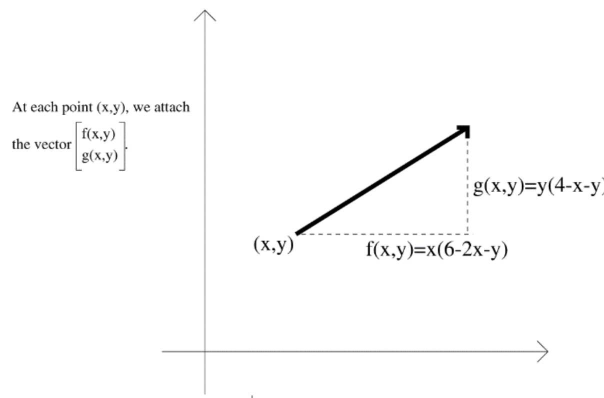
Math E-21c – Ordinary Differential Equations – Lecture #14

Nonlinear systems

Most differential equations and systems of differential equations one encounters in practice are nonlinear. For example, a biologist might model the populations $x(t)$ and $y(t)$ of two interacting species of animals by the following nonlinear system:

$$\left\{\begin{aligned}\frac{dx}{dt} &= x(6-2x-y)\\ \frac{dy}{dt} &= y(4-x-y)\end{aligned}\right\}$$

where the populations are measured in thousands. (To understand the rationale behind these equations, read: J.D. Murray, *Mathematical Biology*, Chapter 3: Continuous Models for Interacting Populations, Springer-Verlag, 1989.)



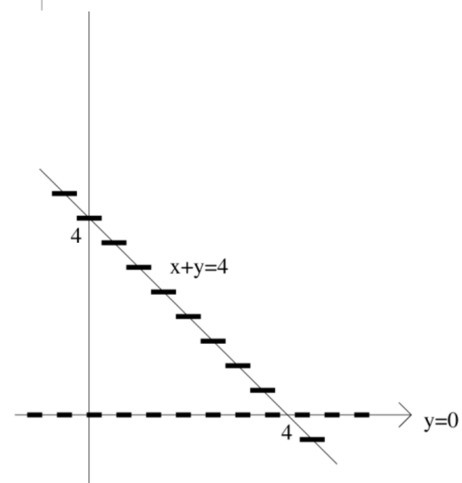
For given initial values x_0 and y_0 this system has a unique solution (a rigorous proof of this fact is beyond the scope of this course), but it turns out that there is no closed formula for this solution. Still, we can gain a good understanding of the evolution of this system and its long-term behavior by taking a qualitative graphical approach. We can write the system as:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(6-2x-y) \\ y(4-x-y) \end{bmatrix},$$

that is, the solutions are the flow lines of the vector field:

$$\begin{bmatrix} x(6-2x-y) \\ y(4-x-y) \end{bmatrix}.$$

We could use a computer to generate this vector field (or a corresponding direction field), but it turns out that even without the aid of a computer it is not hard to analyze the long-term behavior of the system.



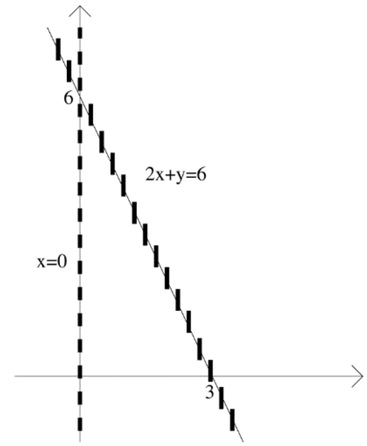
To facilitate this discussion, let us write $f(x,y) = x(6-2x-y)$ and $g(x,y) = y(4-x-y)$. Our task is to draw a rough sketch of the vector field

$$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} x(6-2x-y) \\ y(4-x-y) \end{bmatrix}$$

We might start by finding the horizontal and vertical vectors in the field (when $g(x,y)=0$ or $f(x,y)=0$, respectively).

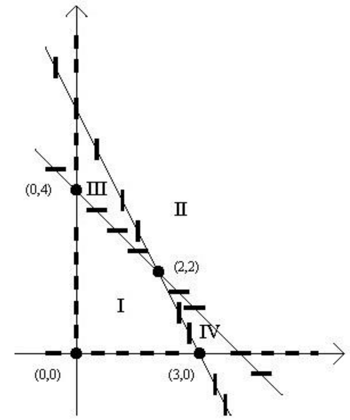
Now $g(x,y) = y(4-x-y) = 0$ when $y=0$ or $4-x-y=0$, that is, when $y=0$ or $x+y=4$.

The horizontal line segments indicate that the vectors of the field are horizontal there; at this point, we don't worry about the direction.



Next, we find out where the vectors are vertical, that is, where $f(x,y) = x(6-2x-y) = 0$.

If we draw the last two figures on the same axes, then we can see the four points where both $f(x,y)=0$ and $g(x,y)=0$.



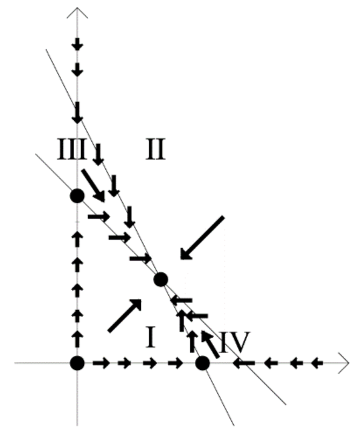
These are the **equilibrium solutions** of the system: If the system is initially in one of these states, then it will remain unchanged (since $\frac{dx}{dt}=0$ and $\frac{dy}{dt}=0$).

To find the equilibrium solution in the first quadrant, we solve the system

$$\begin{Bmatrix} 2x+y=6 \\ x+y=4 \end{Bmatrix}.$$

The equilibria are $(2,2)$, $(0,4)$, $(3,0)$, and $(0,0)$.

The curves where $f(x,y)=0$ and $g(x,y)=0$ are sometimes called the *nullclines* of the system. What happens in the four regions enclosed by the nullclines, labeled (I) to (IV) in the diagram? Since neither $f(x,y)$ nor $g(x,y)$ will ever be zero inside one of these regions, the signs of $f(x,y)$ and $g(x,y)$ will remain unchanged throughout a given region (since the functions $f(x,y)=x(6-2x-y)$ and $g(x,y)=y(4-x-y)$ are continuous). All we need to do is determine these signs at one sample point in each region.



We can represent our work in a table:

Region	Sample Point	Sign of $f(x,y)$	Sign of $g(x,y)$	Vector $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$
I	$(1,1)$	+	+	up and right
II	$(3,3)$	–	–	down and left
III	$(0.1,4)$	+	–	down and right
IV	$(3,0.1)$	–	+	up and left

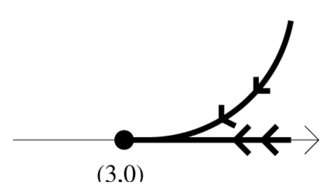
Now we can also fill in the direction of the vectors on the nullclines: it has to be compatible with the directions in the adjacent regions.

What does this analysis tell us about the long-term behavior of this system? Let us consider various scenarios:

If the point (x_0, y_0) representing the initial populations is located in region III, then the trajectory will move to the right and down, and it cannot "escape" from region III since the vectors along the boundaries point "the other way." The trajectory will approach the equilibrium $(2,2)$.

A similar reasoning shows that a trajectory starting in region IV will approach the equilibrium $(2,2)$.

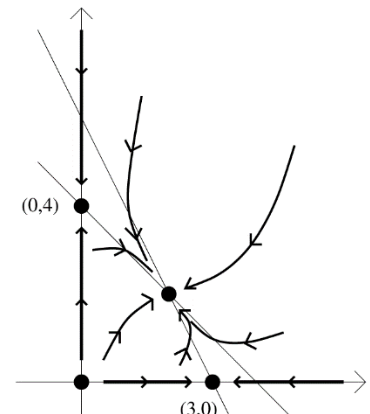
A trajectory starting in region I has three "options": It can approach the equilibrium point $(2,2)$ while remaining in region I at all times, or it can "cross over" into regions III or IV. The final outcome will always be the same: the trajectory will approach $(2,2)$.



Trajectories cannot merge.

A trajectory starting in region II has the three options just discussed (for region I), but besides that it may seem conceivable that a trajectory could "merge" with the *x*-axis or the *y*-axis, approaching the equilibrium $(3,0)$ and $(0,4)$, respectively. Note, however, that there is already a (straight-line) trajectory approaching $(3,0)$ from the right. But trajectories cannot merge since the trajectory for a given initial value is unique, for positive and negative *t* (think about it!).

Let us summarize: as long as there are some animals from each species present initially (that is, x_0 and y_0 are both positive), then the system will eventually approach the equilibrium state $(2,2)$. If $x_0 \neq 0$ and $y_0 = 0$, then the system will approach $(3,0)$; if $x_0 = 0$ and $y_0 \neq 0$, then it will approach $(0,4)$.

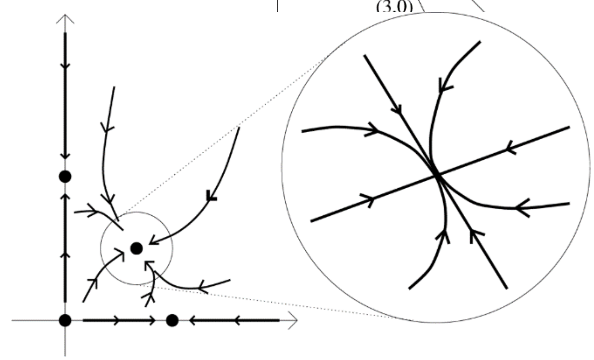


To the right we sketch a phase portrait for this system, for the first quadrant:

We say that $(2,2)$ is a **stable equilibrium**, meaning that all trajectories starting near $(2,2)$ will approach $(2,2)$ as *t* goes to infinity (more precisely: there is a disc centered at $(2,2)$ such that all trajectories with initial value within this disc will approach $(2,2)$ as *t* goes to infinity).

Linearization

In applications one is often interested in the behavior of a dynamical system near an equilibrium state. If we zoom in on the phase portrait above near the equilibrium point $(2,2)$, we see a picture that looks a lot like one of the phase portraits we found when we studied linear systems (the case of two negative eigenvalues).



To study the behavior of a nonlinear dynamical system near an equilibrium point, we can ***linearize*** the system. We will first explain this approach in general and then return to the example discussed above. Consider a system

$$\begin{Bmatrix} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{Bmatrix}$$

with an equilibrium solution (a,b), that is, $f(a,b) = g(a,b) = 0$. In multivariable calculus, you learned that the linear approximation of a function $f(x,y)$ near a point (a,b) is given by

$$f(x,y) \cong f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b).$$

To understand this formula, note that the rate of change of *f* in the *x*-direction near the point (a,b) is approximately $\dfrac{\partial f}{\partial x}(a,b)$, so that

$$f(x,b) \cong f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a).$$

Likewise, the rate of change of *f* in the *y*-direction near (a,b) is approximately $\dfrac{\partial f}{\partial y}(a,b)$, so that

$$f(x,y) \cong f(x,b) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b) \cong f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b)$$

To linearize the system $\left\{\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned}\right\}$ near an equilibrium point (a,b) means to replace the functions $f(x,y)$ and $g(x,y)$ by their linear approximations. Keeping in mind that $f(a,b)=0$ and $g(a,b)=0$, this approximation is

$$\left\{\begin{aligned}\frac{dx}{dt} &= \frac{\partial f}{\partial x}(a,b)\cdot(x-a)+\frac{\partial f}{\partial y}(a,b)\cdot(y-b)\\ \frac{dy}{dt} &= \frac{\partial g}{\partial x}(a,b)\cdot(x-a)+\frac{\partial g}{\partial y}(a,b)\cdot(y-b)\end{aligned}\right\} \quad \text{or} \quad \begin{bmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{bmatrix}=\begin{bmatrix}\frac{\partial f}{\partial x}(a,b) & \frac{\partial f}{\partial y}(a,b)\\ \frac{\partial g}{\partial x}(a,b) & \frac{\partial g}{\partial y}(a,b)\end{bmatrix}\begin{bmatrix}x-a\\ y-b\end{bmatrix}$$

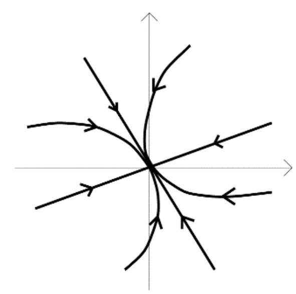
We can use the substitution $u=x-a$ and $v=y-b$ to simplify further:

$$\begin{bmatrix}\frac{du}{dt}\\ \frac{dv}{dt}\end{bmatrix}=\underbrace{\begin{bmatrix}\frac{\partial f}{\partial x}(a,b) & \frac{\partial f}{\partial y}(a,b)\\ \frac{\partial g}{\partial x}(a,b) & \frac{\partial g}{\partial y}(a,b)\end{bmatrix}}_{\mathbf{J}}\begin{bmatrix}u\\ v\end{bmatrix}$$

The matrix **J** is called the <u>Jacobian matrix</u> of the system at the point (a,b). Consider the example discussed above, where

$$\begin{aligned}
&f(x,y)=6x-2x^2-xy \qquad && \frac{\partial f}{\partial x}=6-4x-y \qquad && \frac{\partial f}{\partial y}=-x\\
&g(x,y)=4y-xy-y^2 && \frac{\partial g}{\partial x}=-y && \frac{\partial g}{\partial y}=4-x-2y
\end{aligned}$$

at the point $(2,2)$, so that



$$\mathbf{J}(2,2)=\begin{bmatrix}\frac{\partial f}{\partial x}(2,2) & \frac{\partial f}{\partial y}(2,2)\\ \frac{\partial g}{\partial x}(2,2) & \frac{\partial g}{\partial y}(2,2)\end{bmatrix}=\begin{bmatrix}-4 & -2\\ -2 & -2\end{bmatrix}.$$

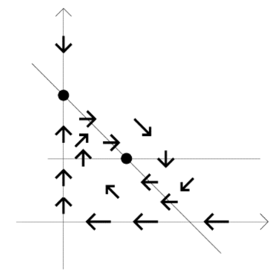
We find the eigenvalues $\lambda_{1,2}=-3\pm\sqrt{5}$ with associated eigenspaces

$$E_{-3+\sqrt{5}}=\operatorname{span}\begin{bmatrix}2\\ -1-\sqrt{5}\end{bmatrix} \quad \text{and} \quad E_{-3-\sqrt{5}}=\operatorname{span}\begin{bmatrix}2\\ -1+\sqrt{5}\end{bmatrix}.$$

Note that the phase portrait of the linearized system looks a lot like the phase portrait of the original system near the equilibrium point; in this introductory course we cannot make this relationship precise. Let us just state some important facts, without proof. Let **J** be the matrix of the linearized system. Then:

- If both eigenvalues of **J** have a negative real part, then (a,b) is a stable equilibrium of the original system.
- If **J** has at least one eigenvalue with a positive real part, then (a,b) is not a stable equilibrium of the original system.

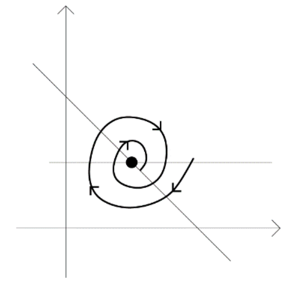
Example: Consider the system $\begin{Bmatrix} \dfrac{dx}{dt} = x(y-1) \\ \dfrac{dy}{dt} = y(2-x-y) \end{Bmatrix}$. Note that $(1,1)$ is an equilibrium solution of this system.



Is this equilibrium stable?

Answer: The phase plane analysis is inconclusive in this case. We cannot tell whether the trajectories spiral inwards, spiral outwards, or are closed.

Alternatively, we can linearize near $(1,1)$.



A routine computation shows that $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ with eigenvalues $\lambda_{1,2} = \dfrac{-1 \pm i\sqrt{3}}{2}$.

It follows that $(1,1)$ is a stable equilibrium; the trajectories starting near that point spiral inward, approaching $(1,1)$.

Summary

In this section we discuss two methods that help us analyze a system of the form

$$\begin{Bmatrix} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{Bmatrix}$$

Phase Plane

The trajectories of the system are the flow lines of the vector field

$$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}.$$

- To get a sense for this vector field, we first sketch the nullclines $f(x,y)=0$ (where the vectors are vertical) and $g(x,y)=0$ (where the vectors are horizontal).
- Next we identify the equilibria, where $f(x,y)=0$ and $g(x,y)=0$.
- Then we can use the sample points in the regions between nullclines to determine the direction of the vectors.
- Use the rough vector field drawn in the previous three steps to draw some representative trajectories, and predict the long-term behavior for the various initial values, if possible.

Linearization

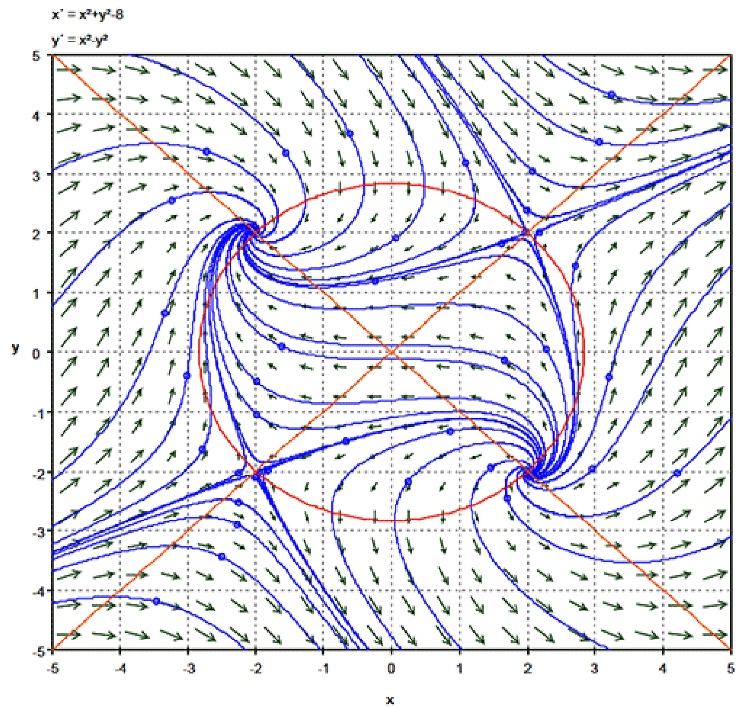
Suppose (a,b) is an equilibrium of the system, that is, $f(a,b)=0$ and $g(a,b)=0$. Replacing the functions $f(x,y)$ and $g(x,y)$ by their linear approximations near (a,b), we obtain the linearized system

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial f}{\partial x}(a,b) & \dfrac{\partial f}{\partial y}(a,b) \\ \dfrac{\partial g}{\partial x}(a,b) & \dfrac{\partial g}{\partial y}(a,b) \end{bmatrix}}_{\mathbf{J}} \begin{bmatrix} u \\ v \end{bmatrix}$$

where $u = x - a$ and $v = y - b$. Then the phase portrait of this linearized system "looks a lot like" the phase portrait of the original system near (a,b). In particular, if the real parts of both eigenvalues of $\mathbf{J}$ are negative, then (a,b) is a stable equilibrium of the original system. If the real part of at least one eigenvalue of $\mathbf{J}$ is positive, then (a,b) isn't a stable equilibrium of the original system. The matrix $\mathbf{J}$ is called the Jacobian matrix of the system at the point (a,b).

Example: Use phase plane analysis to describe the trajectories/flow of the system

$$\begin{cases} \frac{dx}{dt} = f(x,y) = x^2 + y^2 - 8 \\ \frac{dy}{dt} = g(x,y) = x^2 - y^2 \end{cases}.$$



Start by finding and drawing the nullclines.

Horizontal Nullclines (HNC) are curves along which the vector field is purely horizontal, i.e. its vertical component is zero. These occur where $g(x,y) = x^2 - y^2 = (x-y)(x+y) = 0$, i.e. the lines $y = x$ and $y = -x$.

Vertical Nullclines (VNC) are curves along which the vector field is purely vertical, i.e. its horizontal component is zero. These occur where $f(x,y) = x^2 + y^2 - 8 = 0$, i.e. the circle $x^2 + y^2 = 8$.

This enables us to identify and solve for the equilibria which occur at the intersection of horizontal and vertical nullclines. We next mark these nullclines with, respectively, horizontal dashes and vertical dashes taking care not to "crowd the equilibrium." We delay closer examination of the equilibria until we do the Jacobian analysis, i.e. linearization around each equilibrium. The four equilibria are at $(2,2)$, $(2,-2)$, $(-2,2)$, and $(-2,-2)$.

Next, we pick points on the nullclines and determine the direction of the vector field along each segment of the nullclines. The then use "interpolation" to "continuity" to draw a good sampler of vectors in each sector bounded by nullclines. This enables us to (in pencil!) speculate about the general flow everywhere.

Finally, we analyze each of the equilibria using the Jacobian matrix: With $\mathbf{F}(x,y) = \langle f(x,y), g(x,y) \rangle$, the Jacobian at any point is $\mathbf{J_F} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$.

In particular:

$\mathbf{J_F}(2,2) = \begin{bmatrix} 4 & 4 \\ 4 & -4 \end{bmatrix}$ yields the characteristic polynomial $p(\lambda) = \lambda^2 - 32 = 0$ with eigenvalues $\lambda = \pm 4\sqrt{2}$. This translates into one growth direction and one decay direction (they're actually perpendicular because the matrix is symmetric, but that's a result from the Spectral Theorem in Linear Algebra). This agrees with what we speculated from the initial sketch of trajectories.

$\mathbf{J_F}(2,-2) = \begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix}$ yields the characteristic polynomial $p(\lambda) = (\lambda - 4)^2 + 16 = 0$ with eigenvalues $\lambda = 4 \pm 4i$.

The fact that the eigenvalues are complex indicates rotation, and the fact that the real part is positive indicates growth, i.e. outward (unstable) spirals. The vector field shows these to be counterclockwise outward spirals, and this agrees with what we speculated from the initial sketch of trajectories.

$\mathbf{J_F}(-2,+2)=\begin{bmatrix}-4 & 4\\ -4 & -4\end{bmatrix}$ yields the characteristic polynomial $p(\lambda)=(\lambda+4)^2+16=0$ with eigenvalues $\lambda=-4\pm 4i$. The fact that the eigenvalues are complex indicates rotation, and the fact that the real part is negative indicates decay, i.e. inward (stable) spirals. The vector field shows these to be clockwise inward spirals, and this agrees with what we speculated from the initial sketch of trajectories.

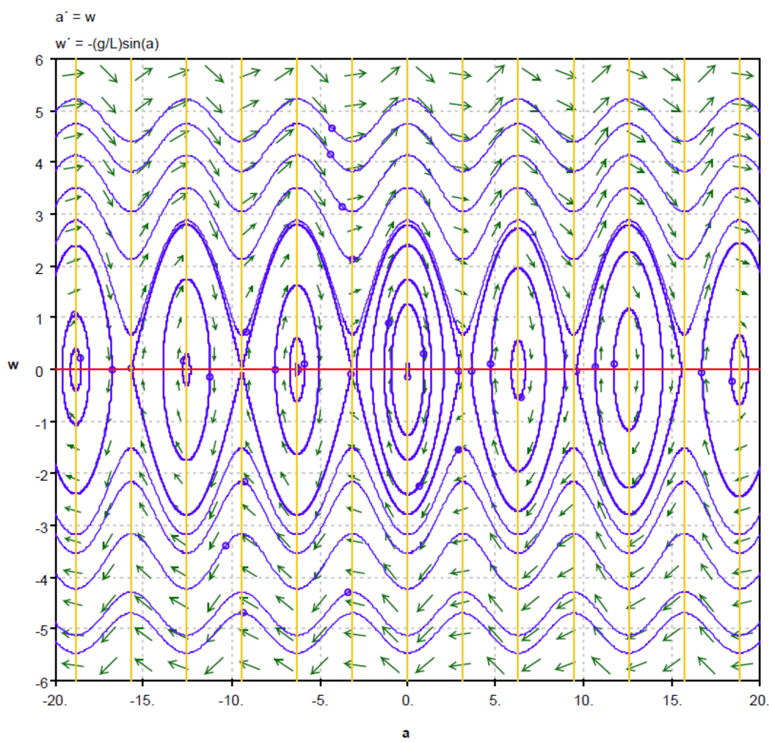
$\mathbf{J_F}(-2,-2)=\begin{bmatrix}-4 & -4\\ -4 & 4\end{bmatrix}$ yields the characteristic polynomial $p(\lambda)=\lambda^2-32=0$ with eigenvalues $\lambda=\pm 4\sqrt{2}$. This translates into one growth direction and one decay direction. This agrees with what we speculated from the initial sketch of trajectories.

Pendulum Example: The dynamics of a frictionless pendulum of length L are given by the system

$$\begin{Bmatrix}\dfrac{d\alpha}{dt}=\omega\\ \dfrac{d\omega}{dt}=-\dfrac{g}{L}\sin\alpha\end{Bmatrix}$$

where α is the angle the rod of the pendulum makes with the vertical line, $\omega=\dfrac{d\alpha}{dt}$ is the angular velocity, and g is the gravitational constant.

The vertical nullcline will be $\omega=0$ and there will be horizontal nullclines whenever $\sin\alpha=0$, i.e. when $\alpha=n\pi$ for all integers n. This gives equilibria at $(\alpha,\omega)=(n\pi,0)$. The Jacobian matrix at these equilibria will be $J_F(n\pi,0)=\begin{bmatrix}0 & 1\\ (-1)^{n+1}\frac{g}{L} & 0\end{bmatrix}$, so when n is an even integer, it's $\begin{bmatrix}0 & 1\\ -\frac{g}{L} & 0\end{bmatrix}$ which has complex eigenvalues with real part 0. This corresponds to periodic (clockwise) orbits in the vicinity of those equilibria. [Technically, all we can say is that the linear approximation would be periodic, but the actual orbits also have this property.] When n is an odd integer, the Jacobian matrix at those equilibria is $\begin{bmatrix}0 & 1\\ \frac{g}{L} & 0\end{bmatrix}$ which has one positive (real) eigenvalue and one negative (real) eigenvalue. This corresponds to hyperbola-like orbits in the vicinity of those equilibria.

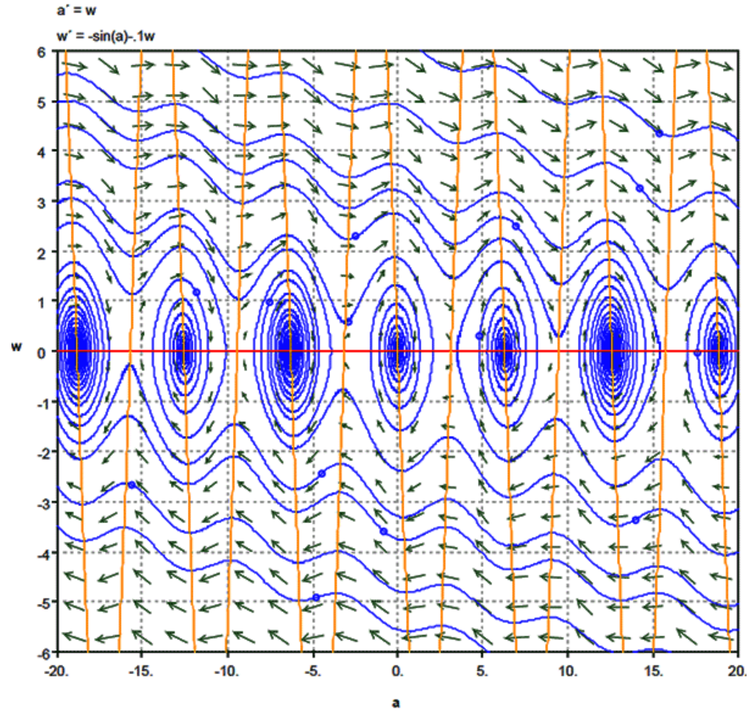


The phase portrait shows these alternating types of equilibria. The period orbits correspond to the pendulum having low energy and oscillating back and forth. The other orbits (except for the borderline cases) correspond to the pendulum having high energy and swinging over the top repeatedly without oscillation.

We can modify this slightly to account for friction.

This results in the system $\left\{ \begin{aligned} \frac{d\alpha}{dt} &= \omega \\ \frac{d\omega}{dt} &= -\frac{g}{L}\sin\alpha - c\omega \end{aligned} \right\}$

for some positive scalar c. If, for example, we choose $c = 0.1$ and do the same phase plane analysis, we'll have the same equilibria, but the trajectories will be fundamentally different – indicative of the pendulum dying down over time.



Higher dimensional Illustrations

One thing that makes two-variable nonlinear systems relatively easy to analyze is that trajectories cannot intersect at a nonzero angle, i.e. they cannot cross. This effectively "confines" trajectories and allows us to make qualitative predictions relatively easily. In higher dimensions, there is no such confinement and the possibilities are far more interesting. Even relatively simple systems can exhibit such things as a mix of periodic trajectories and aperiodic trajectories and very sensitive dependence on initial conditions, i.e. "chaos". This is also known as the "butterfly effect," i.e. the possibility that in a weather model the perturbation of initial conditions associated with a butterfly flapping its wings somewhere could well result in a catastrophic weather event after some time has passed. This observation has far-reaching consequences. In particular, if very simple models exhibit such sensitive dependence on initial conditions, then longer-term predictability in some systems may be an impossibility. No matter how accurate the model is, it may simply be impossible to apply it for longer than a relatively short period of time.

One famous example – arguably the first – was produced somewhat accidentally by Edward Lorenz while analyzing fluid flow in the context of meteorology. A simplified version involving only three variables was rather world-shaking. This example showed the existence of the "***Lorenz attractor***" – a subset in 3-dimensions toward which nearby trajectories converged but within which trajectories exhibited seemingly endless variation even though the system was completely deterministic. [Ref: https://en.wikipedia.org/wiki/Lorenz_system]

Another model of some relevance is one proposed in a 1927 paper by W.O. Kermack and A.G. McKendrick entitled "Mathematical Theory of Epidemics". After some considerable analysis they produced a simple nonlinear model in three variables involving several parameters. The system is:

$$\left\{ \begin{aligned} \frac{dx}{dt} &= -\kappa xy \\ \frac{dy}{dt} &= \kappa xy - ly \\ \frac{dz}{dt} &= ly \end{aligned} \right\} \text{ with constraint } x + y + z = N .$$

The authors go on to show that their model quite accurately matched the mortality data from the plague in the island of Bombay over the period from December 1905 to July 1906. In contrast with the Lorenz attractor, this epidemiological model yields stable predictions.
[Ref: http://math.rwinters.com/E21c/KermackMcKendrick1927-epidemics.pdf]

Notes by Robert Winters