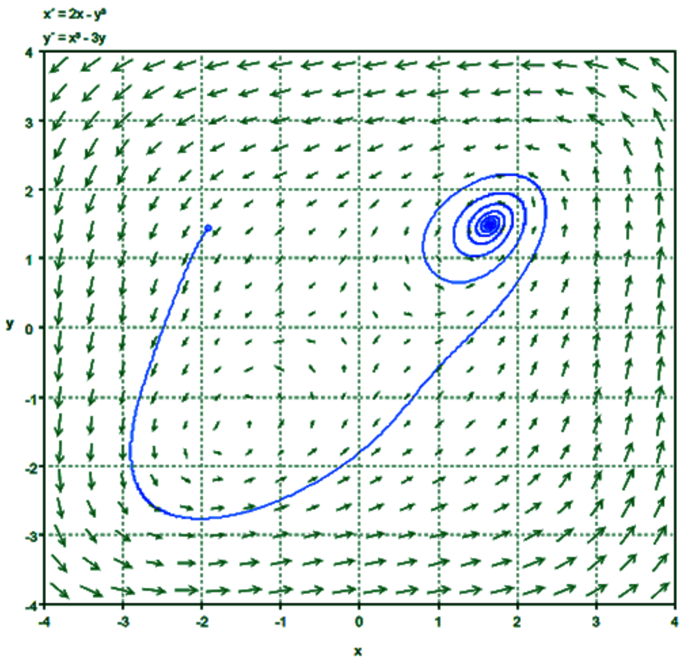
Math E-21c – Ordinary Differential Equations – Lecture #11

Vector fields, Continuous Dynamical Systems, and Systems of 1st Order Linear Differential Equations

Definition: A **vector field** in $\mathbf{R}^n$ is an assignment of a vector to every point in $\mathbf{R}^n$ (with the possible exception of some *singular points*). This can be viewed as a function $\mathbf{F}(x_1,\cdots,x_n)=\begin{bmatrix} f_1(x_1,\cdots,x_n) \\ \vdots \\ f_n(x_1,\cdots,x_n)\end{bmatrix}$ where $f_i(x_1,\cdots,x_n)$ is the *i*-th component of the vector assigned to the point $(x_1,\cdots,x_n)$. We can also write this more succinctly as $\mathbf{F}(\mathbf{x})=\begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x})\end{bmatrix}$. In practice we usually assume some reasonable properties such as that the component functions are continuous or differentiable except perhaps at a finite number of *singular points*. This is actually an ***autonomous vector field*** in the sense that the vector field is static, i.e. it does not change in time.



We can also consider ***nonautonomous vector fields***, i.e. $\mathbf{F}(x_1,\cdots,x_n,t)=\begin{bmatrix} f_1(x_1,\cdots,x_n,t) \\ \vdots \\ f_n(x_1,\cdots,x_n,t)\end{bmatrix}$ where the vector assigned to any given point also depends explicitly on (the time) t. We will primarily consider the nonautonomous case.

If we view the vector assigned to each point as a ***velocity vector*** associated with some smoothly varying system, a reasonable question to ask is this: Given a starting point $\mathbf{x}_0$ (the initial condition), can we find a parameterized curve $\mathbf{x}(t)$ such that $\mathbf{x}(0)=\mathbf{x}_0$ and the velocity vector at any point on this parameterized curve matches the underlying vector field, i.e. $\frac{d\mathbf{x}}{dt}=\mathbf{F}(\mathbf{x}(t))$.

This is equivalent to a system of (autonomous) first-order differential equations, i.e. $\left\{\begin{matrix} \frac{dx_1}{dt}=f_1(x_1,\cdots,x_n) \\ \vdots \\ \frac{dx_n}{dt}=f_n(x_1,\cdots,x_n)\end{matrix}\right\}$.

We are interested in knowing how a system defined in this way evolves over time for any given initial condition. This describes what we call a **continuous dynamical system**. We call the set of all such solution curves the **flow** of the dynamical system.

If you imagine a vector field as describing a flowing liquid, then these parameterized curves simply describe what happens if you drop a particle into the flow and see where it goes as it is carried by the flow. This is a good way to think about a continuous dynamical system even when the variables are describing such things as populations or economic variables rather than geometric coordinates. We'll still refer to the solutions as the flow of the system even though there's nothing physical about this flow.

We are typically interested in the long-term behavior of such a system, but we often would also like to predict exactly where the particle will be after a specified time t, i.e. formulas for how the component functions evolve in time. In general, if the component functions of the underlying vector field are nonlinear, it's very difficult to find a tidy formula for how the system evolves over time. The linear case, on the other hand, is completely solvable using matrix methods.

Reduction of Order

A homogeneous *n*th order linear ordinary differential equation can easily be represented as a system of 1st order ordinary differential equations simply by assigning names to the derivatives up to order $n-1$. For example, for the linear, time-invariant, 2nd order ODE $\ddot{x}+3\dot{x}+2x=0$, we can simply write $y=\dot{x}=\frac{dx}{dt}$. Then

$\frac{dy}{dt}=\ddot{x}=-2x-3\dot{x}=-2x-3y$. So $\left\{\begin{array}{l} \dfrac{dx}{dt}= \qquad\quad y \\ \dfrac{dy}{dt}=-2x-3y \end{array}\right\}$.

For the 3rd order ODE $\dddot{x}+2\ddot{x}-4\dot{x}+7x=0$, we can let $y=\dot{x}$ and $z=\ddot{x}$. Then

$\frac{dz}{dt}=\dddot{x}=-7x+4\dot{x}-2\ddot{x}=-7x+4y-2z$, so $\left\{\begin{array}{l} \dfrac{dx}{dt}= \qquad\quad y \\ \dfrac{dy}{dt}= \qquad\qquad\quad z \\ \dfrac{dz}{dt}=-7x+4y-2z \end{array}\right\}$.

Reduction of order is applicable for any linear ODE, and not just for the constant coefficient case or even the homogeneous case. For example, for the ODE $\ddot{x}+3t\dot{x}+(2-t^2)x=e^{-t}$ we can let $y=\dot{x}=\frac{dx}{dt}$ as before and turn this 2nd order ODE into the system $\left\{\begin{array}{l} \dfrac{dx}{dt}= \qquad\quad y \\ \dfrac{dy}{dt}=(t^2-2)x-3ty+e^{-t} \end{array}\right\}$ with an underlying nonautonomous (time-varying) vector field.

Even if, in theory, solutions exist for some complicated autonomous and even nonautonomous systems, we may not be able to express these solutions in terms of elementary functions and formulas. There is at least one category for which, as we'll see, it is always possible to find explicit representations of solutions. These are the autonomous systems where the component functions defining the underlying vector field are linear.

__**Definition**__: A ***linear continuous dynamical system*** is a system of first-order differential equations of the form $\left\{\begin{array}{c} \frac{dx_1}{dt}=a_{11}x_1+\cdots+a_{1n}x_n \\ \vdots \\ \frac{dx_n}{dt}=a_{n1}x_1+\cdots+a_{nn}x_n \end{array}\right\}$. If $\mathbf{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then $\dfrac{d\mathbf{x}}{dt}=\begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}=\begin{bmatrix} a_{11}x_1+\cdots+a_{1n}x_n \\ \vdots \\ a_{n1}x_1+\cdots+a_{nn}x_n \end{bmatrix}=\begin{bmatrix} a_{11} & \cdots & a_{11} \\ \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{11} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}=\mathbf{Ax}$ where $\mathbf{A}=\begin{bmatrix} a_{11} & \cdots & a_{11} \\ \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{11} \end{bmatrix}$ is the matrix of coefficients. That is, $\dfrac{d\mathbf{x}}{dt}=\mathbf{Ax}$ where **A** is an $n\times n$ real matrix.

Situation: You want to solve a system of first-order linear differential equations of the form $\dfrac{d\mathbf{x}}{dt}=\mathbf{Ax}$ given some initial condition $\mathbf{x}(0)=\mathbf{x}_0$. How is this most efficiently accomplished?

Example 1: The simplest linear continuous dynamical system is the single equation $\frac{dx}{dt}=kx$ with initial condition $x(0)=x_0$. This is something we solved in basic calculus and yields exponential growth or decay (depending on whether $k>0$ or $k<0$). Specifically, we write $\frac{1}{x}\frac{dx}{dt}=k$ and integrate both sides to get

$\ln|x(t)| = kt + c$ for some arbitrary constant c. [Many people choose to do this calculation as $\frac{dx}{x} = kdt$ and integrate both sides to get $\int \frac{dx}{x} = \int kdt \quad \Rightarrow \quad \ln|x| = kt + c$.] In any case, exponentiating both sides gives $|x(t)| = e^{kt+c} = e^c e^{kt} = ae^{kt}$, and we can remove the absolute value by allowing the constant a to be either positive or negative, so we get $x(t) = ae^{kt}$. Using the initial condition $x(0) = x_0$ we see that $x(0) = a = x_0$, so the solution is $\boxed{x(t) = x_0 e^{kt}}$.

Uncoupled systems: We call a system **uncoupled** (or unlinked) if the rates of change of each of the variables do not depend on any of the other variables. In the linear case, this would mean a system of the form $\begin{Bmatrix} \frac{dx_1}{dt} = k_1 x_1 \\ \vdots \\ \frac{dx_n}{dt} = k_n x_n \end{Bmatrix}$ with initial conditions $x_1(0), \dots, x_n(0)$. Note that such a system can be expressed in matrix form as $\frac{d\mathbf{x}}{dt} = \mathbf{Dx}$ where $\mathbf{D}$ is the diagonal matrix $\mathbf{D} = \begin{bmatrix} k_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & k_n \end{bmatrix}$. Solving this system is nothing more than solving the previous problem repeatedly with different rate constants and corresponding initial conditions. We get the solution $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} x_1(0)e^{k_1 t} \\ \vdots \\ x_n(0)e^{k_n t} \end{bmatrix} = \begin{bmatrix} e^{k_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{k_n t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix}$. Note that when $t = 0$ the matrix is just the identity matrix which simply reflects the fact that $t = 0$ corresponds to the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$. Of greater interest is the fact that this time-varying matrix evolves over time to produce the flow emanating from any given initial condition. It is for this reason that we refer to this matrix as the **evolution matrix** for this uncoupled system. If we refer to this matrix as $[e^{t\mathbf{D}}]$, a notation that is perhaps best not taken too literally, then the system $\frac{d\mathbf{x}}{dt} = \mathbf{Dx}$ with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ has solution $\mathbf{x}(t) = [e^{t\mathbf{D}}]\mathbf{x}(0)$.

Note: The unusual notation of $[e^{t\mathbf{D}}]$ can, in fact, be understood in terms of a generalization of the idea of power series – though this is not particularly useful or consequential. Specifically, if you recall the Maclaurin Series representation for $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^k}{k!} + \cdots$ and if we formally replace t by the matrix $t\mathbf{D}$ and 1 by the Identity matrix $\mathbf{I}$, we might write:

$$e^{t\mathbf{D}} = \mathbf{I} + t\mathbf{D} + \frac{(t\mathbf{D})^2}{2!} + \frac{(t\mathbf{D})^3}{3!} + \cdots + \frac{(t\mathbf{D})^k}{k!} + \cdots$$

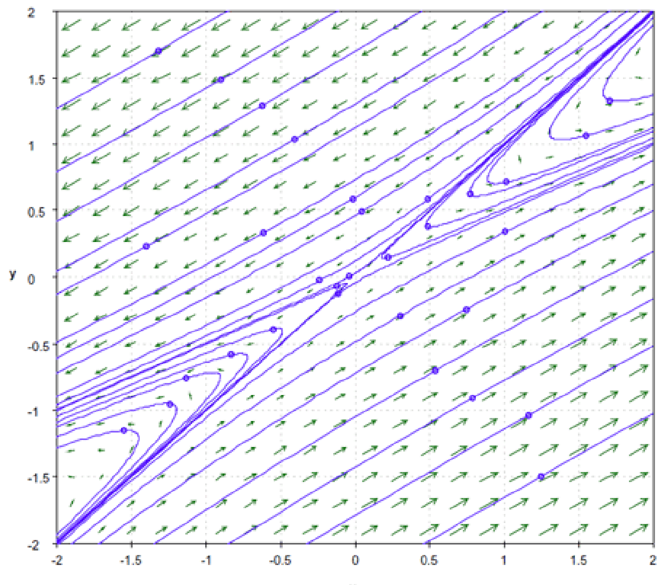
$$= \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & & \\ & \ddots & \\ & & \lambda_n t \end{bmatrix} + \begin{bmatrix} \frac{(\lambda_1 t)^2}{2!} & & \\ & \ddots & \\ & & \frac{(\lambda_n t)^2}{2!} \end{bmatrix} + \cdots + \begin{bmatrix} \frac{(\lambda_1 t)^k}{k!} & & \\ & \ddots & \\ & & \frac{(\lambda_n t)^k}{k!} \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + \cdots + \frac{(\lambda_1 t)^k}{k!} + \dots & & \\ & \ddots & \\ & & 1 + \cdots + \frac{(\lambda_n t)^k}{k!} + \dots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

A **coupled system**, i.e. a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where the matrix A is not diagonal, can often be solved by changing coordinates so that relative to some new basis (of *eigenvectors*) the system has a diagonal matrix. The tool at the heart of these methods is diagonalization or, in the case where a matrix cannot be diagonalized, finding an appropriate change of basis relative to which the underlying linear transformation has the simplest possible matrix representation (Jordan Canonical Form). The introduction of corresponding "evolution matrices" is a useful formalism for handling these general cases.

In order to properly understand this we'll have to review some basic ideas from Linear Algebra – specifically the idea or a basis, coordinates relative to a basis, changing bases, linear transformations, and the matrix of a linear transformation relative to a basis.

Problem: Solve the system $\left\{ \begin{array}{l} \dfrac{dx}{dt} = 5x - 6y \\ \dfrac{dy}{dt} = 3x - 4y \end{array} \right\}$ with initial conditions $x(0) = 3,\ y(0) = 1$.



We can use the PPLANE tool to generate some ideas on how to solve this problem. The figure (right) shows the underlying vector field as well as some trajectories (solution curves for various initial conditions). Observe that the origin is an **equilibrium**, i.e. the vector field vanishes at the origin, so if we started with initial condition at the origin, that would give a constant solution, i.e. there would be no movement.

The figure also indicates that there are two very prominent lines passing through the origin that seem to divide up the plane into fundamentally different trajectories. These special lines are the key to "decoupling" the system. What exactly characterizes these special lines? The essential feature of both lines is that at every point the position vector and the velocity vector assigned to that point are **parallel**. In the example, for one line the vectors are pointing outward (growth) and for the other they are pointing inward (decay).

The idea that will be productive here is to **change coordinates** in a manner that makes use of these special directions as new coordinate axes. The fact that at any position $\mathbf{x}$ along one of these special directions the velocity vector is parallel to the position vector can be succinctly captured by noting that $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \parallel \mathbf{x}$ or that $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.

Definition: Given an $n \times n$ matrix $\mathbf{A}$, a vector $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ is called a **characteristic vector** or ***eigenvector*** of $\mathbf{A}$ and the corresponding scalar λ is called its **characteristic value** or ***eigenvalue***. It's not hard to show that if a vector $\mathbf{x}$ is an eigenvector with eigenvalue λ, then any scalar multiple of $\mathbf{x}$ is also an eigenvector with the same eigenvalue λ.

Finding eigenvalues and eigenvectors

In the above problem, the defining matrix for the system is $\mathbf{A} = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$. We can write the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ as $\mathbf{A}\mathbf{x} = \lambda\mathbf{I}\mathbf{x}$ where **I** is the Identity matrix. This can then be expressed as $\lambda\mathbf{I}\mathbf{x} - \mathbf{A}\mathbf{x} = (\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$. In terms of Linear Algebra, we say that the vector $\mathbf{x}$ must be in the **null space** (or ***kernel***) of the matrix $\lambda\mathbf{I} - \mathbf{A}$. It should be clear that the zero vector $\mathbf{x} = \mathbf{0}$ always solves this equation, so the question is whether there are other solutions. If the matrix $\lambda\mathbf{I}-\mathbf{A}$ was invertible, then the answer would be NO. So the only way that we can have nontrivial solutions is if the matrix $\lambda\mathbf{I}-\mathbf{A}$ is NOT invertible. It is a fundamental fact from Linear Algebra that an $n\times n$ matrix is invertible if and only its determinant is not zero, so this means that we must have $p_{\mathbf{A}}(\lambda)=\det\left(\lambda\mathbf{I}-\mathbf{A}\right)=0$ where $p_{\mathbf{A}}(\lambda)$ is called the **characteristic polynomial**. The roots of the characteristic polynomial are the characteristic values or eigenvalues. Once we determine the eigenvalues we can take them one at a time to determine the respective eigenvectors.

For the matrix $\mathbf{A}=\begin{bmatrix}5 & -6\\ 3 & -4\end{bmatrix}$ we have $\lambda\mathbf{I}-\mathbf{A}=\begin{bmatrix}\lambda-5 & 6\\ -3 & \lambda+4\end{bmatrix}$, and the characteristic polynomial is $p_{\mathbf{A}}(\lambda)=\lambda^2-\lambda-2=(\lambda-2)(\lambda+1)$. This gives the eigenvalues $\lambda_1=2$ and $\lambda_2=-1$. Taking $\lambda_1=2$, we have $\lambda\mathbf{I}-\mathbf{A}=\begin{bmatrix}-3 & 6\\ -3 & 6\end{bmatrix}$ and we seek a vector $\mathbf{x}=\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ such that $\begin{bmatrix}-3 & 6\\ -3 & 6\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. We must therefore have $-3\alpha+6\beta=0$ or, more simply, $\alpha=2\beta$. Because any scalar multiple of an eigenvector is also an eigenvector, we have the freedom to arbitrarily choose $\beta=1$ which gives $\alpha=2$, so we get the eigenvector $\mathbf{v}_1=\begin{bmatrix}2\\ 1\end{bmatrix}$.

Similarly, with $\lambda_2=-1$ we'll have $\begin{bmatrix}-6 & 6\\ -3 & 3\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$, so $-3\alpha+3\beta=0$ or, more simply, $\alpha=\beta$. If we choose $\beta=1$ this gives $\alpha=1$, and we get the eigenvector $\mathbf{v}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$. We have $\left\{\begin{array}{l}\mathbf{A}\mathbf{v}_1=\lambda_1\mathbf{v}_1=-2\mathbf{v}_1\\ \mathbf{A}\mathbf{v}_2=\lambda_2\mathbf{v}_2=\mathbf{v}_2\end{array}\right\}$.

If we compare these to the diagram of the vector field and trajectories we see that these are precisely those special directions that we sought.

Solving systems using eigenvectors without matrices

We can now use these special directions to solve the system of differential equations. Guided by what we saw with the vector field, we seek solutions of the form $\mathbf{x}(t)=c_1(t)\mathbf{v}_1+c_2(t)\mathbf{v}_2$ where $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2\}$ are the eigenvectors. Essentially we are using these vectors as a **basis for a new coordinate system** where the coordinates (relative to this basis) are $\{c_1,c_2\}$.

Differentiation gives $\dfrac{d\mathbf{x}}{dt}=\dfrac{dc_1}{dt}\mathbf{v}_1+\dfrac{dc_2}{dt}\mathbf{v}_2=\mathbf{A}\mathbf{x}=\mathbf{A}\left(c_1\mathbf{v}_1+c_2\mathbf{v}_2\right)=c_1\mathbf{A}\mathbf{v}_1+c_2\mathbf{A}\mathbf{v}_2=c_1\lambda_1\mathbf{v}_1+c_2\lambda_2\mathbf{v}_2$.

So $\dfrac{dc_1}{dt}=\lambda_1 c_1$ and $\dfrac{dc_2}{dt}=\lambda_1 c_2$. These are easily solved to give $c_1(t)=c_1(0)e^{\lambda_1 t}$ and $c_2(t)=c_2(0)e^{\lambda_2 t}$ where $\mathbf{x}(0)=c_1(0)\mathbf{v}_1+c_2(0)\mathbf{v}_2$ yields the new coordinates of the initial condition. So the solution becomes $\boxed{\mathbf{x}(t)=c_1(0)e^{\lambda_1 t}\mathbf{v}_1+c_2(0)e^{\lambda_2 t}\mathbf{v}_2}$ or, more simply, $\boxed{\mathbf{x}(t)=c_1 e^{\lambda_1 t}\mathbf{v}_1+c_2 e^{\lambda_2 t}\mathbf{v}_2}$.

With the initial conditions $x(0)=3,\ y(0)=1$ we have $\mathbf{x}(0)=\begin{bmatrix}3\\ 1\end{bmatrix}=c_1\begin{bmatrix}2\\ 1\end{bmatrix}+c_2\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}2c_1+c_2\\ c_1+c_2\end{bmatrix}$ which yields $c_1=2$ and $c_2=-1$, so the unique solution to the system for these initial conditions is:

$$\boxed{\mathbf{x}(t)=2e^{-2t}\mathbf{v}_1-e^{t}\mathbf{v}_2=2e^{-2t}\begin{bmatrix}2\\ 1\end{bmatrix}-e^{t}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}4e^{-2t}-e^{t}\\ 2e^{-2t}-e^{t}\end{bmatrix}=\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}}$$

Note that there is decay in the direction associated with the eigenvector $\mathbf{v}_1=\begin{bmatrix}2\\ 1\end{bmatrix}$ and growth in the direction associated with the eigenvector $\mathbf{v}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$, and this agrees with what we saw in the sample trajectories in the figure. In short, positive eigenvalues correspond to growth and negative eigenvalues correspond to decay.

Reformulating things in terms of matrices

We can also express the solution as $\mathbf{x}(t)=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}\begin{bmatrix}2e^{-2t}\\ -e^{t}\end{bmatrix}=\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}e^{-2t}&0\\0&e^{t}\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}$. The matrix $\mathbf{S}=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}=\begin{bmatrix}2&1\\1&1\end{bmatrix}$ is known as a "change of basis" matrix motivated by the observation that $\mathbf{x}=c_1\mathbf{v}_1+c_2\mathbf{v}_2=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\mathbf{S}\left[\mathbf{x}\right]_{\mathscr{B}}$ where $\left[\mathbf{x}\right]_{\mathscr{B}}=\begin{bmatrix}c_1\\c_2\end{bmatrix}$ are the coordinates of the vector $\mathbf{x}$ relative to the basis $\mathscr{B}=\left\{\mathbf{v}_1,\mathbf{v}_2\right\}$. Indeed, if $\mathbf{x}=\mathbf{S}\left[\mathbf{x}\right]_{\mathscr{B}}$, then $\left[\mathbf{x}\right]_{\mathscr{B}}=\mathbf{S}^{-1}\mathbf{x}$, so the solution may expressed as $\mathbf{x}(t)=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}\begin{bmatrix}2e^{-2t}\\ -e^{t}\end{bmatrix}=\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}e^{-2t}&0\\0&e^{t}\end{bmatrix}\begin{bmatrix}2&1\\1&1\end{bmatrix}^{-1}\begin{bmatrix}3\\1\end{bmatrix}=\left(\mathbf{S}\left[e^{t\mathbf{D}}\right]\mathbf{S}^{-1}\right)\mathbf{x}(0)$ where $\mathbf{D}=\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}=\begin{bmatrix}-2&0\\0&1\end{bmatrix}$ and $\left[e^{t\mathbf{D}}\right]=\begin{bmatrix}e^{\lambda_1 t}&0\\0&e^{\lambda_2 t}\end{bmatrix}=\begin{bmatrix}e^{-2t}&0\\0&e^{t}\end{bmatrix}$. If we <u>define</u> the evolution matrix as $\left[e^{t\mathbf{A}}\right]=\mathbf{S}\left[e^{t\mathbf{D}}\right]\mathbf{S}^{-1}$, then the solution is expressed simply as $\mathbf{x}(t)=\left[e^{t\mathbf{A}}\right]\mathbf{x}(0)$, i.e. solutions are completely determined by applying the time-varying evolution matrix to the vector representing the initial conditions. The solution trajectories are essentially "drawn" by varying *t*.

Let's try reformulating things from the start in terms of matrices.

Solving systems using diagonalization and evolution matrices

Given an $n\times n$ matrix **A**, suppose **S** is a change of basis matrix corresponding to either diagonalization or reduction to Jordan Canonical Form (*more on this later*). We will have $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\mathbf{B}$ in this case, where **B** is diagonal or otherwise in simplest form. We then calculate $\mathbf{A}=\mathbf{S}\mathbf{B}\mathbf{S}^{-1}$, and substitution gives $\frac{d\mathbf{x}}{dt}=\mathbf{S}\mathbf{B}\mathbf{S}^{-1}\mathbf{x}$.

Multiplying on the left by $\mathbf{S}^{-1}$ and using the basic calculus fact that $\frac{d}{dt}(\mathbf{M}\mathbf{x})=\mathbf{M}\frac{d\mathbf{x}}{dt}$ for any (constant) matrix **M**, we have $\mathbf{S}^{-1}\frac{d\mathbf{x}}{dt}=\frac{d(\mathbf{S}^{-1}\mathbf{x})}{dt}=\mathbf{B}(\mathbf{S}^{-1}\mathbf{x})$. If we write $\mathbf{u}=\mathbf{S}^{-1}\mathbf{x}=\left[\mathbf{x}\right]_{\mathscr{B}}$, where $\mathscr{B}$ is the new, preferred basis, then in these new coordinates the system becomes $\frac{d\mathbf{u}}{dt}=\mathbf{B}\mathbf{u}$, but now the system will be much more straightforward to solve – especially in the case where $\mathbf{B}=\mathbf{D}$, a diagonal matrix.

The diagonalizable case

In the case where **B** is a diagonal matrix with the eigenvalues of **A** on the diagonal, the system is just

$$\frac{d\mathbf{u}}{dt}=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}\mathbf{u}\ \text{ or }\ \left\{\begin{matrix}\dfrac{du_1}{dt}=\lambda_1 u_1\\ \vdots\\ \dfrac{du_n}{dt}=\lambda_n u_n\end{matrix}\right\}.$$

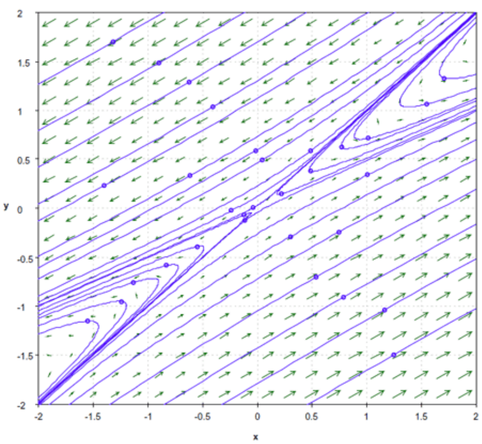
This has the solution $\left\{\begin{matrix}u_1(t)=e^{\lambda_1 t}u_1(0)\\ \vdots\\ u_n(t)=e^{\lambda_n t}u_n(0)\end{matrix}\right\}$ or $\mathbf{u}(t)=\begin{bmatrix}u_1(t)\\ \vdots\\ u_n(t)\end{bmatrix}=\begin{bmatrix}e^{\lambda_1 t} & & 0\\ & \ddots & \\ 0 & & e^{\lambda_n t}\end{bmatrix}\begin{bmatrix}u_1(0)\\ \vdots\\ u_n(0)\end{bmatrix}=[e^{t\mathbf{B}}]\mathbf{u}(0)$.

To revert back to the original coordinates, we write $\mathbf{x}=\mathbf{S}\mathbf{u}$, so $\mathbf{x}(t)=\mathbf{S}\mathbf{u}(t)=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{u}(0)=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. If we denote the evolution matrix for the system in its original coordinates as $[e^{t\mathbf{A}}]$ where $\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)$, then the previous calculation gives the simple relation $[e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$.

In other words, the evolution matrices for the solution are in the same relationship as the matrices **A** and **B**, namely $\mathbf{A}=\mathbf{SBS}^{-1}$. This pattern is very easy to remember, and this same pattern will again be the case where **B** is not diagonal but where the corresponding evolution matrix is still relatively easy to calculate.

$$\mathbf{A}=\mathbf{SBS}^{-1} \quad \Rightarrow \quad [e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\text{, and the solution of the original system will be } \boxed{\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)}.$$

Problem: Solve the system $\begin{Bmatrix} \dfrac{dx}{dt}=5x-6y \\ \dfrac{dy}{dt}=3x-4y \end{Bmatrix}$ with initial conditions $x(0)=3,\ y(0)=1$.



Solution: As we showed earlier, this system can be written as $\dfrac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x}$ where $\mathbf{A}=\begin{bmatrix}5 & -6\\ 3 & -4\end{bmatrix}$ and $\mathbf{x}(0)=\begin{bmatrix}3\\1\end{bmatrix}$. We found the eigenvalues by considering the matrix $\lambda\mathbf{I}-\mathbf{A}=\begin{bmatrix}\lambda-5 & 6\\ -3 & \lambda+4\end{bmatrix}$, and characteristic polynomial $p_{\mathbf{A}}(\lambda)=\lambda^2-\lambda-2=(\lambda-2)(\lambda+1)$. This gave the eigenvalues $\lambda_1=2$ and $\lambda_2=-1$ with corresponding eigenvectors $\mathbf{v}_1=\begin{bmatrix}2\\1\end{bmatrix}$ and $\mathbf{v}_2=\begin{bmatrix}1\\1\end{bmatrix}$. The change of basis matrix is $\mathbf{S}=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$ and with the new basis (of eigenvectors) $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2\}$ we have $[\mathbf{A}]_{\mathcal{B}}=\mathbf{S}^{-1}\mathbf{AS}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}=\mathbf{D}$, a diagonal matrix. [There is no need to carry out the multiplication of the matrices if $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2\}$ is known to be is a basis of eigenvectors. It will always yield a diagonal matrix with the eigenvalues on the diagonal.]

The evolution matrix for this diagonal matrix is $[e^{t\mathbf{D}}]=\begin{bmatrix}e^{2t} & 0\\ 0 & e^{-t}\end{bmatrix}$, and the solution of the system is:

$$\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)=\mathbf{S}[e^{t\mathbf{D}}]\mathbf{S}^{-1}\mathbf{x}(0)=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{2t} & 0\\ 0 & e^{-t}\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}2e^{2t} & e^{-t}\\ e^{2t} & e^{-t}\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}$$

$$=\begin{bmatrix}4e^{2t}-e^{-t}\\ 2e^{2t}-e^{-t}\end{bmatrix}=2e^{2t}\begin{bmatrix}2\\1\end{bmatrix}-e^{-t}\begin{bmatrix}1\\1\end{bmatrix}=2e^{2t}\mathbf{v}_1-e^{-t}\mathbf{v}_2$$

The story is fundamentally the same for an $n\times n$ matrix **A** and its corresponding system 1st order ODEs – as long as the eigenvalues are all real and distinct. Next week we'll also take up the situation where the eigenvalues are either complex or repeated (algebraic multiplicity greater than 1).

Notes by Robert Winters