Calculate $\frac{dx}{dt}=a$ and substitute into the ODE to get $\frac{dx}{dt}-5x=a-5(at+b)=(a-5b)-5at=3$ (for <u>all</u> t). We can solve this by choosing $a-5b=0$ and $-5a=0$. So $\boxed{a=0}$ and $-5b=3$, so $\boxed{b=-\frac{3}{5}}$ and a particular solution is therefore $\boxed{x_p(t)=-\frac{3}{5}}$ which we could have guessed directly.

(3) By linearity, all solutions are therefore of the form $x(t)=x_h(t)+x_p(t)=Ae^{5t}-\frac{3}{5}$. This agrees with our previous result.

(4) Substitution of the initial condition then gives $\boxed{x(t)=\frac{1}{5}(23e^{5t}-3)}$ as before.

Analogy with solving inhomogeneous systems of linear equations

Suppose we want to solve a consistent, inhomogeneous system of linear algebraic equations. In matrix form, if the system is represented as $\mathbf{Ax}=\mathbf{b}$ where $\mathbf{A}$ is an $m\times n$ matrix, and if $\mathbf{x}_h$ represents all solutions to the homogeneous equation $\mathbf{Ax}=\mathbf{0}$ and $\mathbf{x}_p$ is a single solution to $\mathbf{Ax}=\mathbf{b}$, then all solutions to $\mathbf{Ax}=\mathbf{b}$ will be of the form $\mathbf{x}=\mathbf{x}_h+\mathbf{x}_p$. Typically, these homogeneous solutions are lines, planes or higher-dimensional analogues (subspaces) passing through the origin. This just says that the inhomogeneous solutions are parallel translates of these subspaces.

Example: Find all solutions of the linear system $\left\{\begin{matrix} x \quad -2z=3 \\ 2x-y-5z=2 \\ 3x-y-7z=5\end{matrix}\right\}$. We can solve this most easily by row reduction to get an equivalent system from which we can readily express all solutions. Specifically, we have:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 2 & -1 & -5 & 2 \\ 3 & -1 & -7 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left\{\begin{matrix} x-2z=3 \\ y+z=4 \\ z \text{ arbitrary}\end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x=3+2t \\ y=4-t \\ z=t\end{matrix}\right\} \Rightarrow \boxed{\begin{bmatrix} x \\ y \\ z\end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0\end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 1\end{bmatrix}}$$

If we were to solve the corresponding homogeneous linear system $\left\{\begin{matrix} x \quad -2z=0 \\ 2x-y-5z=0 \\ 3x-y-7z=0\end{matrix}\right\}$, the process is similar:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & -1 & -5 & 0 \\ 3 & -1 & -7 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left\{\begin{matrix} x-2z=0 \\ y+z=0 \\ z \text{ arbitrary}\end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x=2t \\ y=t \\ z=t\end{matrix}\right\} \Rightarrow \boxed{\begin{bmatrix} x \\ y \\ z\end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 1\end{bmatrix}}$$

The only difference is that the inhomogeneous solutions differ from the homogeneous solutions by a particular solution (which corresponds to $t=0$).

Back to solving differential equations

Example #1: Solve the initial value problem $\frac{dy}{dx}+xy=2x$; $\quad y(0)=5$.

(1) First, we solve the homogeneous equation $\frac{dy}{dx}+xy=0$. This will always be separable. We get $\frac{dy}{dx}=-xy$ and $\frac{dy}{y}=-xdx$, so $\int\frac{dy}{y}=-\int xdx \quad\Rightarrow\quad \ln|y|=-\frac{1}{2}x^2+C \quad\Rightarrow\quad y_h=Ae^{-\frac{1}{2}x^2}$.

(2) Next, we seek a particular solution to $\frac{dy}{dx}+xy=2x$. The Method of Undetermined Coefficients is a good choice here based on the relatively simple functions involved. If we try a solution of the form $y_p=ax^2+bx+c$ (which is actually more general that we really need), we have $\frac{dy}{dx}=2ax+b$, so substitution into the ODE gives:

$$(2ax+b)+x(ax^2+bx+c)=ax^3+bx^2+(2a+c)x+b=2x$$

So we must have $a=0, b=0, 2a+c=2, b=0 \quad\Rightarrow\quad a=0, b=0, c=2 \quad\Rightarrow\quad y_p=2$.

(3) So, the general solution must be $y=y_h+y_p=Ae^{-\frac{1}{2}x^2}+2$. The initial value gives $y(0)=A+2=5$, so $A=3$ and the unique solution to the initial value problem is $\boxed{y=3e^{-\frac{1}{2}x^2}+2}$.

<u>Note</u>: This problem could also have been solved using the integrating factor $e^{\frac{1}{2}x^2}$ derived by the method already discussed. This would give $e^{\frac{1}{2}x^2}\frac{dy}{dx}+xe^{\frac{1}{2}x^2}y=\frac{d}{dx}\left(e^{\frac{1}{2}x^2}y\right)=2xe^{\frac{1}{2}x^2} \quad\Rightarrow\quad e^{\frac{1}{2}x^2}y=2e^{\frac{1}{2}x^2}+C \quad\Rightarrow\quad y=2+Ce^{-\frac{1}{2}x^2}$, as above.

Example #2: Solve the initial value problem $\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x=\sin t,\ x(0)=1,\ x'(0)=2=\dot{x}(0)$.

This problem cannot be done using an integrating factor as that's really a method specific to 1st order linear equations. So we proceed using our methods based on linearity.

(1) First we seek <u>homogeneous solutions</u>, i.e. solutions of $\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x=0$. We're getting a little ahead of ourselves here, but for a linear ODE with constant coefficients we begin by seeking exponential solutions of the form $x=e^{rt}$. The logic behind this choice will be developed soon, but differentiation gives $\frac{dx}{dt}=re^{rt}$ and $\frac{d^2x}{dt^2}=r^2e^{rt}$. Substitution into the ODE gives $r^2e^{rt}-3re^{rt}+2e^{rt}=(r^2-3r+2)e^{rt}=0$. This can only vanish when $r^2-3r+2=(r-1)(r-2)=0$, so either $r=1$ or $r=2$. Therefore $x_1(t)=e^t$ and $x_2(t)=e^{2t}$ are solutions.

<u>Now here's where linearity becomes especially useful</u>. If $T[x(t)]=0$ is the form of the homogeneous equation (so $T[x_1(t)]=0$ and $T[x_2(t)]=0$ for all *t*), then any function of the form $c_1x_1(t)+c_2x_2(t)$ will also satisfy the homogeneous equation, i.e. $T[c_1x_1+c_2x_2]=c_1T(x_1)+c_2T(x_2)=c_1\cdot 0+c_2\cdot 0=0$. So $\boxed{x_h(t)=c_1e^t+c_2e^{2t}}$ will give homogeneous solutions for any scalars c_1,c_2. Though we have not yet shown it, the fact is that these give <u>all</u> of the homogeneous solutions.

(2) Now let's concentrate on getting a <u>particular solution</u> to the original inhomogeneous equation. If you think about what kinds of functions might be such that when combined with its 1st and 2nd derivatives in the manner prescribed by the ODE to yield the function $\sin x$, it should be pretty clear that something of the form $x_p=A\sin t+B\cos t$ is a likely candidate. We have $\left\{\begin{aligned} x_p &= A\sin t+B\cos t\\ x_p' &= -B\sin t+A\cos t\\ x_p'' &= -A\sin t-B\cos t\end{aligned}\right\}$, so:

$$x_p''-3x_p'+2x_p=(-A+3B+2A)\sin t+(-B-3A+2B)\cos t=(A+3B)\sin t+(-3A+B)\cos t=\sin t$$

This implies that $\left\{\begin{aligned} A+3B&=1\\ -3A+B&=0\end{aligned}\right\} \Rightarrow \begin{bmatrix}1&3\\-3&1\end{bmatrix}\begin{bmatrix}A\\B\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow \begin{bmatrix}A\\B\end{bmatrix}=\frac{1}{10}\begin{bmatrix}1&-3\\3&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\frac{1}{10}\\ \frac{3}{10}\end{bmatrix} \Rightarrow \left\{\begin{aligned} A&=\tfrac{1}{10}\\ B&=\tfrac{3}{10}\end{aligned}\right\}$.

So $x_p(t) = \frac{1}{10}\sin t + \frac{3}{10}\cos t$ is a particular solution.

(3) Therefore all solutions are of the form $\boxed{x(t) = c_1e^t + c_2e^{2t} + \frac{1}{10}\sin t + \frac{3}{10}\cos t}$.

Finally, to solve the given initial value problem, note that $x'(t) = c_1e^t + 2c_2e^{2t} + \frac{1}{10}\cos t - \frac{3}{10}\sin t$, so:

$$\left\{\begin{array}{l} x(0) = c_1 + c_2 + \frac{3}{10} = 1 \\ x'(0) = c_1 + 2c_2 + \frac{1}{10} = 2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} c_1 + c_2 = \frac{7}{10} \\ c_1 + 2c_2 = \frac{19}{10} \end{array}\right\} \Rightarrow c_2 = \frac{6}{5}, c_1 = -\frac{1}{2} \Rightarrow \boxed{x(t) = -\frac{1}{2}e^t + \frac{6}{5}e^{2t} + \frac{1}{10}\sin t + \frac{3}{10}\cos t}$$

Input-Response formulation for linear ODEs

A linear ODE of the form $x^{(n)}(t) + p_{n-1}(t)x^{(n-1)}(t) + \cdots + p_1(t)x'(t) + p_0(t)x(t) = q(t)$ where $p_{n-1}(t), \ldots, p_1(t), p_0(t), q(t)$ are functions of the independent variable *t* can be expressed in the form $T(x(t)) = g(t)$ where *T* is a **linear operator** of the form $T = \frac{d^n}{dt^n} + p_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + p_1(t)\frac{d}{dt} + p_0(t)\cdot$. The last term refers to multiplication by $p_0(t)$. A useful way of formulating such an ODE is to think of the left-hand-side as corresponding to "the system" and the inhomogeneity $q(t)$ on the right-hand-side as corresponding to the "input signal" or, more simply, the "input." The general solution of the ODE is then referred to as the "output signal" or "response." Some motivating examples are in order.

Banking: If we let $x = x(t)$ represent how much money (in dollars) we have in a bank account after *t* years with a fixed interest rate I, the simple model for this is $\frac{dx}{dt} = Ix$ and we have solved this to get $x(t) = Ae^{It}$. If we have only the initial deposit $x(0) = x_0$, then we'll have $x(t) = x_0e^{It}$. Note, however, that we can write the ODE as $\frac{dx}{dt} - Ix = 0$, a homogeneous 1st order linear ODE. This corresponds to the situation where you make your deposit and then go home and let the system grow your money without further intervention.

Now let's suppose that you make deposits and withdrawals according to some function $q(t)$ (in dollars/year). If we add this rate into our model, we have $\frac{dx}{dt} = Ix + q(t)$, or $\boxed{\frac{dx}{dt} - Ix = q(t)}$. Note how this intervention (or input) corresponds to the inhomogeneity of this linear ODE. The "system" will carry on as before but will be subject to the input associated with the deposits and withdrawals. The "response" to all this internal and external activity will be the output $x(t)$, i.e. how much money you'll have in the bank at any given time.

Newton's Law of Cooling (diffusion): Suppose we have an enclosed space such as a building or a cooler chest and that the temperature at any given time *t* in the interior space is measured as some function $x(t)$ and that the initial temperature is $x(0) = x_0$. If the outside temperature is given by some function $y(t)$ (possibly constant or possibly variable), then we might expect the interior temperature to change depending on the quality of the insulation and on the difference between outside and inside temperatures. That is, the rate of change of temperature might be modeled as $\frac{dx}{dt} = F(y - x)$ for some function *F*. We would expect that when $y > x$ the temperature would increase, i.e. that $\frac{dx}{dt} > 0$; when $y < x$ the temperature would decrease, i.e. that $\frac{dx}{dt} < 0$; and that when the outside temperatures are the same there would be no change in temperature, i.e. $\frac{dx}{dt} = 0$. The simplest model for this would be $\boxed{\frac{dx}{dt} = k(y - x)}$ for some positive constant *k* (called the coupling constant) that depends on the level of insulation. We can rewrite this as $\boxed{\frac{dx}{dt} + kx = ky}$. In this form, we can think of the homogeneous equation $\frac{dx}{dt} + kx = 0$ as representing how this system would be governed if the outside temperature remained constant (at 0) and the interior temperature gradually rose or fell to that level. The inhomogeneous equation $\frac{dx}{dt}+k\,x(t)=k\,y(t)$ would govern how the interior temperature would respond to the input $y(t)$ (or $ky(t)$) in the case where the outside temperature varied according to some known pattern, e.g. the sinusoidal temperature change that might be associated with either a day/night cycle or seasonal cycle, or perhaps some other temperature variation.

Hooke's Law: A simple model for a frictionless mass-spring system is given by Hooke's Law $F=-kx$ where F represents an applied force, x represents the displacement of the mass from the equilibrium position, and where k is the spring constant that corresponds to the stiffness of the spring. If we combine this with Newton's 2nd Law that $F=ma$ where m is the mass, $v=\frac{dx}{dt}$ is the velocity, and $a=\frac{dv}{dt}=\frac{d^2x}{dt^2}$ is the acceleration of the mass, we have $ma=-kx$ or $m\frac{d^2x}{dt^2}=-kx$. We can rewrite this as $\frac{d^2x}{dt^2}+\frac{k}{m}x=0$. If these is some friction in the system, a simple model suggests that this friction would grow proportionally to the velocity, i.e. there would be an additional force $F_f=-cv$ opposing the motion. The revised equation becomes $m\frac{d^2x}{dt^2}=-kx-cv$ or $\frac{d^2x}{dt^2}+\frac{c}{m}\frac{dx}{dt}+\frac{k}{m}x=0$. Physicists often favor the "dot notation" for time derivatives with $\dot{x}=\frac{dx}{dt}$ and $\ddot{x}=\frac{d^2x}{dt^2}$, so the equation may also be expressed as $\ddot{x}+\frac{c}{m}\dot{x}+\frac{k}{m}x=0$.

Now imagine that you mess with this spring system by "driving" the system with an additional external force $E(t)$. The model might then look like $m\frac{d^2x}{dt^2}=-kx-cv+E(t)$ and if we write $E(t)=mq(t)$ for simplicity, the ODE becomes $\frac{d^2x}{dt^2}+\frac{c}{m}\frac{dx}{dt}+\frac{k}{m}x=\frac{E(t)}{m}=q(t)$ or $\boxed{\ddot{x}+\frac{c}{m}\dot{x}+\frac{k}{m}x=q(t)}$. Once again, the inhomogeneity corresponds to the external input imposed on the system, and the homogeneous ODE would govern how the system would evolve without this intervention.

As we've seen previously, a good approach to solving all of these linear ODEs is to use linear methods that involve finding all homogeneous solutions (the system), finding one particular solution, and combining these to determine the overall response $x(t)$.

Example (diffusion): Suppose a closed container has an initial interior temperature of 32°F at 10:00am and that the outside temperature (also in °F) rises steadily according to $y(t)=60+6t$ where time t is measured in hours. Further suppose that Newton's Law of Cooling applies where the coupling constant is $k=\frac{1}{3}$. (a) How will the interior temperature vary in time, and (b) at what time will the interior temperature reach 60°F?

Solution: The temperature will be governed by $\frac{dx}{dt}=\frac{1}{3}(y-x)$ or $\frac{dx}{dt}+\frac{1}{3}x=\frac{1}{3}y(t)=\frac{1}{3}(60+6t)=20+2t$, so the inhomogeneous ODE is $\frac{dx}{dt}+\frac{1}{3}x=20+2t$. This can be solved using an integrating factor, but let's use linearity.

(1) The homogeneous equation $\frac{dx}{dt}+\frac{1}{3}x=0$ easily yields the solutions of the form $\boxed{x_h(t)=ce^{-\frac{1}{3}t}}$. It's worth noting that over time any such homogeneous solution will tend toward 0 and become negligible. For this reason we often refer to this as a <u>transient</u>. In the short term it may be relevant, but in the long term it is not.

(2) We can use undetermined coefficients to find a particular solution. The nature of the inhomogeneity $q(t)=20+2t$ suggests that we seek a solution of the form $x_p(t)=A+Bt$. We have $\frac{dx_p}{dt}(t)=B$, so we must have $B+\frac{1}{3}(A+Bt)=(B+\frac{1}{3}A)+\frac{1}{3}Bt=20+2t$, so $B+\frac{1}{3}A=20$ and $\frac{1}{3}B=2$. This gives $B=6$ and $A=42$, so $x_p(t)=42+6t$. Once the transients have become negligible, this is all that will remain. For this reason we might refer to this as the "steady state" solution.

(3) The general solution is $x(t) = x_h(t) + x_p(t) = ce^{-\frac{1}{3}t} + 42 + 6t$. If we substitute the initial condition $x(0) = 32$, we have $x(0) = c + 42 = 32$, so $c = -10$ and $\boxed{x(t) = 42 + 6t - 10e^{-\frac{1}{3}t}}$. Note that eventually the interior temperature will be rising at the same rate as the outside temperature but always 18°F cooler.

The interior temperature will reach 60°F at a time *T* when $42 + 6T - 10e^{-\frac{1}{3}T} = 60$ or $6T - 10e^{-\frac{1}{3}T} = 18$. This cannot be solved algebraically, but it's easy to get a numerical solution using a graphing calculator and the *trace* function. It gives a time $T \approx 3.33 \approx 3\text{ hrs, }20\text{ min}$, i.e. about 1:20pm.

Example #2 (exponential input): Solve the initial value problem $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = e^t,\ x(0) = 4,\ x'(0) = 2$.

Solution: This ODE is of the type we might expect from a mass-spring system, though the external driving force is not especially realistic (relentlessly exponential in a single direction). It is nonetheless good for illustrating the methods, and the exponential input will be very relevant in the days and weeks to come. For simplicity, let's write the ODE as $x'' + 3x' + 2x = e^t$.

(1) For the homogeneous solutions, look for exponential solutions $x = e^{rt}$ to the equation $x'' + 3x' + 2x = 0$. This gives $r^2e^{rt} + 3re^{rt} + 2e^{rt} = (r^2 + 3r + 2)e^{rt} = 0$, so $r^2 + 3r + 2 = (r+1)(r+2) = 0 \Rightarrow r = -1, r = -2$. Individual homogeneous solutions are $x_1(t) = e^{-t}$ and $x_2(t) = e^{-2t}$, and by linearity any solution of the form $x_h(t) = c_1e^{-t} + c_2e^{-2t}$ will satisfy the homogeneous ODE. It's not hard to prove that these give all homogeneous solutions if we think of the 2nd order homogenous linear ODE as a composition of two 1st order linear ODEs and use the fact that we can always solve such equations. [See if you can complete the argument.]

Note that, in this case, the homogeneous solutions are transient.

(2) Once again, undetermined coefficients provide the simplest way to find a particular solution in this case. The obvious choice is to try a solution of the form $x = Ae^t$. This gives $x' = Ae^t, x'' = Ae^t$, and we get that $Ae^t + 3Ae^t + 2Ae^t = 6Ae^t = e^t \Rightarrow A = \frac{1}{6}$, so our particular solution is $x_p(t) = \frac{1}{6}e^t$.

(3) Our general solution is then $\boxed{x(t) = c_1e^{-t} + c_2e^{-2t} + \frac{1}{6}e^t}$. We compute $x'(t) = -c_1e^{-t} - 2c_2e^{-2t} + \frac{1}{6}e^t$, and the initial conditions give $\left\{\begin{array}{l} x(0) = c_1 + c_2 + \frac{1}{6} = 4 \\ x'(0) = -c_1 - 2c_2 + \frac{1}{6} = 2 \end{array}\right\} \Rightarrow \left\{\begin{array}{r} c_1 + c_2 = \frac{23}{6} \\ -c_1 - 2c_2 = \frac{11}{6} \end{array}\right\} \Rightarrow c_1 = \frac{19}{2}, c_2 = -\frac{17}{3}$. So the unique solution to the initial value problem is $\boxed{x(t) = \underbrace{\frac{19}{2}e^{-t} - \frac{17}{3}e^{-2t}}_{transient} + \underbrace{\frac{1}{6}e^t}_{steady-state}}$.

Example #3 (sinusoidal input): Find the general solution to the ODE $\frac{dx}{dt} + 2x = \cos 3t$

Solution: As with all 1st order linear equations, solving using an integrating factor is always an option, though it could lead to some difficult integration. In this example, the integrating factor is e^{2t} which gives $e^{2t}\frac{dx}{dt} + 2e^{2t}x = \frac{d}{dt}(e^{2t}x) = e^{2t}\cos 3t$. Integration gives $e^{2t}x(t) = \int e^{2t}\cos 3t + C$ and $x(t) = e^{-2t}\left[\int e^{2t}\cos 3t + C\right]$. The integration can be done using integration by parts (twice) and some additional algebra.

<u>If we solve this using linearity</u>:

(1) $\frac{dx}{dt} + 2x = 0$ gives the homogeneous solutions $x_h(t) = ce^{-2t}$

(2) For a particular solution, try $x = a\cos 3t + b\sin 3t$. We calculate $x' = 3b\cos 3t - 3a\sin 3t$, and substitution gives $x' + 2x = (2a + 3b)\cos 3t + (-3a + 2b)\sin 3t = \cos 3t$, so

$$\begin{Bmatrix} 2a+3b=1 \\ -3a+2b=0 \end{Bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \tfrac{1}{13}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{13} \\ \frac{3}{13} \end{bmatrix}$$, so $x_p(t) = \tfrac{2}{13}\cos 3t + \tfrac{3}{13}\sin 3t$ or $x_p(t) = \tfrac{1}{13}(2\cos 3t + 3\sin 3t)$.

(3) The general solution is therefore $\boxed{x(t) = \underset{transient}{ce^{-2t}} + \underbrace{\tfrac{1}{13}(2\cos 3t + 3\sin 3t)}_{steady-state}}$.

A little trigonometry

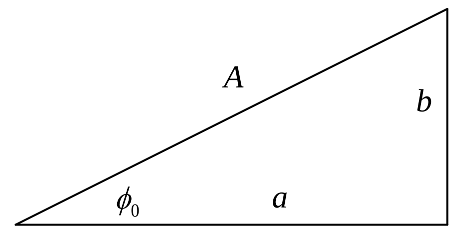
Any expression of the form $a\cos\omega t + b\sin\omega t$ actually represents a single sinusoidal curve with frequency ω and an appropriate translation (phase shift), i.e. a function of the form $A\cos(\omega t - \phi_0)$. We can see this quickly using the sum of angle formula for cosine:

$$A\cos(\omega t - \phi_0) = A\cos\omega t\cos\phi_0 + A\sin\omega t\sin\phi_0 = a\cos\omega t + b\sin\omega t$$

We must therefore have $\begin{Bmatrix} A\cos\phi_0 = a \\ A\sin\phi_0 = b \end{Bmatrix}$.

This is most easily understood with a right triangle as shown.

From this we see that $A = \sqrt{a^2+b^2}$ and $\tan\phi_0 = \frac{b}{a}$.



In our example with $x_p(t) = \tfrac{1}{13}(2\cos 3t + 3\sin 3t)$ we would get $A = \tfrac{1}{13}\sqrt{2^2+3^2} = \frac{\sqrt{13}}{13} = \frac{1}{\sqrt{13}}$ and $\tan\phi_0 = \frac{3}{2}$. This gives $\phi_0 \cong 56.31°$ or $\phi_0 \cong 0.9828$ radians. The period of the oscillation would be $\frac{2\pi}{\omega} = \frac{2\pi}{3}$.

Variation of parameters

Another useful method for finding a particular solution to a linear ODE is to take the homogeneous solutions that you've presumably already found and "vary the parameters." This method can be formulated for nth order linear ODEs (and we'll do that eventually), but for now we'll formulate the method for 1st order linear ODEs.

Suppose we are trying to solve the linear ODE $\frac{dx}{dt} + p(t)\cdot x = q(t)$ where $p(t), q(t)$ are functions of the independent variable *t*, and that we have already solved the homogeneous equation $\frac{dx}{dt} + p(t)\cdot x = 0$ to find the homogeneous solutions $x_h(t)$. This equation is separable and can, in principle, always be solved to give $x_h(t) = Ae^{-\int p(t)dt} = Ax_1(t)$. The basic idea is to treat the scalar A as variable, i.e. we "*vary the parameter*."

If we write $x(t) = v(t)x_1(t)$ where $x_1(t)$ as the basic homogeneous solution, we can then calculate that $\frac{dx}{dt} = v(t)\frac{dx_1}{dt} + \frac{dv}{dt}x_1(t)$ and substitute into the ODE to get:

$$v(t)\frac{dx_1}{dt} + \frac{dv}{dt}x_1(t) + p(t)v(t)x_1(t) = v(t)\left(\frac{dx_1}{dx} + p(t)x_1(t)\right) + \frac{dv}{dt}x_1(t) = q(t)$$

Note that since $x_1(t)$ is a solution to the homogeneous equation, the expression in parentheses vanishes. So the resulting equation becomes $\frac{dv}{dt}x_1(t) = q(t)$. This is, in principle, easily solved by writing $\frac{dv}{dt} = \frac{q(t)}{x_1(t)}$ and integrating to get $\boxed{v(t) = \int \frac{q(t)}{x_1(t)}dt}$. We then have the particular solution $x_p(t) = v(t)x_1(t)$.

Example #1: Find the general solution of the 1st order linear ODE $\frac{dy}{dx}+\frac{5}{x}y=7x$.

Solution: The homogeneous equation $\frac{dy}{dx}+\frac{5}{x}y=0$ gives:

$$\frac{dy}{dx}=-\frac{5}{x}y \quad\Rightarrow\quad \frac{dy}{y}=-\frac{5}{x}dx \quad\Rightarrow\quad \int\frac{dy}{y}=-\int\frac{5}{x}dx \quad\Rightarrow\quad \ln|y|=-5\ln|x|+C \quad\Rightarrow\quad y_h(x)=Ax^{-5}$$

So we take $y_h(x)=x^{-5}=\frac{1}{x^5}$ for the purpose of doing variation of parameters to find a particular solution. With $q(x)=7x$, the method as described above gives $v(x)=\int\frac{7x}{x^{-5}}dx=\int 7x^6dx=x^7$. [Note that we don't add an arbitrary constant because we're only trying to find one particular solution.]

So $y_p(x)=v(x)y_h(x)=x^7\cdot x^{-5}=x^2$. The general solution is therefore $\boxed{y(x)=Ax^{-5}+x^2}$ where A is an arbitrary constant.

Example #2 (sinusoidal input): Find the general solution to the ODE $\frac{dx}{dt}+2x=\cos 3t$. [This is the same problem we solved in the previous lecture.]

Solution: Last time we solved the homogeneous ODE to get $x_h(t)=ce^{-2t}$. If we use $x_h(t)=e^{-2t}$ and $q(t)=\cos 3t$ for the variation of parameters, we get $v(t)=\int\frac{q(t)}{x_h(t)}dt=\int\frac{\cos 3t}{e^{-2t}}dt=\int e^{2t}\cos 3t\,dt$.

The integral is found using integration by parts (twice) and some algebra. As a reminder of integration methods, the calculation would go something like this:

$$I=\int e^{2t}\cos 3t\,dt=\tfrac{1}{3}e^{2t}\sin 3t-\tfrac{2}{3}\int e^{2t}\sin 3t\,dt=\tfrac{1}{3}e^{2t}\sin 3t-\tfrac{2}{3}\left[-\tfrac{1}{3}e^{2t}\cos 3t+\tfrac{2}{3}\int e^{2t}\cos 3t\,dt\right]$$

$$=e^{2t}\left(\tfrac{1}{3}\sin 3t+\tfrac{2}{9}\cos 3t\right)-\tfrac{4}{9}I \;\Rightarrow\; \tfrac{13}{9}I=e^{2t}\left(\tfrac{1}{3}\sin 3t+\tfrac{2}{9}\cos 3t\right) \;\Rightarrow\; I=\boxed{\int e^{2t}\cos 3t\,dt=e^{2t}\left(\tfrac{2}{13}\cos 3t+\tfrac{3}{13}\sin 3t\right)}$$

So the particular solution is $x_p(t)=\left(e^{2t}\left(\frac{2}{13}\cos 3t+\frac{3}{13}\sin 3t\right)\right)e^{-2t}=\frac{2}{13}\cos 3t+\frac{3}{13}\sin 3t$ and the general solution is then $\boxed{x(t)=ce^{-2t}+\frac{2}{13}\cos 3t+\frac{3}{13}\sin 3t}$ where c is an arbitrary constant to be determined by initial conditions. You can check that this coincides with the solution we derived last time via other methods. We also have the option of putting this in the form $\boxed{x(t)=ce^{-2t}+\frac{1}{\sqrt{13}}\cos(3t-\phi_0)}$ where $\phi_0\cong 56.31^\circ$ as we showed last time.

Notes by Robert Winters