Math E-21c – Ordinary Differential Equations – Lecture #13

Repeated eigenvalues (with geometric multiplicity less than the algebraic multiplicity)

Suppose we want to solve a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$ where $\mathbf{A}$ is a non-diagonalizable 2×2 real matrix with a repeated eigenvalue λ. In this case, we can always find a change of basis matrix $\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. Why?

Generalized eigenvectors

When the algebraic multiplicity k of an eigenvalue λ of $\mathbf{A}$ is greater than 1, we will usually not be able to find k linearly independent eigenvectors corresponding to this eigenvalue. This is the case where the ***geometric multiplicity*** (the number of linearly independent eigenvectors corresponding to this eigenvalue) is strictly less than the ***algebraic multiplicity*** of this eigenvalue. The next best thing to an eigenvector is often referred to as a "generalized eigenvector".

If, for example, a matrix $\mathbf{A}$ had λ as an eigenvalue with algebraic multiplicity 2, but the geometric multiplicity was 1, we could certainly find an actual eigenvector $\mathbf{v}_1$ such that $\mathbf{Av}_1 = \lambda\mathbf{v}_1$, but we would not be able to produce a 2nd linearly independent eigenvector. However, it can be shown (and we'll demonstrate this in an example) that we will always be able to find a vector $\mathbf{v}_2$ such that $\mathbf{Av}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2$. Another way of stating this is that $(\lambda\mathbf{I}-\mathbf{A})\mathbf{v}_1 = \mathbf{0}$ and $(\lambda\mathbf{I}-\mathbf{A})\mathbf{v}_2 = -\mathbf{v}_1$ (or $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$), so $(\lambda\mathbf{I}-\mathbf{A})^2\mathbf{v}_2 = -(\lambda\mathbf{I}-\mathbf{A})\mathbf{v}_1 = \mathbf{0}$. If an eigenvector is a vector in $\ker(\lambda\mathbf{I}-\mathbf{A})$, then a generalized eigenvector would be in $\ker(\lambda\mathbf{I}-\mathbf{A})^2$.

In the case where the algebraic multiplicity was 3 and the geometric multiplicity was only 1, we'd also seek a vector in $\ker(\lambda\mathbf{I}-\mathbf{A})^3$, namely a vector $\mathbf{v}_3$ such that $\mathbf{Av}_3 = \mathbf{v}_2 + \lambda\mathbf{v}_3$. The idea is that a generalized eigenvector is a vector such that the transformation acts on it by scaling together with a shift by the previously found vector.

It can be shown that this process will <u>always</u> yield k linearly independent vectors corresponding to the eigenvalue λ, the first few vectors of which will be actual eigenvectors of $\mathbf{A}$. If a matrix $\mathbf{A}$ has all real eigenvalues and if we carry out this process for all eigenvalues of $\mathbf{A}$, we'll produce a complete basis $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ where we assume that all vectors corresponding to a given eigenvalue are grouped together and ordered in the way in which they were found.

As in the previous two cases, $\mathbf{A} = \mathbf{SBS}^{-1} \quad\Rightarrow\quad [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$ and it comes down to finding $[e^{t\mathbf{B}}]$. This is perhaps most easily done by explicitly solving the corresponding differential equations. In the new coordinates, this system translates into $\left\{\begin{array}{l} \frac{du_1}{dt} = \lambda u_1 + u_2 \\ \frac{du_2}{dt} = \lambda u_2 \end{array}\right\}$. The second equation is easily solved to get $u_2(t) = e^{\lambda t}u_2(0)$. We can guess a solution for the first equation of the form $u_1(t) = c_1 te^{\lambda t} + c_2 e^{\lambda t}$. Differentiating this and substituting into the first equation, we get $c_1(e^{\lambda t} + \lambda te^{\lambda t}) + c_2\lambda e^{\lambda t} = \lambda(c_1 te^{\lambda t} + c_2 e^{\lambda t}) + e^{\lambda t}u_2(0)$. Comparing like terms, we conclude that $c_1 = u_2(0)$. Substituting $t = 0$, we further conclude that $u_1(0) = c_2$. Putting these results together, we get $u_1(t) = u_2(0)te^{\lambda t} + u_1(0)e^{\lambda t} = e^{\lambda t}u_1(0) + te^{\lambda t}u_2(0)$. We therefore have that

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda t}u_1(0) + te^{\lambda t}u_2(0) \\ e^{\lambda t}u_2(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\mathbf{u}(0)$$

So, $[e^{t\mathbf{B}}]=\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t}\end{bmatrix}$ in this case and the solution is given by $\mathbf{x}(t)=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}=\mathbf{S}\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t}\end{bmatrix}\mathbf{S}^{-1}\mathbf{x}(0)$.

An alternate method of deriving this result was in Problem Set #12:

If $\mathbf{B}=\begin{bmatrix} \lambda & 1 \\ 0 & \lambda\end{bmatrix}$ (there are analogous forms in cases larger than 2×2 matrices), we write $\mathbf{B}=\begin{bmatrix} \lambda & 1 \\ 0 & \lambda\end{bmatrix}=\lambda\mathbf{I}+\mathbf{P}$ where $\mathbf{P}=\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}$. There is a simple relationship between the solutions of the systems $\frac{d\mathbf{x}}{dt}=\mathbf{Bx}$ and $\frac{d\mathbf{u}}{dt}=\mathbf{Pu}$, namely $\mathbf{x}(t)=e^{\lambda t}\mathbf{u}(t)$. This is easily seen by differentiation:

$$\tfrac{d\mathbf{x}}{dt}=\tfrac{d}{dt}[e^{\lambda t}\mathbf{u}(t)]=e^{\lambda t}\tfrac{d\mathbf{u}}{dt}+\lambda e^{\lambda t}\mathbf{u}=e^{\lambda t}\mathbf{Pu}+\lambda e^{\lambda t}\mathbf{u}=e^{\lambda t}(\mathbf{Pu}+\lambda\mathbf{Iu})=e^{\lambda t}(\lambda\mathbf{I}+\mathbf{P})\mathbf{u}=(\lambda\mathbf{I}+\mathbf{P})e^{\lambda t}\mathbf{u}=\mathbf{Bx}$$

together with the fact that $\mathbf{x}(0)=\mathbf{u}(0)$.

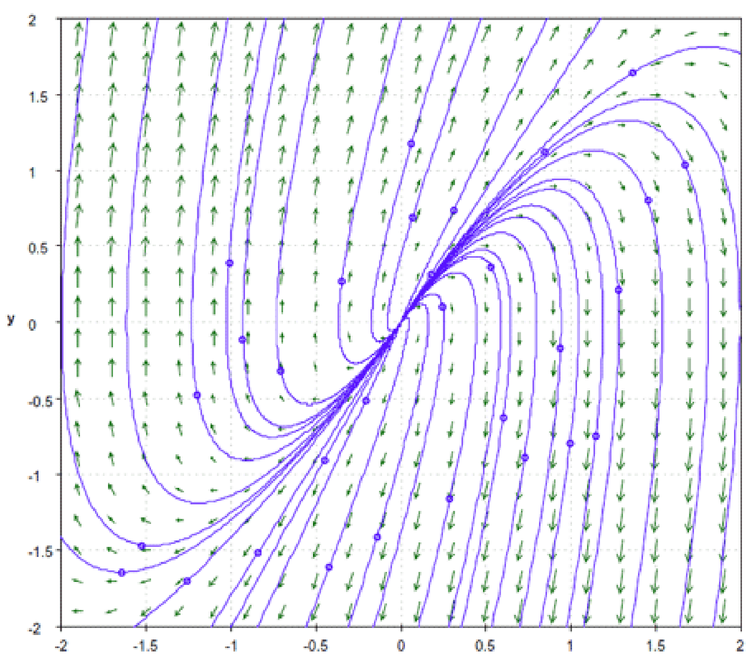
Furthermore, solving $\frac{d\mathbf{u}}{dt}=\mathbf{Pu}$ is simple: If $\mathbf{u}=\begin{bmatrix} u_1 \\ u_2\end{bmatrix}$, then with $\mathbf{P}=\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}$ we have $\begin{Bmatrix} u_1'(t)=u_2 \\ u_2'(t)=0\end{Bmatrix}$.

The second equation gives that $u_2(t)=c_2=u_2(0)$, a constant. The first equation is then $u_1'(t)=u_2(0)$, so $u_1(t)=u_2(0)\cdot t+c_1$. At $t=0$ this gives $u_1(0)=c_1$, so $u_1(t)=u_1(0)+u_2(0)\cdot t$. Together this gives:

$$\mathbf{u}(t)=\begin{bmatrix} u_1(t) \\ u_2(t)\end{bmatrix}=\begin{bmatrix} u_1(0)+u_2(0)\cdot t \\ u_2(0)\end{bmatrix}=\begin{bmatrix} 1 & t \\ 0 & 1\end{bmatrix}\begin{bmatrix} u_1(0) \\ u_2(0)\end{bmatrix}=\begin{bmatrix} 1 & t \\ 0 & 1\end{bmatrix}\mathbf{u}(0)=\left[e^{t\mathbf{P}}\right]\mathbf{u}(0)$$

Therefore $\mathbf{x}(t)=e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1\end{bmatrix}\mathbf{x}(0)=\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t}\end{bmatrix}\mathbf{x}(0)$, so $\left[e^{t\mathbf{B}}\right]=e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1\end{bmatrix}=\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t}\end{bmatrix}$ for $\mathbf{B}=\begin{bmatrix} \lambda & 1 \\ 0 & \lambda\end{bmatrix}$.

Problem: Solve the system $\begin{Bmatrix} \frac{dx}{dt}= & y \\ \frac{dy}{dt}= & -4x+4y\end{Bmatrix}$ with initial conditions $x(0)=3,\ y(0)=2$.



Solution: In matrix form, we have $\frac{d\mathbf{x}}{dt}=\mathbf{Ax}$ where $\mathbf{A}=\begin{bmatrix} 0 & 1 \\ -4 & 4\end{bmatrix}$ and $\mathbf{x}(0)=\begin{bmatrix} 3 \\ 2\end{bmatrix}$. We again start by finding the eigenvalues of the matrix: $\lambda\mathbf{I}-\mathbf{A}=\begin{bmatrix} \lambda & -1 \\ 4 & \lambda-4\end{bmatrix}$, and the characteristic polynomial is $p_{\mathbf{A}}(\lambda)=\lambda^2-4\lambda+4=(\lambda-2)^2$.

This gives the repeated eigenvalue $\lambda=2$ with (algebraic) multiplicity 2. We seek eigenvectors: $\begin{bmatrix} 2 & -1 \\ 4 & -2\end{bmatrix}\begin{bmatrix} \alpha \\ \beta\end{bmatrix}=\begin{bmatrix} 0 \\ 0\end{bmatrix}$ gives the (redundant) equations $2\alpha-\beta=0$ and $4\alpha-2\beta=0$. Therefore $\beta=2\alpha$, so we can choose $\mathbf{v}_1=\begin{bmatrix} 1 \\ 2\end{bmatrix}$ or any scalar multiple of this as an eigenvector, but we are <u>unable to find a second linearly independent eigenvector</u>. (We say that the geometric multiplicity of the $\lambda=2$ eigenvalue is 1.)

The standard procedure in this case is to seek a ***generalized eigenvector*** for this repeated eigenvalue, i.e. a vector $\mathbf{v}_2$ such that $(\lambda\mathbf{I}-\mathbf{A})\mathbf{v}_2$ is not zero, but rather a multiple of the eigenvector $\mathbf{v}_1$. Specifically, we seek a vector such that $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2$. This translates into seeking $\mathbf{v}_2$ such that $(\lambda\mathbf{I}-\mathbf{A})\mathbf{v}_2 = -\mathbf{v}_1$. That is, $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. This gives redundant equations the first of which is $2\alpha - \beta = -1$ or $\beta = 2\alpha + 1$.

If we (arbitrarily) choose $\alpha = 0$, then $\beta = 1$, so $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The fact that $\left\{ \begin{matrix} \mathbf{A}\mathbf{v}_1 = 2\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + 2\mathbf{v}_2 \end{matrix} \right\}$ tells us that with the change of basis matrix $\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, we will have $[\mathbf{A}]_{\mathscr{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \mathbf{B}$.

If we apply the previously outline procedure, we get $[e^{t\mathbf{B}}] = e^{2t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$. The solution to the system is therefore $\mathbf{x}(t) = [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0) = e^{2t}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = e^{2t}\begin{bmatrix} 1 & t \\ 2 & 2t+1 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = e^{2t}\begin{bmatrix} 3-4t \\ 2-8t \end{bmatrix}$

That is, $\left\{ \begin{matrix} x(t) = e^{2t}(3-4t) \\ y(t) = e^{2t}(2-8t) \end{matrix} \right\}$. It's worth noting that this can also be expressed as $\mathbf{x}(t) = e^{2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} - 4te^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The phase portrait in this case has just one invariant (eigenvector) direction. It gives an unstable **node** which can be viewed as a degenerate case of a (clockwise) outward spiral that cannot get past the eigenvector direction.

Similar calculations enable us to deal with cases such as a repeated eigenvalue where the geometric multiplicity is 1 and the algebraic multiplicity is 3 (or even worse).

Finally, an actual system may exhibit several of these qualities – one or more complex pairs of eigenvalues, repeated eigenvalues, and distinct real eigenvalues. The Jordan Canonical Form of the matrix for such a system can be analyzed block by block and each of the above solutions applied within each block to determine the evolution matrix for the entire system.

<u>Summarizing the Main Idea</u>:

Given a system of 1st order linear differential equations $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ with initial conditions $\mathbf{x}(0)$, we use eigenvalue-eigenvector analysis to find an appropriate basis $\mathscr{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ for $\mathbf{R}^n$ and a change of basis matrix $\mathbf{S} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}$ such that in coordinates relative to this basis ($\mathbf{u} = \mathbf{S}^{-1}\mathbf{x}$) the system is in a standard form with a known solution. Specifically, we find a standard matrix $\mathbf{B} = [\mathbf{A}]_{\mathscr{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, transform the system into $\frac{d\mathbf{u}}{dt} = \mathbf{B}\mathbf{u}$, solve it as $\mathbf{u}(t) = [e^{t\mathbf{B}}]\mathbf{u}(0)$ where $[e^{t\mathbf{B}}]$ is the ***evolution matrix*** for **B**, then transform back to the original coordinates to get $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$ where $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$ is the ***evolution matrix*** for **B**. That is $\mathbf{x}(t) = [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. This is easier to do than it is to explain, so here are a few illustrative examples:

Moral of the Story: It's always possible to find a special basis relative to which a given linear system is in its simplest possible form. The new basis provides a way to decompose the given problem into several simple, standard problems which can be easily solved. Any complication in the algebraic expressions for the solution is the result of changing back to the original coordinates.

The standard 2×2 **cases are**:

Diagonalizable with eigenvalues λ_1, λ_2: $\mathbf{B}=\mathbf{D}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$ $[e^{t\mathbf{B}}]=[e^{t\mathbf{D}}]=\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}$

Complex pair of eigenvalues $\lambda = a\pm ib$: $\mathbf{B}=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$ $[e^{t\mathbf{B}}]=e^{at}\begin{bmatrix}\cos bt & -\sin bt\\ \sin bt & \cos bt\end{bmatrix}$

Repeated eigenvalue λ **with** $\text{GM}<\text{AM}$: $\mathbf{B}=\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}$ $[e^{t\mathbf{B}}]=\begin{bmatrix}e^{\lambda t} & te^{\lambda t}\\ 0 & e^{\lambda t}\end{bmatrix}=e^{\lambda t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$

In general, you should expect to encounter systems more complicated than these 2×2 examples.

Fundamental Matrices

In the case where initial conditions are not specified, it is often simpler to express solutions not in terms of the **evolution matrix** but rather in terms of the corresponding **fundamental matrix**. Specifically, when we standardize a system of the form $\frac{d\mathbf{x}}{dt}=\mathbf{Ax}$ in terms of a matrix B via the $\mathbf{S}^{-1}\mathbf{AS}=\mathbf{B}$ protocol, then $\mathbf{A}=\mathbf{SBS}^{-1}$, $[e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$, and $\mathbf{x}(t)=[e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. However, if the initial conditions are not specified, we can simply express $\mathbf{S}^{-1}\mathbf{x}(0)=\mathbf{c}$ where $\mathbf{c}=\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}$ is a constant vector. Then $\mathbf{x}(t)=[e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{c}=\mathbf{M}(t)\mathbf{c}$ where $\mathbf{M}(t)=\mathbf{S}[e^{t\mathbf{B}}]$ is a time-varying matrix known as the ***fundamental matrix*** for this system.

Nonlinear systems

Though we have spent considerable time on systems that can be expressed as $\frac{d\mathbf{x}}{dt}=\mathbf{Ax}$ where **A** is an $n\times n$ matrix (and especially the case of a 2×2 matrix), the reality is that most 1st order systems of ODEs are not of this form. That is, the functions on the right hand sides for the systems are not linear functions. In fact, the system may not even be autonomous. It's still possible to understand these more complicated systems, but the analysis is not nearly as straightforward – even in the case of autonomous (time independent) systems. We'll take up some of these methods in the next lecture, but there's at least one nonlinear case that we can handle by relating it directly to the linear case.

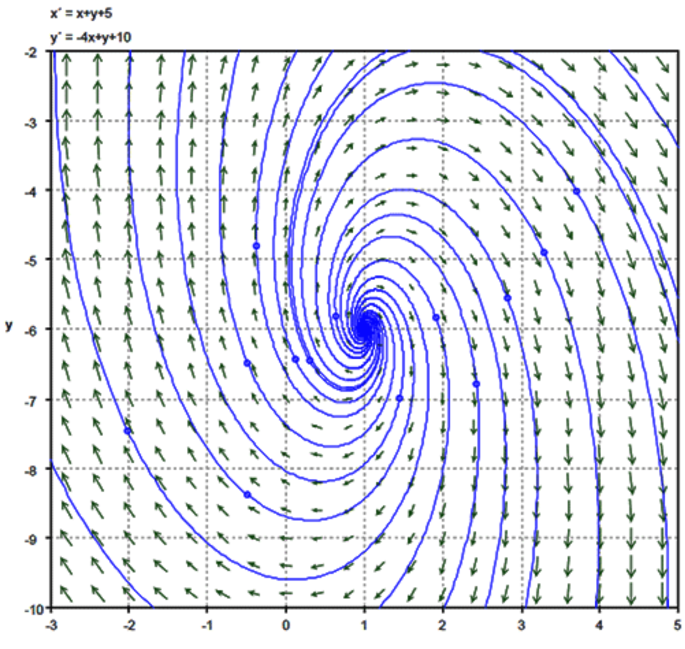
Problem: Solve $\left\{\begin{aligned}\frac{dx}{dt}&=x+y+5\\ \frac{dy}{dt}&=-4x+y+10\end{aligned}\right\}$ with initial conditions $x(0)=2,\ y(0)=-1$.

Solution: Though the functions on the right-hand side may look simple (first order polynomial expressions), they are not linear. However, we can express this (translated) system as:

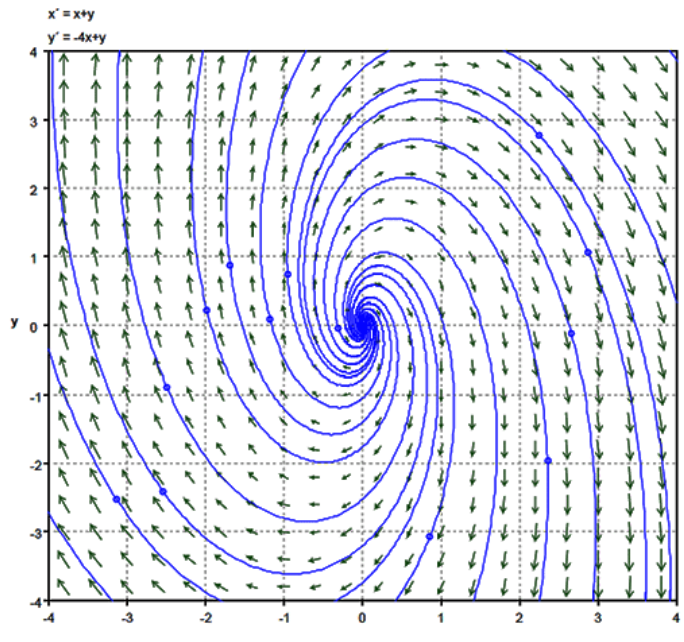
$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} x+y+5 \\ -4x+y+10 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \mathbf{Ax} + \mathbf{b} \text{ where } \mathbf{A} = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \text{ and } \mathbf{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

If we use the PPLANE tool to look at the underlying vector fields and the flow of each system, we observe that they appear identical except that the equilibrium is at $(0,0)$ for the linear case, and elsewhere for the nonlinear case (see next page). This suggests that we first find the equilibrium. At an equilibrium we'll have $\frac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{b} = \mathbf{0}$, so $\mathbf{Ax} = -\mathbf{b}$. We can usually (though not always!) solve for the equilibrium as $\mathbf{x}_p = -\mathbf{A}^{-1}\mathbf{b}$ unless the matrix **A** is not invertible. Fortunately, in this case A is invertible and we calculate $\mathbf{x}_p = -\mathbf{A}^{-1}\mathbf{b} = -\frac{1}{5}\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$. The key idea is to change to a new coordinate system by translating the axes to center them at this equilibrium point. That is, let $\mathbf{u} = \mathbf{x} - \mathbf{x}_p$. This gives $\mathbf{x} = \mathbf{u} + \mathbf{x}_p$ and we calculate:

$$\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{b} = \mathbf{A}(\mathbf{u} + \mathbf{x}_p) + \mathbf{b} = \mathbf{Au} + (\mathbf{Ax}_p + \mathbf{b}) = \mathbf{Au} + \mathbf{0} = \mathbf{Au}$$



$$\begin{Bmatrix} \frac{dx}{dt} = x+y+5 \\ \frac{dy}{dt} = -4x+y+10 \end{Bmatrix}$$



$$\begin{Bmatrix} \frac{dx}{dt} = x+y \\ \frac{dy}{dt} = -4x+y \end{Bmatrix}$$

So $\frac{d\mathbf{u}}{dt} = \mathbf{Au}$, and we know how to solve any such system, i.e. $\mathbf{u}(t) = [e^{t\mathbf{A}}]\mathbf{u}(0)$.

We can then observe that $\mathbf{u}(0) = \mathbf{x}(0) - \mathbf{x}_p$, so $\mathbf{x}(t) = \mathbf{x}_p + \mathbf{u}(t) = \mathbf{u}(t) = \mathbf{x}_p + [e^{t\mathbf{A}}][\mathbf{x}(0) - \mathbf{x}_p]$.

In our example, $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$, so $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda-1 & -1 \\ 4 & \lambda-1 \end{bmatrix}$, $p_{\mathbf{A}}(\lambda) = (\lambda-1)^2 + 4$ and the (complex) eigenvalues are $\lambda = 1+2i$ and $\bar{\lambda} = 1+2i$.

The eigenvalue $\lambda = 1+2i$ gives (complex) eigenvector $\mathbf{w} = \begin{bmatrix} 1 \\ 2i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \mathbf{u} + i\mathbf{v}$, and if we use the basis $\mathscr{B} = \{\mathbf{v}, \mathbf{u}\} = \left\{ \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ with change-of-basis matrix $\mathbf{S} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{S}^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, we'll have $[\mathbf{A}]_{\mathscr{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{B} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ with evolution matrix $[e^{t\mathbf{B}}] = e^t \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix}$. Since $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$ we'll have $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$. Since $\mathbf{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, we have $\mathbf{u}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, so we get:

$$\mathbf{x}(t) = \mathbf{x}_p + [e^{t\mathbf{A}}][\mathbf{x}(0) - \mathbf{x}_p] = \mathbf{x}_p + \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}[\mathbf{x}(0) - \mathbf{x}_p] = \begin{bmatrix} 1 \\ -6 \end{bmatrix} + \frac{1}{2}e^t \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -6 \end{bmatrix} + \frac{1}{2}e^t \begin{bmatrix} \sin 2t & \cos 2t \\ 2\cos 2t & -2\sin 2t \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 + e^t(\frac{5}{2}\sin 2t + \cos 2t) \\ -6 + e^t(5\cos 2t - 2\sin 2t) \end{bmatrix}$$

So $\left\{ \begin{array}{l} x(t) = 1 + e^t(\frac{5}{2}\sin 2t + \cos 2t) \\ y(t) = -6 + e^t(5\cos 2t - 2\sin 2t) \end{array} \right\}$, but the qualitative picture showing outward spirals coming out from the shifted equilibrium is the primary point.

There's a straightforward extension of this idea that brings together the recent topics involving linear systems of ODEs and some of our previous topics in which we used linearity properties to construct solutions to nth order linear equations in terms of homogeneous and particular solutions.

Consider a nonlinear system that can be expressed in the form $\boxed{\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{F}(t)}$ where $\mathbf{A}$ is an $n \times n$ (constant) matrix. This can also be expressed in the form: $\frac{d\mathbf{x}}{dt} - \mathbf{A}\mathbf{x} = \mathbf{F}(t)$. The case where $\mathbf{F}(t) = \mathbf{0}$ can then be properly characterized as a homogeneous system. This should be reminiscent of first-order linear equations of the form $\frac{dx}{dt} - ax = f(t)$ which could be solved either using linearity methods (homogeneous plus particular) or by using an integrating factor. Can we do something similar for such a nonlinear system?

Suppose $\mathbf{x}_p(t)$ is a particular solution of $\frac{d\mathbf{x}}{dt} - \mathbf{A}\mathbf{x} = \mathbf{F}(t)$, and let $\mathbf{x}(t)$ be any other solution. Consider the vector-valued function $\mathbf{x}(t) - \mathbf{x}_p(t)$. We calculate:

$$\frac{d}{dt}(\mathbf{x}(t) - \mathbf{x}_p(t)) - \mathbf{A}(\mathbf{x}(t) - \mathbf{x}_p(t)) = \left(\frac{d\mathbf{x}}{dt} - \mathbf{A}\mathbf{x} \right) - \left(\frac{d\mathbf{x}_p}{dt} - \mathbf{A}\mathbf{x}_p \right) = \mathbf{F}(t) - \mathbf{F}(t) = \mathbf{0}$$

So $\mathbf{x}(t) - \mathbf{x}_p(t)$ is a solution of the (homogeneous) linear system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$. If we express the general solution of this linear system as $\mathbf{x}_h(t)$, then we have $\mathbf{x}(t) - \mathbf{x}_p(t) = \mathbf{x}_h(t)$ and therefore $\boxed{\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)}$. Since we can always explicitly solve the homogeneous system, everything then comes down to finding a particular solution, and for this we'll see that previous methods such as the **Method of Undetermined Coefficients** and **Variation of Parameters** can be adapted to this more general situation.

A nonautonomous example

The case of a nonautonomous (time-dependent) system is generally more difficult to analyze, but in some cases it is straightforward. We can adapt methods developed earlier in the course to solve these problems completely.

Example: Find the general solution for the system $\left\{\begin{array}{l}\frac{dx}{dt}=f(x,y,t)=5x-6y+t+1\\ \frac{dy}{dt}=g(x,y,t)=3x-4y+t\end{array}\right\}$.

Solution: This system may be expressed as $\frac{d\mathbf{x}}{dt}=\begin{bmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{bmatrix}=\begin{bmatrix}5 & -6\\ 3 & -4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\begin{bmatrix}t+1\\ t\end{bmatrix}=\mathbf{Ax}+\mathbf{F}(t)$. We can rewrite this as $\frac{d\mathbf{x}}{dt}-\mathbf{Ax}=\mathbf{F}(t)$ which is reminiscent of the first-order linear equations we solved earlier in the course – only now with vector-valued functions instead of scalar-valued functions. We can adapt our earlier linearity methods in which we split the problem into (a) finding all homogeneous solutions, (b) find a particular solution, and (c) adding these to produce all possible solutions.

Solving for the ***homogeneous solutions*** where $\frac{d\mathbf{x}}{dt}-\mathbf{Ax}=\mathbf{0}$ is precisely the same as solving $\frac{d\mathbf{x}}{dt}=\mathbf{Ax}$ which we have done. Specifically, for $\mathbf{A}=\begin{bmatrix}5 & -6\\ 3 & -4\end{bmatrix}$, the eigenvalues were $\lambda_1=2$ with eigenvector $\mathbf{v}_1=\begin{bmatrix}2\\ 1\end{bmatrix}$, and $\lambda_2=-1$ with eigenvector $\mathbf{v}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$. This yields all solutions of the form $\mathbf{x}_h(t)=c_1e^{2t}\begin{bmatrix}2\\ 1\end{bmatrix}+c_2e^{-t}\begin{bmatrix}1\\ 1\end{bmatrix}$.

We can use an adapted version of the Method of Undetermined Coefficients to find a ***particular solution***. Specifically, for $\mathbf{F}(t)=\begin{bmatrix}t+1\\ t\end{bmatrix}$, we might try a solution of the form $\mathbf{x}_p(t)=\begin{bmatrix}at+b\\ ct+d\end{bmatrix}$. Substitution gives $\frac{d\mathbf{x}}{dt}=\begin{bmatrix}a\\ c\end{bmatrix}=\begin{bmatrix}(5a-6c+1)t+(5b-6d+1)\\ (3a-4c+1)t+(3b-4d)\end{bmatrix}$ or $\begin{bmatrix}(5a-6c+1)t+(-a+5b-6d+1)\\ (3a-4c+1)t+(3b-c-4d)\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ and these yield the system of equations $\left\{\begin{array}{r}5a-6c=-1\\ 3a-4c=-1\\ -a+5b-6d=-1\\ 3b-c-4d=0\end{array}\right\}$ with solutions $a=1$, $b=-3$, $c=1$, $d=-\frac{5}{2}$. So $\mathbf{x}_p(t)=\begin{bmatrix}t-3\\ t-\frac{5}{2}\end{bmatrix}$.

Therefore, all solutions are of the form $\mathbf{x}(t)=\mathbf{x}_h(t)+\mathbf{x}_p(t)=c_1e^{2t}\begin{bmatrix}2\\ 1\end{bmatrix}+c_2e^{-t}\begin{bmatrix}1\\ 1\end{bmatrix}+\begin{bmatrix}t-3\\ t-\frac{5}{2}\end{bmatrix}$.

Note: The earlier nonautonomous system $\left\{\begin{array}{l}\dfrac{dx}{dt}=x+y+5\\ \dfrac{dy}{dt}=-4x+y+10\end{array}\right\}$ with initial conditions $x(0)=2,\ y(0)=-1$ could also have been solved in this manner. We express this system as: $\frac{d\mathbf{x}}{dt}=\begin{bmatrix}1 & 1\\ -4 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\begin{bmatrix}5\\ 10\end{bmatrix}=\mathbf{Ax}+\mathbf{b}$ where $\mathbf{A}=\begin{bmatrix}1 & 1\\ -4 & 1\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}5\\ 10\end{bmatrix}$ and $\mathbf{x}(0)=\begin{bmatrix}2\\ -1\end{bmatrix}$. We previously found the equilibrium to be $\mathbf{x}_p=\begin{bmatrix}1\\ -6\end{bmatrix}$, a fixed point that is, in fact, <u>a particular solution</u>. The homogeneous solutions are just the solutions of $\frac{d\mathbf{x}}{dt}=\mathbf{Ax}$ which are $\mathbf{x}_h(t) = e^t \begin{bmatrix} \sin 2t & \cos 2t \\ 2\cos 2t & -2\sin 2t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Therefore, all solutions are of the form $\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t) = e^t \begin{bmatrix} \sin 2t & \cos 2t \\ 2\cos 2t & -2\sin 2t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -6 \end{bmatrix}$. If we substitute the initial conditions $\mathbf{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, we can determine the constants and arrive at the unique solution $\mathbf{x}(t) = \begin{bmatrix} 1 + e^t(\frac{5}{2}\sin 2t + \cos 2t) \\ -6 + e^t(5\cos 2t - 2\sin 2t) \end{bmatrix}$.

Variation of Parameters

Suppose we have a nonautonomous and inhomogeneous system in the form $\boxed{\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{F}(t)}$ where $\mathbf{P}(t)$ is an $n \times n$ matrix that may have time-dependent entries (though this includes the simpler case where this is a constant matrix). If we are able to find the general solution of the (homogeneous) system $\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x}$ in the form $\mathbf{x}_h(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{M}(t)\mathbf{c}$ where $\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is a constant vector, we can try to "vary the parameters" in search of a particular solution to the inhomogeneous system $\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{F}(t)$. That is, we seek a solution of the form $\mathbf{x}(t) = v_1(t)\mathbf{x}_1(t) + \cdots + v_n(t)\mathbf{x}_n(t) = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{x}_1(t) & \cdots & \mathbf{x}_n(t) \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{bmatrix} = \mathbf{M}(t)\mathbf{v}(t)$.

Differentiation gives $\frac{d\mathbf{x}}{dt} = \frac{d}{dt}\left[\mathbf{M}(t)\mathbf{v}(t)\right] = \mathbf{M}'(t)\mathbf{v}(t) + \mathbf{M}(t)\mathbf{v}'(t)$.

If $\mathbf{x}_h(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{M}(t)\mathbf{c}$ gives the homogeneous solutions, then $\frac{d\mathbf{x}_h}{dt} = \mathbf{M}'(t)\mathbf{c}$ because **c** is a constant vector. Further note that because $\frac{d\mathbf{x}_h}{dt} = \mathbf{P}(t)\mathbf{x}_h$, we have $\mathbf{M}'(t)\mathbf{c} = \mathbf{P}(t)\mathbf{M}(t)\mathbf{c}$ or, more simply $\mathbf{M}'(t) = \mathbf{P}(t)\mathbf{M}(t)$. Substitution gives $\mathbf{M}'(t)\mathbf{v}(t) + \mathbf{M}(t)\mathbf{v}'(t) = \cancel{\mathbf{P}(t)\mathbf{M}(t)\mathbf{v}(t)} + \mathbf{M}(t)\mathbf{v}'(t) = \cancel{\mathbf{P}(t)\mathbf{M}(t)\mathbf{v}(t)} + \mathbf{F}(t)$, so $\mathbf{M}(t)\mathbf{v}'(t) = \mathbf{F}(t)$. Because the columns of the matrix $\mathbf{M}(t)$ are linearly independent solutions, this matrix must be invertible. Therefore $\boxed{\mathbf{v}'(t) = \left[\mathbf{M}(t)\right]^{-1}\mathbf{F}(t)}$ and we can (hopefully) integrate each of the component functions to determine $\{v_1(t), \cdots, v_n(t)\}$.

If we apply this method to the system $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t+1 \\ t \end{bmatrix} = \mathbf{A}\mathbf{x} + \mathbf{F}(t)$, we'll have

$\mathbf{M}(t) = \mathbf{S}[e^{t\mathbf{A}}] = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix}$ for the fundamental matrix.

Its inverse is $\left[\mathbf{M}(t)\right]^{-1} = e^{-t} \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{2t} & 2e^{2t} \end{bmatrix} = \begin{bmatrix} e^{-2t} & -e^{-2t} \\ -e^{t} & 2e^{t} \end{bmatrix}$.

So $\mathbf{v}'(t) = \left[\mathbf{M}(t)\right]^{-1}\mathbf{F}(t) = \begin{bmatrix} e^{-2t} & -e^{-2t} \\ -e^{t} & 2e^{t} \end{bmatrix}\begin{bmatrix} t+1 \\ t \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ te^{t} - e^{t} \end{bmatrix}$. Integration gives $\mathbf{v}(t) = \begin{bmatrix} -\frac{1}{2}e^{-2t} \\ te^{t} - 2e^{t} \end{bmatrix}$.

Therefore, a particular solution will be $\mathbf{x}(t) = \mathbf{M}(t)\mathbf{v}(t) = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix}\begin{bmatrix} -\frac{1}{2}e^{-2t} \\ te^{t} - 2e^{t} \end{bmatrix} = \begin{bmatrix} -1+t-2 \\ -\frac{1}{2}+t-2 \end{bmatrix} = \begin{bmatrix} t-3 \\ t-\frac{5}{2} \end{bmatrix}$ which agrees with the result we obtained using undetermined coefficients.

Please see the first several pages of the Nonlinear Systems and Linearization Supplement, in particular horizontal and vertical nullclines, equilibria, and how to use these to produce a qualitative picture of the underlying vector fields from which we can sketch qualitative solutions (phase plane analysis). These may also be found in the Lecture #14 Notes. We'll cover the topics in that Supplement next week as well as some additional ideas and applications.

Notes by Robert Winters