Math E-21c – Ordinary Differential Equations – Lecture #12

Definition: A ***linear continuous dynamical system*** is a system of first-order differential equations of the form $\left\{\begin{matrix} \frac{dx_1}{dt} = a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ \frac{dx_n}{dt} = a_{n1}x_1 + \cdots + a_{nn}x_n \end{matrix}\right\}$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{11} \\ \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{11} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{Ax}$ where

$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{11} \\ \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{11} \end{bmatrix}$ is the matrix of coefficients. That is, $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$ where **A** is an $n \times n$ real matrix.

The emphasis in this week's lecture will be on the **matrix formalism for solving such a system** – specifically in the cases of distinct real or complex eigenvalues.

Uncoupled systems: We call a system **uncoupled** (or unlinked) if the rates of change of each of the variables do not depend on any of the other variables. In the linear case, this would mean a system of the form $\left\{\begin{matrix} \frac{dx_1}{dt} = k_1x_1 \\ \vdots \\ \frac{dx_n}{dt} = k_nx_n \end{matrix}\right\}$ with initial conditions $x_1(0), \ldots, x_n(0)$. Note that such a system can be expressed in matrix form as

$\frac{d\mathbf{x}}{dt} = \mathbf{Dx}$ where **D** is the diagonal matrix $\mathbf{D} = \begin{bmatrix} k_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & k_n \end{bmatrix}$. Solving this system is nothing more than solving each ODE separately with different rate constants and corresponding initial conditions. We get the solution $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} x_1(0)e^{k_1t} \\ \vdots \\ x_n(0)e^{k_nt} \end{bmatrix} = \begin{bmatrix} e^{k_1t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{k_nt} \end{bmatrix}\begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix}$. Note that when $t = 0$ the matrix is just the identity matrix which simply reflects the fact that $t = 0$ corresponds to the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$. Of greater interest is the fact that this time-varying matrix evolves over time to produce the flow emanating from any given initial condition. It is for this reason that we refer to this matrix as the **evolution matrix** for this uncoupled system. If we refer to this matrix as $[e^{t\mathbf{D}}]$, a notation that is perhaps best not taken too literally, then the system $\frac{d\mathbf{x}}{dt} = \mathbf{Dx}$ with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ has solution $\mathbf{x}(t) = [e^{t\mathbf{D}}]\mathbf{x}(0)$.

A **coupled system**, i.e. a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$ where the matrix A is not diagonal, can often be solved by changing coordinates so that relative to some new basis (of *eigenvectors*) the system has a diagonal matrix. The tool at the heart of these methods is diagonalization or, in the case where a matrix cannot be diagonalized, finding an appropriate change of basis relative to which the underlying linear transformation has the simplest possible matrix representation (Jordan Canonical Form). The introduction of corresponding "evolution matrices" is a useful formalism for handling these general cases.

In order to properly understand this we'll use some basic ideas from Linear Algebra – specifically the idea or a basis, coordinates relative to a basis, changing bases, linear transformations, and the matrix of a linear transformation relative to a basis.

Definition: Given an $n \times n$ matrix $\mathbf{A}$, a vector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$ is called a **characteristic vector** or ***eigenvector*** of $\mathbf{A}$ and the corresponding scalar λ is called its **characteristic value** or ***eigenvalue***. It's not hard to show that if a vector $\mathbf{x}$ is an eigenvector with eigenvalue λ, then any scalar multiple of $\mathbf{x}$ is also an eigenvector with the same eigenvalue λ.

Finding eigenvalues and eigenvectors

We can write the equation $\mathbf{Ax} = \lambda\mathbf{x}$ as $\mathbf{Ax} = \lambda\mathbf{Ix}$ where **I** is the Identity matrix. This can then be expressed as $\lambda\mathbf{Ix} - \mathbf{Ax} = (\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$. In terms of Linear Algebra, we say that the vector $\mathbf{x}$ must be in the **null space** (or ***kernel***) of the matrix $\lambda\mathbf{I} - \mathbf{A}$, i.e. $\mathbf{x} \in \ker(\lambda\mathbf{I} - \mathbf{A})$. It should be clear that the zero vector $\mathbf{x} = \mathbf{0}$ always solves this equation, so the question is whether there are other solutions. If the matrix $\lambda\mathbf{I} - \mathbf{A}$ was invertible, then the answer would be NO. So the only way that we can have nontrivial solutions is if the matrix $\lambda\mathbf{I} - \mathbf{A}$ is NOT invertible. It is a fundamental fact from Linear Algebra that an $n \times n$ matrix is invertible if and only its determinant is not zero, so this means that we must have $p_{\mathbf{A}}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = 0$ where $p_{\mathbf{A}}(\lambda)$ is called the **characteristic polynomial**. The roots of the characteristic polynomial are the characteristic values or eigenvalues. Once we determine the eigenvalues we can take them one at a time to determine the respective eigenvectors.

Solving systems using diagonalization and evolution matrices

Given an $n \times n$ matrix **A**, suppose **S** is a change of basis matrix corresponding to either diagonalization or reduction to Jordan Canonical Form (*more on this later*). We will have $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{B}$ in this case, where **B** is diagonal or otherwise in simplest form. We then calculate $\mathbf{A} = \mathbf{SBS}^{-1}$, and substitution gives $\frac{d\mathbf{x}}{dt} = \mathbf{SBS}^{-1}\mathbf{x}$.

Multiplying on the left by $\mathbf{S}^{-1}$ and using the basic calculus fact that $\frac{d}{dt}(\mathbf{Mx}) = \mathbf{M}\frac{d\mathbf{x}}{dt}$ for any (constant) matrix **M**, we have $\mathbf{S}^{-1}\frac{d\mathbf{x}}{dt} = \frac{d(\mathbf{S}^{-1}\mathbf{x})}{dt} = \mathbf{B}(\mathbf{S}^{-1}\mathbf{x})$. If we write $\mathbf{u} = \mathbf{S}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$, where $\mathcal{B}$ is the new, preferred basis, then in these new coordinates the system becomes $\frac{d\mathbf{u}}{dt} = \mathbf{Bu}$, but now the system will be much more straightforward to solve – especially in the case where $\mathbf{B} = \mathbf{D}$, a diagonal matrix.

The diagonalizable case

In the case where **B** is a diagonal matrix with the eigenvalues of **A** on the diagonal, the system is just

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \mathbf{u} \text{ or } \left\{ \begin{array}{c} \frac{du_1}{dt} = \lambda_1 u_1 \\ \vdots \\ \frac{du_n}{dt} = \lambda_n u_n \end{array} \right\}.$$

This has the solution $\left\{ \begin{array}{c} u_1(t) = e^{\lambda_1 t} u_1(0) \\ \vdots \\ u_n(t) = e^{\lambda_n t} u_n(0) \end{array} \right\}$ or $\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} u_1(0) \\ \vdots \\ u_n(0) \end{bmatrix} = [e^{t\mathbf{B}}]\mathbf{u}(0)$.

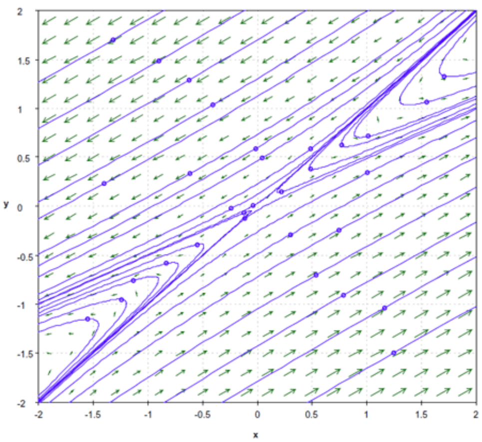
To revert back to the original coordinates, we write $\mathbf{x} = \mathbf{Su}$, so $\mathbf{x}(t) = \mathbf{Su}(t) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{u}(0) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. If we denote the evolution matrix for the system in its original coordinates as $[e^{t\mathbf{A}}]$ where $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$, then the previous calculation gives the simple relation $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$.

In other words, the evolution matrices for the solution are in the same relationship as the matrices **A** and **B**, namely $\mathbf{A} = \mathbf{SBS}^{-1}$. This pattern is very easy to remember, and this same pattern will again be the case where **B** is not diagonal but where the corresponding evolution matrix is still relatively easy to calculate.

$\mathbf{A} = \mathbf{SBS}^{-1} \quad \Rightarrow \quad [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$, and the solution of the original system will be $\boxed{\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)}$.

Problem: Solve the system $\begin{Bmatrix} \frac{dx}{dt} = 5x - 6y \\ \frac{dy}{dt} = 3x - 4y \end{Bmatrix}$ with initial conditions $x(0) = 3,\ y(0) = 1$.

Solution: This system can be written as $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. We find the eigenvalues by considering the matrix $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 5 & 6 \\ -3 & \lambda + 4 \end{bmatrix}$, and characteristic polynomial $p_{\mathbf{A}}(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$.



This gives the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The change of basis matrix is $\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and with the new basis (of eigenvectors) $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ we have $[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \mathbf{D}$, a diagonal matrix.

[There is no need to carry out the multiplication of the matrices if $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is known to be is a basis of eigenvectors. It will always yield a diagonal matrix with the eigenvalues on the diagonal.]

The evolution matrix for this diagonal matrix is $[e^{t\mathbf{D}}] = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}$, and the solution of the system is:

$$\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0) = \mathbf{S}[e^{t\mathbf{D}}]\mathbf{S}^{-1}\mathbf{x}(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4e^{2t} - e^{-t} \\ 2e^{2t} - e^{-t} \end{bmatrix} = 2e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2e^{2t}\mathbf{v}_1 - e^{-t}\mathbf{v}_2$$

The story is fundamentally the same for an $n \times n$ matrix **A** and its corresponding system of 1st order ODEs – as long as the eigenvalues are all real and distinct. We still have to take up the situation where the eigenvalues are either complex or repeated (algebraic multiplicity greater than 1).

The complex eigenvalue case

Suppose we want to solve a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where **A** is an 2×2 real matrix with a complex conjugate pair of eigenvalues $\lambda = a + ib$ and $\bar{\lambda} = a - ib$. There are several reasonable ways to proceed, but they all come down to determining the evolution matrix $[e^{t\mathbf{A}}]$ so that we can solve for $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$.

First, put the system into normal form. There are two good choices for how to do this.

<u>Using complex eigenvectors as a basis</u>:

First, since the eigenvalues are distinct, we can proceed formally the same way as in the real eigenvalue case. Specifically for the eigenvalue $\lambda = a + ib$ we find a complex eigenvector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are vectors with real components. It's straightforward to show that the eigenvalue $\bar{\lambda} = a - ib$ will have $\hat{\mathbf{w}} = \mathbf{u} - i\mathbf{v}$ as an eigenvector. This means that $\begin{Bmatrix} \mathbf{A}\mathbf{w} = \lambda\mathbf{w} \\ \mathbf{A}\hat{\mathbf{w}} = \bar{\lambda}\hat{\mathbf{w}} \end{Bmatrix}$, so if we formally take the <u>complex basis</u> $\mathcal{B} = \{\mathbf{w}, \hat{\mathbf{w}}\}$ and the change-of-basis matrix $\mathbf{S}=\begin{bmatrix}\mathbf{w} & \hat{\mathbf{w}}\end{bmatrix}$ we'll have $\left[\mathbf{A}\right]_{\mathscr{B}}=\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\mathbf{D}=\begin{bmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{bmatrix}$ and $\mathbf{A}=\mathbf{S}\mathbf{D}\mathbf{S}^{-1}$ and $[e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{D}}]\mathbf{S}^{-1}$ where $[e^{t\mathbf{D}}]=\begin{bmatrix}e^{\lambda t} & 0\\ 0 & e^{\bar{\lambda}t}\end{bmatrix}$ and solutions $\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)=\mathbf{S}[e^{t\mathbf{D}}]\mathbf{S}^{-1}\mathbf{x}(0)$. We'll illustrate this in the example that follows.

Using a preferred real basis:

Alternatively, $\begin{Bmatrix}\mathbf{A}\mathbf{w}=\lambda\mathbf{w}\\ \mathbf{A}\hat{\mathbf{w}}=\bar{\lambda}\hat{\mathbf{w}}\end{Bmatrix}$ can be rewritten in terms of $\mathbf{w}=\mathbf{u}+i\mathbf{v}$ and $\hat{\mathbf{w}}=\mathbf{u}-i\mathbf{v}$ as follows:

$$\begin{Bmatrix}\mathbf{A}(\mathbf{u}+i\mathbf{v})=\mathbf{A}\mathbf{u}+i\mathbf{A}\mathbf{v}=(a+ib)(\mathbf{u}+i\mathbf{v})=(a\mathbf{u}-b\mathbf{v})+i(b\mathbf{u}+a\mathbf{v})\\ \mathbf{A}(\mathbf{u}-i\mathbf{v})=\mathbf{A}\mathbf{u}-i\mathbf{A}\mathbf{v}=(a-ib)(\mathbf{u}-i\mathbf{v})=(a\mathbf{u}-b\mathbf{v})-i(b\mathbf{u}+a\mathbf{v})\end{Bmatrix}$$

Addition gives $2\mathbf{A}\mathbf{u}=2(a\mathbf{u}-b\mathbf{v})$ or $\mathbf{A}\mathbf{u}=a\mathbf{u}-b\mathbf{v}$. Subtraction gives $2i\mathbf{A}\mathbf{v}=2i(b\mathbf{u}+a\mathbf{v})$ or $\mathbf{A}\mathbf{v}=b\mathbf{u}+a\mathbf{v}$. In terms of the basis $\mathscr{B}=\{\mathbf{v},\mathbf{u}\}$ (note the reversed order), we can write these as $\begin{Bmatrix}\mathbf{A}\mathbf{v}=\ \ a\mathbf{v}+b\mathbf{u}\\ \mathbf{A}\mathbf{u}=-b\mathbf{v}+a\mathbf{u}\end{Bmatrix}$. This means that if we use the basis $\mathscr{B}=\{\mathbf{v},\mathbf{u}\}$ with change of basis matrix $\mathbf{S}=\begin{bmatrix}\mathbf{v} & \mathbf{u}\end{bmatrix}$, we'll have $\left[\mathbf{A}\right]_{\mathscr{B}}=\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\mathbf{B}=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$, a standard ***rotation-dilation*** matrix. We will again have $\mathbf{A}=\mathbf{S}\mathbf{B}\mathbf{S}^{-1}$ and $[e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$, but what is $[e^{t\mathbf{B}}]$?

In fact, if $\mathbf{B}=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$ then $\boxed{[e^{t\mathbf{B}}]=e^{at}\begin{bmatrix}\cos bt & -\sin bt\\ \sin bt & \cos bt\end{bmatrix}}$, a time-varying rotation matrix with exponential scaling. For any initial condition (except the zero vector), this yields a trajectory that spirals out in the case where $\mathrm{Re}(\lambda)=a>0$ (look to the original vector field to see whether it's clockwise or counterclockwise), or a trajectory that spirals inward toward **0** in the case where $\mathrm{Re}(\lambda)=a<0$.

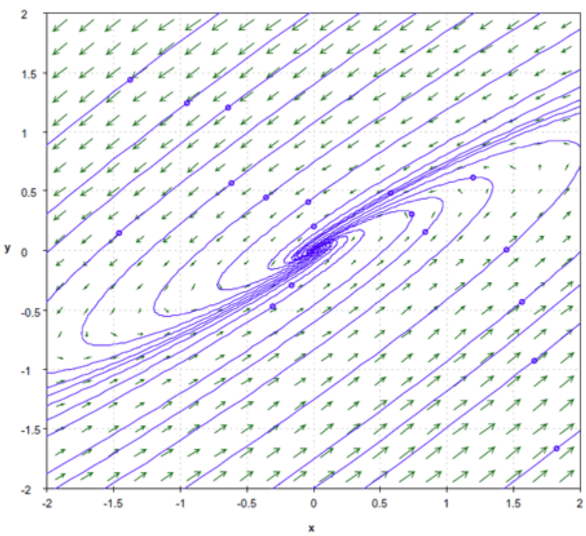
To derive this expression for $[e^{t\mathbf{B}}]$, make another coordinate change with complex eigenvectors starting with $\mathbf{B}=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$. We know this has the same eigenvalues of **A**, namely $\lambda=a+ib$ and $\bar{\lambda}=a-ib$. Use $\lambda=a+ib$ to get the complex eigenvector $\mathbf{w}=\begin{bmatrix}1\\ -i\end{bmatrix}$. The eigenvalue $\bar{\lambda}=a-ib$ will then give the eigenvector $\hat{\mathbf{w}}=\begin{bmatrix}1\\ i\end{bmatrix}$.

Using the (complex) change of basis matrix $\mathbf{P}=\begin{bmatrix}1 & 1\\ -i & i\end{bmatrix}$, we have that $\mathbf{P}^{-1}\mathbf{B}\mathbf{P}=\mathbf{D}=\begin{bmatrix}a+ib & 0\\ 0 & a-ib\end{bmatrix}$. It follows that (using Euler's Formula as needed):

$$[e^{t\mathbf{B}}]=\mathbf{P}[e^{t\mathbf{D}}]\mathbf{P}^{-1}=\begin{bmatrix}1 & 1\\ -i & i\end{bmatrix}\begin{bmatrix}e^{(a+ib)t} & 0\\ 0 & e^{(a-ib)t}\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & i\\ 1 & -i\end{bmatrix}=e^{at}\begin{bmatrix}\frac{e^{ibt}+e^{-ibt}}{2} & -\frac{e^{ibt}-e^{-ibt}}{2i}\\ \frac{e^{ibt}-e^{-ibt}}{2i} & \frac{e^{ibt}+e^{-ibt}}{2}\end{bmatrix}=e^{at}\begin{bmatrix}\cos bt & -\sin bt\\ \sin bt & \cos bt\end{bmatrix}.$$

These calculations enable us to write down an easy-to-calculate expression for the solution of the corresponding system of linear ODEs, namely $\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)$ where $[e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}=e^{at}\mathbf{S}\begin{bmatrix}\cos bt & -\sin bt\\ \sin bt & \cos bt\end{bmatrix}\mathbf{S}^{-1}$. However, the more important result is the ability to qualitatively describe the trajectories for this system by knowing only the real part of the eigenvalues of the matrix **A** and the direction of the corresponding vector field (clockwise vs. counterclockwise).

Problem: Solve the system $\left\{ \begin{array}{l} \dfrac{dx}{dt} = 2x - 5y \\ \dfrac{dy}{dt} = 2x - 4y \end{array} \right\}$ with initial conditions $x(0) = 0,\ y(0) = 1$.



Solution: In matrix form, we have $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We again start by finding the eigenvalues of the matrix: $\lambda \mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 2 & 5 \\ -2 & \lambda + 4 \end{bmatrix}$, and the characteristic polynomial is $p_{\mathbf{A}}(\lambda) = \lambda^2 + 2\lambda + 2 = (\lambda+1)^2 + 1$.

This gives the complex eigenvalue pair $\lambda = -1 + i$ and $\bar{\lambda} = -1 - i$. We seek a complex eigenvector for the first of these: $\begin{bmatrix} -3+i & 5 \\ -2 & 3+i \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives the (redundant) equations $(-3+i)\alpha + 5\beta = 0$ and $-2\alpha + (3+i)\beta = 0$.

The first of these can be written as $5\beta = (3-i)\alpha$, and an easy solution to this is where $\alpha = 5, \beta = 3 - i$. (We could also have used the second equation – which is a scalar multiple of the first. The eigenvector might then have been different, but ultimately we'll get the same result.) This gives the complex eigenvector

$$\mathbf{w} = \begin{bmatrix} 5 \\ 3-i \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \mathbf{u} + i\mathbf{v}.$$

<u>If we choose to use the complex basis</u> $\mathcal{B} = \{\mathbf{w}, \hat{\mathbf{w}}\}$ and change-of-basis matrix $\mathbf{S} = \begin{bmatrix} \mathbf{w} & \hat{\mathbf{w}} \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 3-i & 3+i \end{bmatrix}$ (not recommended) we'll get a solution of the form:

$$\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0) = \mathbf{S}[e^{t\mathbf{D}}]\mathbf{S}^{-1}\mathbf{x}(0) = \begin{bmatrix} 5 & 5 \\ 3-i & 3+i \end{bmatrix}\begin{bmatrix} e^{(-1+i)t} & 0 \\ 0 & e^{(-1-i)t} \end{bmatrix}\mathbf{S}^{-1}\mathbf{x}(0) = e^{-t}\begin{bmatrix} 5 & 5 \\ 3-i & 3+i \end{bmatrix}\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= e^{-t}\left(c_1 e^{it}\begin{bmatrix} 5 \\ 3-i \end{bmatrix} + c_2 e^{-it}\begin{bmatrix} 5 \\ 3+i \end{bmatrix} \right)$$ where c_1 and c_2 are complex constants.

This gives $\mathbf{x}(0) = \left(c_1\begin{bmatrix} 5 \\ 3-i \end{bmatrix} + c_2\begin{bmatrix} 5 \\ 3+i \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ which translates into:

$$\left\{ \begin{array}{c} c_1 + c_2 = 0 \\ 3(c_1 + c_2) + i(-c_1 + c_2) = 1 \end{array} \right\} \Rightarrow c_2 = -c_1 \Rightarrow 2c_2 = -i \Rightarrow c_2 = -\tfrac{1}{2}i, c_1 = \tfrac{1}{2}i$$

So $$\mathbf{x}(t) = e^{-t}\left(\tfrac{1}{2}ie^{it}\begin{bmatrix} 5 \\ 3-i \end{bmatrix} - \tfrac{1}{2}ie^{-it}\begin{bmatrix} 5 \\ 3+i \end{bmatrix} \right) = e^{-t}\begin{bmatrix} -5\left(\dfrac{e^{it} - e^{-it}}{2i} \right) \\ \left(\dfrac{e^{it} + e^{-it}}{2} \right) - 3\left(\dfrac{e^{it} - e^{-it}}{2i} \right) \end{bmatrix} = e^{-t}\begin{bmatrix} -5\sin t \\ \cos t - 3\sin t \end{bmatrix}.$$

<u>If we instead choose to use the real basis</u> $\mathcal{B} = \{\mathbf{v}, \mathbf{u}\}$ (recommended), the new system will have standard matrix $[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \mathbf{B}$ where a is the real part of the complex eigenvalue and b is its imaginary part. We showed that $[e^{t\mathbf{B}}] = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$. In this example, $a = -1$ and $b = 1$, $\mathbf{S} = \begin{bmatrix} \mathbf{v} & \mathbf{u} \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$,

$\mathbf{S}^{-1}=\frac{1}{5}\begin{bmatrix}3 & -5\\1 & 0\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}-1 & -1\\1 & -1\end{bmatrix}$, and $[e^{t\mathbf{B}}]=e^{-t}\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}$. The solution to the system is therefore:

$$\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)=\tfrac{1}{5}e^{-t}\begin{bmatrix}0 & 5\\-1 & 3\end{bmatrix}\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}\begin{bmatrix}3 & -5\\1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$

$$=\frac{e^{-t}}{5}\begin{bmatrix}5\sin t & 5\cos t\\-\cos t+3\sin t & \sin t+3\cos t\end{bmatrix}\begin{bmatrix}-5\\0\end{bmatrix}=e^{-t}\begin{bmatrix}-5\sin t\\ \cos t-3\sin t\end{bmatrix}.$$ That is, $\left\{\begin{aligned}x(t)&=-5e^{-t}\sin t\\ y(t)&=e^{-t}\cos t-3e^{-t}\sin t\end{aligned}\right\}$.

Repeated eigenvalues (with geometric multiplicity less than the algebraic multiplicity)

Suppose we want to solve a system of the form $\frac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a non-diagonalizable 2×2 real matrix with a repeated eigenvalue λ. In this case, we can always find a change of basis matrix $\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\mathbf{B}=\begin{bmatrix}\lambda & 1\\0 & \lambda\end{bmatrix}$. Why?

Generalized eigenvectors

When the algebraic multiplicity k of an eigenvalue λ of $\mathbf{A}$ is greater than 1, we will usually not be able to find k linearly independent eigenvectors corresponding to this eigenvalue. This is the case where the ***geometric multiplicity*** (the number of linearly independent eigenvectors corresponding to this eigenvalue) is strictly less than the ***algebraic multiplicity*** of this eigenvalue. The next best thing to an eigenvector is often referred to as a "generalized eigenvector".

If, for example, a matrix $\mathbf{A}$ had λ as an eigenvalue with algebraic multiplicity 2, but the geometric multiplicity was 1, we could certainly find an actual eigenvector $\mathbf{v}_1$ such that $\mathbf{A}\mathbf{v}_1=\lambda\mathbf{v}_1$, but we would not be able to produce a 2nd linearly independent eigenvector. However, it can be shown (and we'll demonstrate this in an example) that we will always be able to find a vector $\mathbf{v}_2$ such that $\mathbf{A}\mathbf{v}_2=\mathbf{v}_1+\lambda\mathbf{v}_2$. Another way of stating this is that $(\lambda\mathbf{I}-\mathbf{A})\mathbf{v}_1=\mathbf{0}$ and $(\lambda\mathbf{I}-\mathbf{A})\mathbf{v}_2=-\mathbf{v}_1$ (or $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2=\mathbf{v}_1$), so $(\lambda\mathbf{I}-\mathbf{A})^2\mathbf{v}_2=-(\lambda\mathbf{I}-\mathbf{A})\mathbf{v}_1=\mathbf{0}$. If an eigenvector is a vector in $\ker(\lambda\mathbf{I}-\mathbf{A})$, then a generalized eigenvector would be in $\ker(\lambda\mathbf{I}-\mathbf{A})^2$.

In the case where the algebraic multiplicity was 3 and the geometric multiplicity was only 1, we'd also seek a vector in $\ker(\lambda\mathbf{I}-\mathbf{A})^3$, namely a vector $\mathbf{v}_3$ such that $\mathbf{A}\mathbf{v}_3=\mathbf{v}_2+\lambda\mathbf{v}_3$. The idea is that a generalized eigenvector is a vector such that the transformation acts on it by scaling together with a shift by the previously found vector.

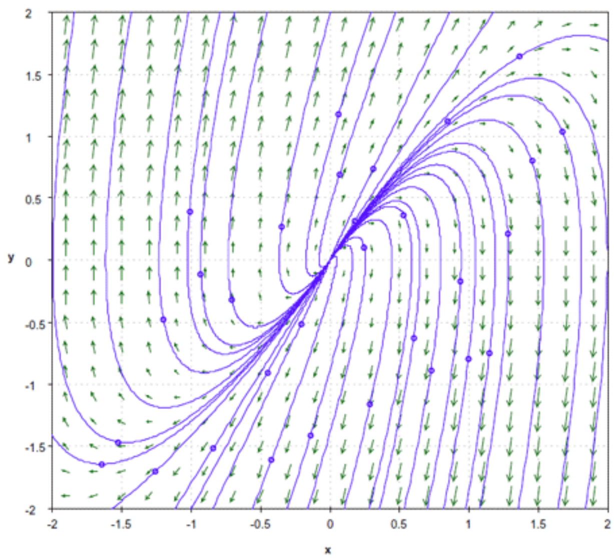
It can be shown that this process will <u>always</u> yield k linearly independent vectors corresponding to the eigenvalue λ, the first few vectors of which will be actual eigenvectors of $\mathbf{A}$. If a matrix $\mathbf{A}$ has all real eigenvalues and if we carry out this process for all eigenvalues of $\mathbf{A}$, we'll produce a complete basis $\mathcal{B}=\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ where we assume that all vectors corresponding to a given eigenvalue are grouped together and ordered in the way in which they were found.

As in the previous two cases, $\mathbf{A}=\mathbf{S}\mathbf{B}\mathbf{S}^{-1} \;\Rightarrow\; [e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$ and it comes down to finding $[e^{t\mathbf{B}}]$. This is perhaps most easily done by explicitly solving the corresponding differential equations. In the new coordinates, this system translates into $\left\{\begin{aligned}\frac{du_1}{dt}&=\lambda u_1+u_2\\ \frac{du_2}{dt}&=\lambda u_2\end{aligned}\right\}$. The second equation is easily solved to get $u_2(t)=e^{\lambda t}u_2(0)$. We can guess a solution for the first equation of the form $u_1(t)=c_1te^{\lambda t}+c_2e^{\lambda t}$. Differentiating this and substituting into the first equation, we get $c_1(e^{\lambda t}+\lambda te^{\lambda t})+c_2\lambda e^{\lambda t}=\lambda(c_1te^{\lambda t}+c_2e^{\lambda t})+e^{\lambda t}u_2(0)$. Comparing like terms, we conclude that $c_1 = u_2(0)$. Substituting $t=0$, we further conclude that $u_1(0) = c_2$. Putting these results together, we get $u_1(t) = u_2(0)te^{\lambda t} + u_1(0)e^{\lambda t} = e^{\lambda t}u_1(0) + te^{\lambda t}u_2(0)$. We therefore have that

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda t}u_1(0) + te^{\lambda t}u_2(0) \\ e^{\lambda t}u_2(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\mathbf{u}(0) = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\mathbf{u}(0)$$

So, $[e^{t\mathbf{B}}] = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$ in this case and the solution is given by $\mathbf{x}(t) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1} = \mathbf{S}\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\mathbf{S}^{-1}\mathbf{x}(0)$.

An alternate method of deriving this result may be found in the homework exercises.



Problem: Solve the system $\begin{Bmatrix} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -4x + 4y \end{Bmatrix}$ with initial conditions $x(0) = 3,\ y(0) = 2$.

Solution: In matrix form, we have $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. We again start by finding the eigenvalues of the matrix: $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & -1 \\ 4 & \lambda - 4 \end{bmatrix}$, and the characteristic polynomial is $p_{\mathbf{A}}(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$. This gives the repeated eigenvalue $\lambda = 2$ with (algebraic) multiplicity 2. We seek eigenvectors: $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives the (redundant) equations $2\alpha - \beta = 0$ and $4\alpha - 2\beta = 0$. Therefore $\beta = 2\alpha$, so we can choose $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or any scalar multiple of this as an eigenvector, but we are <u>unable to find a second linearly independent eigenvector</u>. (We say that the geometric multiplicity of the $\lambda = 2$ eigenvalue is 1.)

The standard procedure in this case is to seek a ***generalized eigenvector*** for this repeated eigenvalue, i.e. a vector $\mathbf{v}_2$ such that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_2$ is not zero, but rather a multiple of the eigenvector $\mathbf{v}_1$. Specifically, we seek a vector such that $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2$. This translates into seeking $\mathbf{v}_2$ such that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_2 = -\mathbf{v}_1$. That is, $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. This gives redundant equations the first of which is $2\alpha - \beta = -1$ or $\beta = 2\alpha + 1$.

If we (arbitrarily) choose $\alpha = 0$, then $\beta = 1$, so $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The fact that $\begin{Bmatrix} \mathbf{A}\mathbf{v}_1 = 2\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + 2\mathbf{v}_2 \end{Bmatrix}$ tells us that with the change of basis matrix $\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, we will have $[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \mathbf{B}$ and $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$ and $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$. So:

$$\mathbf{x}(t) = [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = e^{2t}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= e^{2t}\begin{bmatrix} 1 & t \\ 2 & 2t+1 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = e^{2t}\begin{bmatrix} 3 - 4t \\ 2 - 8t \end{bmatrix}$$

That is, $\begin{Bmatrix} x(t) = e^{2t}(3-4t) \\ y(t) = e^{2t}(2-8t) \end{Bmatrix}$. It's worth noting that this can also be expressed as $\mathbf{x}(t) = e^{2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} - 4te^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The phase portrait in this case has just one invariant (eigenvector) direction. It gives an unstable **node** which can be viewed as a degenerate case of a (clockwise) outward spiral that cannot get past the eigenvector direction.

Similar calculations enable us to deal with cases such as a repeated eigenvalue where the geometric multiplicity is 1 and the algebraic multiplicity is 3 (or even worse).

Finally, an actual system may exhibit several of these qualities – one or more complex pairs of eigenvalues, repeated eigenvalues, and distinct real eigenvalues. The Jordan Canonical Form of the matrix for such a system can be analyzed block by block and each of the above solutions applied within each block to determine the evolution matrix for the entire system.

Summarizing the Main Idea:

Given a system of 1st order linear differential equations $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ with initial conditions $\mathbf{x}(0)$, we use eigenvalue-eigenvector analysis to find an appropriate basis $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ for $\mathbf{R}^n$ and a change of basis matrix $\mathbf{S} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}$ such that in coordinates relative to this basis ($\mathbf{u} = \mathbf{S}^{-1}\mathbf{x}$) the system is in a standard form with a known solution. Specifically, we find a standard matrix $\mathbf{B} = [\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, transform the system into $\frac{d\mathbf{u}}{dt} = \mathbf{B}\mathbf{u}$, solve it as $\mathbf{u}(t) = [e^{t\mathbf{B}}]\mathbf{u}(0)$ where $[e^{t\mathbf{B}}]$ is the ***evolution matrix*** for **B**, then transform back to the original coordinates to get $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$ where $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$ is the ***evolution matrix*** for **B**. That is $\mathbf{x}(t) = [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$.

Moral of the Story: It's always possible to find a special basis relative to which a given linear system is in its simplest possible form. The new basis provides a way to decompose the given problem into several simple, standard problems which can be easily solved. Any complication in the algebraic expressions for the solution is the result of changing back to the original coordinates.

The standard 2×2 cases are:

Diagonalizable with eigenvalues λ_1, λ_2: $\quad \mathbf{B} = \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad [e^{t\mathbf{B}}] = [e^{t\mathbf{D}}] = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$

Complex pair of eigenvalues $\lambda = a \pm ib$: $\quad \mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad [e^{t\mathbf{B}}] = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$

Repeated eigenvalue λ with $\mathrm{GM} < \mathrm{AM}$: $\quad \mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad [e^{t\mathbf{B}}] = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

In general, you should expect to encounter systems more complicated than these 2×2 examples.

The basic protocol can be summarized simply as:
Standardize, **Solve**, the **Switch Back** to the original coordinates.

Notes by Robert Winters