Math E-21c – Ordinary Differential Equations – Lecture #10

In today's lecture we define Convolution and apply it along with unit impulse response to determine the Zero State Response (ZSR) for a time-invariant *n*-th order linear ODE, i.e. an ODE of the form $p(D)[x(t)] = f(t)$. In this context we are only interested in solutions for $t > 0$. These methods are particularly useful for dealing with input functions $f(t)$ defined by data rather than simply by familiar elementary functions.

Definition (again): The ***Laplace transform*** of a function $f(t)$ is defined by $\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} e^{-st} f(t)dt$ where the new (complex) variable s is such that its real part $\text{Re}(s) \gg 0$ (the integral would otherwise not converge). Note that the lower limit of the integral indicates that $t = 0$ is included and is intended to address potential discontinuities and delta functions. We use the convention that a function of t will be represented by a lower case name and its Laplace transform by the corresponding upper case name, e.g. $\mathscr{L}[x(t)] = X(s)$.

Unit Impulse Response

Unit impulse response refers to the solution of the ODE $p(D)[x(t)] = \delta(t)$ with rest initial conditions. The solution is also known as the ***weight function*** for the given differential operator $p(D)$. It is the simplest to tackle algebraically and we'll use it soon along with convolution to solve Initial Value Problems. We generally denote the unit impulse response (weight function) by $w(t)$. It's Laplace Transform $W(s)$ is called the ***transfer function***.

Example 1: Find the unit impulse response and the unit step response for the operator $p(D) = D + 3I$.

Solution: For the unit impulse response we solve $\dot{w} + 3w = \delta(t)$ with rest initial conditions. Transforming both sides gives $p(s)W(s) = (s+3)W(s) = 1$, so $W(s) = \dfrac{1}{p(s)} = \dfrac{1}{s+3}$. This is just $\mathscr{L}(e^{-3t})$, so $\boxed{w(t) = e^{-3t}}$.

For the unit step response we solve $\dot{v} + 3v = u(t)$ with rest initial conditions. Transforming both sides gives $p(s)V(s) = (s+3)V(s) = \dfrac{1}{s}$, so $V(s) = \dfrac{1}{s(s+3)} = \tfrac{1}{3}\left(\dfrac{1}{s} - \dfrac{1}{s+3}\right)$. It follows that $\boxed{v(t) = \tfrac{1}{3}(1 - e^{-3t})}$.

Example 2: Find the unit impulse response for the operator $p(D) = D^2 + \omega^2$ where ω is a given positive constant (natural frequency for a harmonic oscillator).

Solution: For the unit impulse response we solve $\ddot{w} + \omega^2 w = \delta(t)$ with rest initial conditions. Transforming both sides gives $p(s)W(s) = (s^2 + \omega^2)W(s) = 1$, so $W(s) = \dfrac{1}{p(s)} = \dfrac{1}{s^2 + \omega^2}$. Adjusting the coefficients to write this as $W(s) = \dfrac{1}{\omega}\left(\dfrac{\omega}{s^2 + \omega^2}\right)$ we deduce from our table of transforms that $\boxed{w(t) = \tfrac{1}{\omega}\sin(\omega t)}$.

ZIR + ZSR

Given an *n*-th order linear ODE $p(D)[x(t)] = f(t)$ with initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$, …, $x^{(n-1)}(t_0) = x_0^{(n-1)}$, we refer to the case where $x(t_0) = 0$ and $\dot{x}(t_0) = 0$, …, $x^{(n-1)}(t_0) = 0$ as the **zero state**. If we solve $p(D)[x(t)] = f(t)$ for the zero state, we refer to this solution $x_{ZSR}(t)$ as the **zero state response (ZSR)**.

If we seek homogeneous solutions to the ODE $p(D)[x(t)] = 0$ with initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$, …, $x^{(n-1)}(t_0) = x_0^{(n-1)}$, this will have a unique solution $x_{ZIR}(t)$ called the **zero input response (ZIR)**.

The general solution to the ODE $p(D)[x(t)] = f(t)$ will be $x(t) = x_h(t) + x_p(t)$ for some particular solution $x_p(t)$ and homogeneous solutions $x_h(t)$, and we would then use the initial conditions to determine any unknown coefficients. However, note that the zero state response (ZSR) is a particular solution and the zero input response is a (single) homogeneous solution. If we let $x(t) = x_{ZIR}(t) + x_{ZSR}(t)$, note that:

$$\left\{ \begin{array}{c} x(t_0) = x_{ZIR}(t_0) + x_{ZSR}(t_0) = x_{ZIR}(t_0) + 0 = x_{ZIR}(t_0) = x_0 \\ \dot{x}(t_0) = \dot{x}_{ZIR}(t_0) + \dot{x}_{ZSR}(t_0) = \dot{x}_{ZIR}(t_0) + 0 = \dot{x}_{ZIR}(t_0) = \dot{x}_0 \\ \vdots \\ x^{(n-1)}(t_0) = x_{ZIR}{}^{(n-1)}(t_0) + x_{ZSR}{}^{(n-1)}(t_0) = x_{ZIR}{}^{(n-1)}(t_0) + 0 = x_{ZIR}{}^{(n-1)}(t_0) = x_0{}^{(n-1)} \end{array} \right\}$$

so $x(t) = x_h(t) + x_p(t)$ satisfies the initial value problem (IVP) without the need to introduce any additional constants. That is, $x(t) = \mathbf{ZIR} + \mathbf{ZSR}$.

This observation is very helpful when solving initial value problems using Laplace Transform methods – specifically when we use the Unit Impulse Response together with ***convolution*** to solve for the zero state response (ZSR). More on that later.

Example 3: Solve the IVP $\frac{dx}{dt} + 3x = 3\cos 2t$ with initial value $x(0-) = 2$ (the $0-$ is just for emphasis here).

Solution: First, it should be emphasized that for a problem like this our previous methods work well and there is no particular need to use Laplace transform methods. That said, we proceed with two different approaches.

Laplace Direct: For this we simply transform both sides of the equation mindful of the need to incorporate the initial condition as we transform the derivative. This gives:

$sX(s) - 2 + 3X(s) = (s+3)X(s) - 2 = \dfrac{3s}{s^2+4}$, so $(s+3)X(s) = 2 + \dfrac{3s}{s^2+4} = \dfrac{2s^2+3s+8}{s^2+4}$.

Therefore $X(s) = \dfrac{2s^2+3s+8}{(s+3)(s^2+4)} = \dfrac{A}{s+3} + \dfrac{Bs+C}{s^2+4}$.

Clearing fractions gives $2s^2 + 3s + 8 = A(s^2+4) + (s+3)(Bs+C)$

There are several good ways to proceed. First, if we choose convenient points we might first choose $s = -3$ to quickly conclude that $17 = 13A$, so $\boxed{A = \frac{17}{13}}$. You might think the well has run dry, but we are free to use complex numbers. If we choose $s = 2i$ (and as we'll see we won't even have to separately consider its complex conjugate) we get $-8 + 6i + 8 = 6i = (3+2i)(2Bi + C) = (-4B + 3C) + i(6B + 2C)$. We can equate both real and imaginary parts to conclude that $-4B + 3C = 0$ and $6B + 2C = 6$. These give $\boxed{B = \frac{9}{13}}$ and $\boxed{C = \frac{12}{13}}$.

Thus $\boxed{X(s) = \frac{17}{13}\left(\dfrac{1}{s+3}\right) + \frac{9}{13}\left(\dfrac{s}{s^2+4}\right) + \frac{6}{13}\left(\dfrac{2}{s^2+4}\right)}$. So $\boxed{x(t) = \frac{17}{13}e^{-3t} + \frac{9}{13}\cos 2t + \frac{6}{13}\sin 2t}$.

Alternatively, we could simply multiply out and collect terms to get $2s^2 + 3s + 8 = (A+B)s^2 + (3B+C)s + (4A+3C)$ and then use your favorite linear algebra method to derive the same results as above.

ZSR+ZIR (not really recommended here): If we first solve $\frac{dx}{dt} + 3x = 3\cos 2t$ <u>with rest initial conditions</u> we get

$(s+3)X(s) = \dfrac{3s}{s^2+4}$ and $X(s) = \dfrac{3s}{(s+3)(s^2+4)} = -\frac{9}{13}\left(\dfrac{1}{s+3}\right) + \frac{9}{13}\left(\dfrac{s}{s^2+4}\right) + \frac{6}{13}\left(\dfrac{2}{s^2+4}\right)$. So

$\boxed{x_{ZSR}(t) = -\frac{9}{13}e^{-3t} + \frac{9}{13}\cos 2t + \frac{6}{13}\sin 2t}$. Next we seek the zero input response, so we solve $\frac{dx}{dt} + 3x = 0$ with $x(0) = 2$. This quickly gives $\boxed{x_{ZIR}(t) = 2e^{-3t}}$. Combining these gives $\boxed{x(t) = \frac{17}{13}e^{-3t} + \frac{9}{13}\cos 2t + \frac{6}{13}\sin 2t}$.

Some More Calculations (continuing where we left off)

10) ***t*-shift rule**: $\mathcal{L}[f(t-a)] = e^{-as}F(s)$ if $a \geq 0$ and $f(t) = 0$ for $t < 0$.

This may also be expressed as $\mathcal{L}[f_a(t)] = e^{-as}F(s)$ where $f_a(t) = u(t-a)f(t-a) = \begin{Bmatrix} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{Bmatrix}$.

Starting with the definition $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} e^{-st} f(t)dt$, we make the substitution $u = t-a$, $du = dt$ and change the limits on the integral to get:

$$\mathcal{L}[f(t-a)] = \int_{0-}^{\infty} e^{-st} f(t-a)dt = \int_{a-}^{\infty} e^{-st} f(t-a)dt = \int_{0-}^{\infty} e^{-s(u+a)} f(u)du$$

$$= \int_{0-}^{\infty} e^{-as}e^{-su} f(u)du = e^{-as}\int_{0-}^{\infty} e^{-su} f(u)du = e^{-as}F(s)$$

11) $\mathcal{L}[\delta(t-a)] = \mathcal{L}[\delta_a(t)] = e^{-as}$

We calculate $\mathcal{L}[\delta(t-a)] = \int_{0-}^{\infty} e^{-st}\delta(t-a)dt = e^{-st}\Big|_{t=a} = e^{-as}$

This also follows immediately from the *t*-shift rule and the fact that $\mathcal{L}[\delta(t)] = 1$.

12) $\mathcal{L}[u(t-a)] = \mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$

This follows immediately from the *t*-shift rule and the fact that $\mathcal{L}[1] = \mathcal{L}[u(t)] = \frac{1}{s}$.

13) $\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$

This follows by repeated application of the *s*-derivative rule: $\mathcal{L}[t f(t)] = -F'(s)$.

14) $\boxed{\mathcal{L}[e^{at}\cos(\omega t)] = \frac{s-a}{(s-a)^2 + \omega^2}}$ and $\boxed{\mathcal{L}[e^{at}\sin(\omega t)] = \frac{\omega}{(s-a)^2 + \omega^2}}$

These follow immediately from the **s-shift rule**, $\mathcal{L}[e^{rt} f(t)] = F(s-r)$ together with the transforms for the cosine and sine functions, i.e. $\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2 + \omega^2}$ and $\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2}$.

15) $\mathcal{L}[t\cos(\omega t)] = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$ and $\mathcal{L}[t\sin(\omega t)] = \frac{2\omega s}{(s^2 + \omega^2)^2}$

These both follow immediately from the *s*-derivative rule: $\mathcal{L}[t f(t)] = -F'(s)$ by differentiating the transforms of the sine and cosine functions, i.e. $\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2 + \omega^2}$ and $\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2}$.

Convolution

Situation: We need to solve a differential equation of the form $p(D)[x(t)] = f(t)$ with initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, etc.

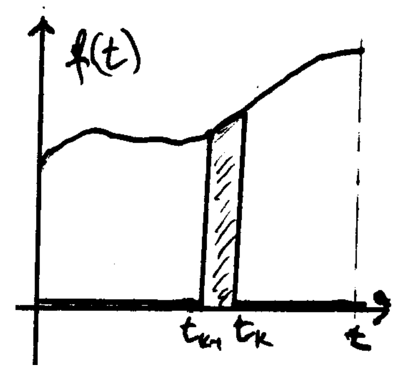
Plan: If we can find the unit impulse response for this system (with rest initial conditions), i.e. the solution to $p(D)[x(t)] = \delta(t)$ with initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, etc., we will develop a method for finding a solution to $p(D)[x(t)] = f(t)$ by thinking of $f(t)$ as a "train of impulses." We will likely use Laplace transform methods to find the unit impulse response. We'll use Riemann Sums ideas to construct an integral by piecing together solutions associated with the impulses. This will be the **convolution** integral.

Developing the convolution integral

(1) We start by solving for the unit impulse function $w(t)$, i.e. the solution to $p(D)[x(t)] = \delta(t)$ with initial conditions $x(0) = 0,\ \dot{x}(0) = 0$, etc. We refer to this by $w(t)$ and reserve $x(t)$ for the solution to $p(D)[x(t)] = f(t)$. If $W(s)$ is the Laplace transform of $w(t)$, we'll have $p(s)W(s) = 1$ where $p(s)$ is the characteristic polynomial, so $W(s) = \dfrac{1}{p(s)}$ and with some partial fractions calculations and the inverse Laplace transform, finding $w(t)$ can be reduced to a relatively simple routine.
We call $w(t)$ the ***weight function***, and we call $W(s)$ the ***transfer function***.

(2) We use **time invariance** to declare that the translated unit impulse response for $p(D)[x(t)] = \delta(t - t_k)$ will be $w(t - t_k)$. We'll use this in the integral to follow.

(3) If we're interested in understanding what's happening during the time interval $[0,t]$, we start by partitioning this interval into many small subintervals $[t_{k-1}, t_k]$. On each of these subintervals, we can use a box function to "switch on" just one small section of the function $f(t)$. That is, if we let $f_k(t) = f(t)[u(t - t_{k-1}) - u(t - t_k)]$ then this function will be identically zero except in the *k*-th subinterval $[t_{k-1}, t_k]$. We can later reassemble the function as $f(t) = \sum f_k(t)$.

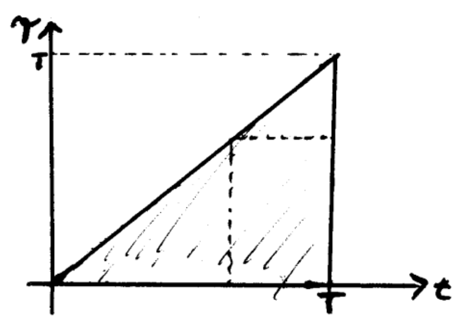


If $f(t)$ is reasonably well behaved (except, perhaps, at finitely many points), we can say that within a given subinterval, $f_k(t) \cong f(t_k) = \int_0^{+\infty} f(t)\delta(t - t_k)dt = \int_{-\infty}^{+\infty} f(t)\delta(t - t_k)dt$, where we use the fact that evaluation of a function at a point is accomplished by integrating against a delta function concentrated at that point.

(4) If the solution to $p(D)[x(t)] = \delta(t - t_k)$ is $w(t - t_k)$, then linearity gives that the solution to $p(D)[x(t)] = f(t_k)\delta(t - t_k)\Delta t_k$ will be $f(t_k)w(t - t_k)\Delta t_k$ where $\Delta t_k = t_k - t_{k-1}$ is the width of the *k*-th subinterval. [We simply multiplied both sides by $f(t_k)\Delta t_k$.]

(5) By linearity (superposition), we can sum to get that the solution to $p(D)[x(t)] \cong \sum_k f(t_k)\delta(t - t_k)\Delta t_k$ must therefore be $x(t) \cong \sum_k f(t_k)w(t - t_k)\Delta t_k = \sum_k f(t_k)w(t - \tau_k)\Delta \tau_k$, where we changed to the variable τ in anticipation of the next step.

(6) If we pass to the limit as the norm of the partition goes to zero, the sum will become an integral and the approximation will become exact, i.e. $x(t) = \int_0^t f(\tau)w(t - \tau)d\tau \equiv (f * w)(t)$, the convolution integral. This provides a solution to $p(D)[x(t)] = f(t)$ for the zero state (rest conditions), and we refer to this solution as the **zero state response (ZSR)**.



Definition (Convolution): Given two functions $w(t)$ and $f(t)$, we define $\boxed{(f * w)(t) = \int_0^t f(\tau)w(t - \tau)d\tau}$.

It's a straightforward exercise to show that the convolution product is commutative, i.e. $f * w = w * f$.

Even More Calculations (continuing where we left off)

16) ***Convolution***: $(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau=\int_0^t g(\tau)f(t-\tau)d\tau=(g*f)(t)$, i.e. convolution is commutative.

This follows by substitution. If we let $u=t-\tau$ (u and τ are the variables here), then $du=-d\tau$ and when we also adjust the integral limits we get:

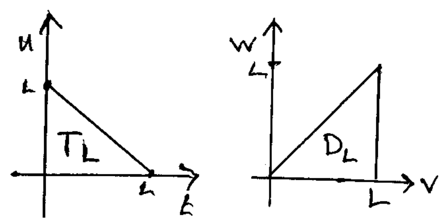
$$(f*g)(t)=\int_{\tau=0}^{\tau=t} f(\tau)g(t-\tau)d\tau=-\int_{u=t}^{u=0} f(t-u)g(u)du=\int_{u=0}^{u=t} g(u)f(t-u)du=(g*f)(t)$$

17) If $\mathcal{L}[f(t)]=F(s)$ and $\mathcal{L}[g(t)]=G(s)$, then $\mathcal{L}[(f*g)(t)]=F(s)G(s)$

The proof of this is a bit complicated in that we have to change variables (and use Jacobian determinants), interpret the domain of integration, and deal with the fact that these are improper integrals. It's easiest if we start with the right-hand side:

$$F(s)G(s)=\left(\int_0^\infty e^{-st}f(t)dt\right)\left(\int_0^\infty e^{-su}g(u)du\right)=\lim_{L\to\infty}\left\{\left(\int_0^L e^{-st}f(t)dt\right)\left(\int_0^L e^{-su}g(u)du\right)\right\}=\lim_{L\to\infty}\left[\iint_{R_L} e^{-s(t+u)}f(t)g(u)dtdu\right]$$

where R_L is the (large) $L\times L$ rectangle with $0\le t\le L$, $0\le u\le L$. Because we will eventually be letting $L\to\infty$, we will get the same result if we instead use the triangular domain T_L (see diagram) bounded by the lines $t=0$, $u=0$, $t+u=L$. That is, $F(s)G(s)=\lim_{L\to\infty}\left[\iint_{T_L} e^{-s(t+u)}f(t)g(u)dtdu\right]$.



If we now make the change of variables $\left\{\begin{matrix} v=t+u \\ w=u \end{matrix}\right\}$ with inverse transformation $\left\{\begin{matrix} t=v-w \\ u=w \end{matrix}\right\}$, the new domain will be D_L (see diagram) bounded by the lines $w=0$, $v=L$, $w=v$. The Jacobian determinant we'll need is $\dfrac{\partial(t,u)}{\partial(v,w)}=\det\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}=1$. Thus, by the Change of Variable Theorem for multiple integrals:

$$\iint_{T_L} e^{-s(t+u)}f(t)g(u)dtdu=\iint_{D_L} e^{-sv}f(v-w)g(w)\cdot 1\cdot dwdv$$

$$=\int_{v=0}^{v=L}\int_{w=0}^{w=v} e^{-sv}f(v-w)g(w)dwdv=\int_{v=0}^{v=L} e^{-sv}\left[\int_{w=0}^{w=v} f(v-w)g(w)dw\right]dv$$

Finally, if we now let $L\to\infty$, we get

$\int_0^\infty e^{-sv}\left[\int_0^\infty f(v-w)g(w)dw\right]dv=\int_0^\infty e^{-sv}\left[(f*g)(v)\right]dv=\left[\mathcal{L}\left[f*g\right]\right](s)$, so $F(s)G(s)=\left[\mathcal{L}\left[f*g\right]\right](s)$.

Note: The fact that the solution to the differential equation $p(D)x=f(t)$ will have the solution $(f*w)(t)$ is also known as ***Green's Formula***.

Note: When applying the convolution method to solving $p(D)x=f(t)$ for more general initial conditions, the solution will be $x(t)=\textbf{ZIR}+\textbf{ZSR}$, where **ZIR** is the **zero input response** and **ZSR** is the **zero state response**.

The usefulness of this transform method is built on the fact that we can relatively easily find the Laplace transform for most everything that appears in a given differential equation of the form $p(D)x=f(t)$, and once we have a table of these transforms we can generally invert the process by inspection. Another essential fact is that the Laplace transform acts linearly, and this allows us to decompose complex problems into a sums of simple problems.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} e^{-st} f(t)dt$ for $\mathrm{Re}(s) \gg 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{rt} f(t)] = F(s-r)$.

4. t-shift rule: $\mathcal{L}[f(t-a)] = e^{-as} F(s)$ if $a \ge 0$ and $f(t) = 0$ for $t < 0$.

5. s-derivative rule: $\mathcal{L}[t f(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0-)$ $\quad$ $\mathcal{L}[f''(t)] = s^2 F(s) - sf(0-) - f'(0-)$

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0-) - s^{n-2} f'(0-) - \cdots - f^{(n-1)}(0-)$$

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $\quad (f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s)$, $w(t)$ the unit impulse response.

   If $q(t)$ is regarded as the input signal in $p(D)x = q(t)$, $W(s) = \frac{1}{p(s)}$.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s}$$

$$\mathcal{L}[\delta(t)] = 1$$

$$\mathcal{L}[\delta(t-a)] = \mathcal{L}[\delta_a(t)] = e^{-as}$$

$$\mathcal{L}[u(t-a)] = \mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

$$\mathcal{L}[t] = \frac{1}{s^2}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$$

$$\mathcal{L}[u(t-a) f(t-a)] = e^{-as} F(s)$$

$$\mathcal{L}[u(t-a) f(t)] = e^{-as} \mathcal{L}[f(t+a)]$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}[t\cos(\omega t)] = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$$

$$\mathcal{L}[t\sin(\omega t)] = \frac{2\omega s}{(s^2 + \omega^2)^2}$$

$$\mathcal{L}[e^{zt}\cos(\omega t)] = \frac{s-z}{(s-z)^2 + \omega^2}$$

$$\mathcal{L}[e^{zt}\sin(\omega t)] = \frac{\omega}{(s-z)^2 + \omega^2}$$

Example 4 (with a familiar input): Solve $\ddot{x}+3\dot{x}+2x=2e^{-t}$, $\quad x(0)=0, \quad \dot{x}(0)=0$ using (a) previous methods, (b) using only the Laplace transform, and (c) using the Laplace transform plus convolution.

"Old-Fashioned" Solution: **(a)** For the homogeneous equation $\ddot{x}+3\dot{x}+2x=0$, the characteristic polynomial is $p(s)=s^2+3s+2=(s+2)(s+1)$ which yields the two roots $s=-2$ and $s=-1$. This gives the two independent solutions e^{-2t} and e^{-t}, and all homogeneous solutions are of the form $x_h(t)=c_1e^{-2t}+c_2e^{-t}$.

In seeking a particular solution $x_p(t)$ that satisfies the inhomogeneous differential equation, we see that the Exponential Response Formula (ERF) won't work – there is resonance. We can, however, use the Resonant Response Formula to get the particular solution $x_p(t)=\dfrac{2te^{-t}}{p'(-1)}=\dfrac{2te^{-t}}{1}=2te^{-t}$, so the general solution is $x(t)=x_h(t)+x_p(t)=c_1e^{-2t}+c_2e^{-t}+2te^{-t}$. Its derivative is $\dot{x}(t)=-2c_1e^{-2t}-c_2e^{-t}-2te^{-t}+2e^{-t}$. Substituting the (rest) initial conditions gives $\left\{\begin{matrix} x(0)=c_1+c_2=0 \\ \dot{x}(0)=-2c_1-c_2+2=0 \end{matrix}\right\}$, and these can be solved to give $c_1=2, c_2=-2$, so the solution is $\boxed{x(t)=2e^{-2t}-2e^{-t}+2te^{-t}}$.

Direct Laplace Transform Solution (b) We need the following Laplace transforms:

(1) $\mathcal{L}(e^{kt})=\dfrac{1}{s-k}$ with region of convergence $\text{Re}(s)>k$, so $\mathcal{L}(e^{-2t})=\dfrac{1}{s+2}$.

(2) If the Laplace transform of $x(t)$ is $X(s)$, then the Laplace transforms of its derivatives are $\mathcal{L}(\dot{x}(t))=sX(s)-x(0-)$ and $\mathcal{L}(\ddot{x}(t))=s^2X(s)-s\,x(0-)-\dot{x}(0-)$. We have rest initial conditions, so these are greatly simplified and, in fact, $\mathcal{L}(p(D)x)=p(s)X(s)$. Specifically, $\mathcal{L}(\ddot{x}+3\dot{x}+2x)=s^2X(s)+3sX(s)+2X(s)=(s^2+3s+2)X(s)=p(s)X(s)$.

If we now transform the entire differential equation, we get $(s^2+3s+2)X(s)=\dfrac{2}{s+1}$.

We then solve for $X(s)=\dfrac{2}{(s+1)(s^2+3s+2)}=\dfrac{2}{(s+2)(s+1)^2}=\dfrac{A}{s+2}+\dfrac{B}{s+1}+\dfrac{C}{(s+1)^2}$.

There are many good ways to find the unknowns *A*, *B*, and *C*. For example, if we multiply through by the common denominator to clear fractions, we get $2=A(s+1)^2+B(s+1)(s+2)+C(s+2)$. Plugging in the specific values $s=-2$ and $s=-1$ quickly yields that $A=2$ and $C=2$. Plugging in, for example, $s=0$ and using the values for *A* and *C* then yields $B=-2$. So $X(s)=\dfrac{2}{s+2}-\dfrac{2}{s+1}+\dfrac{2}{(s+1)^2}$.

Consulting our table of common Laplace transforms, we see that $\dfrac{2}{s+2}=\mathcal{L}(2e^{-2t})$, $\dfrac{2}{s+1}=\mathcal{L}(2e^{-t})$, and $\dfrac{2}{(s+1)^2}=\mathcal{L}(2te^{-t})$, so transforming back gives $\boxed{x(t)=2e^{-2t}-2e^{-t}+2te^{-t}}$.

Solution using Laplace Transform with Convolution (c) We start by finding the unit impulse response, a the solution to $\ddot{x}+3\dot{x}+2x=\delta(t)$ with rest initial conditions $x(0)=0, \dot{x}(0)=0$. Laplace transform gives $p(s)W(s)=1$, so $W(s)=\dfrac{1}{(s+2)(s+1)}=-\dfrac{1}{s+2}+\dfrac{1}{s+1}$. Consulting the Laplace transform table, this yields the <u>weight function</u> $w(t)=\left\{\begin{matrix} 0 & t<0 \\ -e^{-2t}+e^{-t} & t>0 \end{matrix}\right\}=u(t)(-e^{-2t}+e^{-t})$.

With $f(t)=2e^{-t}$, then convolution of the weight function and the given input signal gives:

$$(w*f)(t)=\int_{\tau=0}^{\tau=t} w(\tau)f(t-\tau)d\tau=\int_{\tau=0}^{\tau=t}(-e^{-2\tau}+e^{-\tau})2e^{-(t-\tau)}d\tau$$
$$=\int_{\tau=0}^{\tau=t}2e^{-t}(-e^{-\tau}+1)d\tau=2e^{-t}\left[e^{-\tau}+\tau\right]_{\tau=0}^{\tau=t}=2e^{-t}[e^{-t}-1+t]$$
$$=\boxed{+2e^{-2t}-2e^{-t}+2te^{-t}=x(t)}$$

Alternatively, we could have calculated this as:

$$(f*w)(t)=\int_{\tau=0}^{\tau=t} f(\tau)w(t-\tau)d\tau=\int_{\tau=0}^{\tau=t}2e^{-\tau}(-e^{-2(t-\tau)}+e^{-(t-\tau)})d\tau$$
$$=\int_{\tau=0}^{\tau=t}(-2e^{-2t}e^{\tau}+2e^{-t})d\tau=-2e^{-2t}(e^{t}-1)+2e^{-t}(t-0)=\boxed{+2e^{-2t}-2e^{-t}+2te^{-t}=x(t)}$$

Example 5: Solve the same ODE as above but with non-rest initial conditions:

$$\ddot{x}+3\dot{x}+2x=2e^{-t},\quad x(0)=4,\quad \dot{x}(0)=0$$

Solution: All of the previous steps are the same in deriving the **zero state response (ZSR)**, so we have $\mathbf{ZSR}=+2e^{-2t}-2e^{-t}+2te^{-t}$.

We need only find the **zero input response (ZIR)**. This simply means that we solve $\ddot{x}+3\dot{x}+2x=0$ to get $x_h(t)=c_1e^{-2t}+c_2e^{-t}$ and $\dot{x}_h(t)=-2c_1e^{-2t}-c_2e^{-t}$, so $\left\{\begin{matrix} x_h(0)=c_1+c_2=4 \\ \dot{x}_h(0)=-2c_1-c_2=0 \end{matrix}\right\} \Rightarrow c_1=-4,\ c_2=8$.

So $\mathbf{ZIR}=-4e^{-2t}+8e^{-t}$. Therefore $x(t)=\mathbf{ZSR}+\mathbf{ZIR}=+2e^{-2t}-2e^{-t}+2te^{-t}-4e^{-2t}+8e^{-t}=\boxed{-2e^{-2t}+6e^{-t}+2te^{-t}}$.

Vector fields, Continuous Dynamical Systems, and Systems of 1st Order Linear Differential Equations

Definition: A **vector field** in $\mathbf{R}^n$ is an assignment of a vector to every point in $\mathbf{R}^n$ (with the possible exception of some singular points). This can be viewed as a function $\mathbf{F}(x_1,\cdots,x_n)=\begin{bmatrix} f_1(x_1,\cdots,x_n) \\ \vdots \\ f_n(x_1,\cdots,x_n)\end{bmatrix}$ where $f_i(x_1,\cdots,x_n)$ is the *i*-th component of the vector assigned to the point $(x_1,\cdots,x_n)$. We can also write this more succinctly as $\mathbf{F}(\mathbf{x})=\begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x})\end{bmatrix}$. In practice we usually assume some reasonable properties such as that the component functions are continuous or differentiable except perhaps at a finite number of *singular points*. This is actually an ***autonomous vector field*** in the sense that the vector field is static, i.e. it does not change in time.

We can also consider ***nonautonomous vector fields***, i.e. $\mathbf{F}(x_1,\cdots,x_n,t)=\begin{bmatrix} f_1(x_1,\cdots,x_n,t) \\ \vdots \\ f_n(x_1,\cdots,x_n,t)\end{bmatrix}$ where the vector assigned to any given point also depends explicitly on (the time) *t*. We will, for the most part, consider only the nonautonomous case.

If we view the vector assigned to each point as a ***velocity vector*** associated with some smoothly varying system, a reasonable question to ask is this: Given a starting point $\mathbf{x}_0$ (the initial condition), can we find a parameterized curve $\mathbf{x}(t)$ such that $\mathbf{x}(0)=\mathbf{x}_0$ and the velocity vector at any point on this parameterized curve matches the underlying vector field, i.e. $\frac{d\mathbf{x}}{dt}=\mathbf{F}(\mathbf{x}(t))$. This is equivalent to a system of (time-independent) first-order differential equations, i.e. $\left\{\begin{matrix} \frac{dx_1}{dt} = f_1(x_1,\cdots,x_n) \\ \vdots \\ \frac{dx_n}{dt} = f_n(x_1,\cdots,x_n)\end{matrix}\right\}$. We are interested in knowing how a system defined in this way evolves over time for any given initial condition. This describes what we call a **continuous dynamical system**. We call the set of all such solution curves the **flow** of the dynamical system.

If you imagine a vector field as describing a flowing liquid, then these parameterized curves simply describe what happens if you drop a particle into the flow and see where it goes as it carried by the flow. This is a good way to think about a continuous dynamical system even when the variables are describing such things as populations or economic variables rather than geometric coordinates. We'll still refer to the solutions as the flow of the system even though there's nothing physical about this flow.

We are typically interested in the long-term behavior of such a system, but we often would also like to predict exactly where the particle will be after a specified time *t*, i.e. formulas for how the component functions evolve in time. In general, if the component functions of the underlying vector field are nonlinear, it's very difficult to find a tidy formula for how the system evolves over time. The linear case, on the other hand, is completely solvable using matrix methods.

Reduction of Order

A homogeneous *n*th order linear ordinary differential equation can easily be represented as a system of 1st order ordinary differential equations simply by assigning names to the derivatives up to order $n-1$. For example, for the linear, time-invariant, 2nd order ODE $\ddot{x}+3\dot{x}+2x=0$, we can simply write $y=\dot{x}=\frac{dx}{dt}$.

$$\frac{dy}{dt} = \ddot{x} = -2x - 3\dot{x} = -2x - 3y \text{. So } \begin{Bmatrix} \frac{dx}{dt} = \quad y \\ \frac{dy}{dt} = -2x - 3y \end{Bmatrix}.$$

For the 3rd order ODE $\dddot{x} + 2\ddot{x} - 4\dot{x} + 7x = 0$, we can let $y = \dot{x}$ and $z = \ddot{x}$.

$$\frac{dz}{dt} = \dddot{x} = -7x + 4\dot{x} - 2\ddot{x} = -7x + 4y - 2z \text{, so } \begin{Bmatrix} \frac{dx}{dt} = \quad y \\ \frac{dy}{dt} = \quad z \\ \frac{dz}{dt} = -7x + 4y - 2z \end{Bmatrix}.$$

Reduction of order is applicable for any linear ODE, and not just for the constant coefficient case or even the homogeneous case. For example, for the ODE $\ddot{x} + 3t\dot{x} + (2 - t^2)x = e^{-t}$ we can let $y = \dot{x} = \frac{dx}{dt}$ as before and turn this 2nd order ODE into the system $\begin{Bmatrix} \frac{dx}{dt} = \quad y \\ \frac{dy}{dt} = (t^2 - 2)x - 3ty + e^{-t} \end{Bmatrix}$ with an underlying nonautonomous (time-varying) vector field.

Even if, in theory, solutions exist for some complicated autonomous and even nonautonomous systems, we may not be able to express these solutions in terms of elementary functions and formulas. There is at least one category for which, as we'll see, it is always possible to find explicit representations of solutions. These are the autonomous systems where the component functions defining the underlying vector field are linear.

Definition: A ***linear continuous dynamical system*** is a system of first-order differential equations of the form $\begin{Bmatrix} \frac{dx_1}{dt} = a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ \frac{dx_n}{dt} = a_{n1}x_1 + \cdots + a_{nn}x_n \end{Bmatrix}$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{11} \\ \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{11} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{11} \\ \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{11} \end{bmatrix}$ is the matrix of coefficients. That is, $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where **A** is an $n \times n$ real matrix.

Situation: You want to solve a system of first-order linear differential equations of the form $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ given some initial condition $\mathbf{x}(0) = \mathbf{x}_0$. How is this most efficiently accomplished?

Example 1: The simplest linear continuous dynamical system is the single equation $\frac{dx}{dt} = kx$ with initial condition $x(0) = x_0$. This is something we solved in basic calculus and yields exponential growth or decay (depending on whether $k > 0$ or $k < 0$). Specifically, we write $\frac{1}{x}\frac{dx}{dt} = k$ and integrate both sides to get $\ln|x(t)| = kt + c$ for some arbitrary constant *c*. [Many people choose to do this calculation as $\frac{dx}{x} = kdt$ and integrate both sides to get $\int \frac{dx}{x} = \int kdt \quad \Rightarrow \quad \ln|x| = kt + c$.] In any case, exponentiating both sides gives $|x(t)| = e^{kt+c} = e^c e^{kt} = ae^{kt}$, and we can remove the absolute value by allowing the constant *a* to be either positive or negative, so we get $x(t) = ae^{kt}$. Using the initial condition $x(0) = x_0$ we see that $x(0) = a = x_0$, so the solution is $\boxed{x(t) = x_0 e^{kt}}$.

Uncoupled systems: We call a system uncoupled (or unlinked) if the rates of change of each of the variables do not depend on any of the other variables. In the linear case, this would mean a system of the form $\left\{\begin{matrix} \frac{dx_1}{dt} = k_1 x_1 \\ \vdots \\ \frac{dx_n}{dt} = k_n x_n \end{matrix}\right\}$ with initial conditions $x_1(0), \ldots, x_n(0)$. Note that such a system can be expressed in matrix form as $\frac{d\mathbf{x}}{dt} = \mathbf{Dx}$ where **D** is the diagonal matrix $\mathbf{D} = \begin{bmatrix} k_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & k_n \end{bmatrix}$. Solving this system is nothing more than solving the previous problem repeatedly with different rate constants and corresponding initial conditions. We get the solution $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} x_1(0)e^{k_1 t} \\ \vdots \\ x_n(0)e^{k_n t} \end{bmatrix} = \begin{bmatrix} e^{k_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{k_n t} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$. Note that when $t = 0$ this matrix is just the identity matrix which simply reflects the fact that $t = 0$ corresponds to the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$. Of greater interest is the fact that this time-varying matrix evolves over time to produce the flow emanating from any given initial condition. It is for this reason that we refer to this matrix as the **evolution matrix** for this uncoupled system. If we refer to this matrix as $[e^{t\mathbf{D}}]$, a notation that is perhaps best not taken too literally, then the system $\frac{d\mathbf{x}}{dt} = \mathbf{Dx}$ with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ has solution $\mathbf{x}(t) = [e^{t\mathbf{D}}]\mathbf{x}(0)$.

A **coupled system**, i.e. a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$ where the matrix A is not diagonal, can often be solved by changing coordinates so that relative to some new basis (of eigenvectors) the system has a diagonal matrix. The tool at the heart of these methods is diagonalization or, in the case where a matrix cannot be diagonalized, finding an appropriate change of basis relative to which the underlying linear transformation has the simplest possible matrix representation (Jordan Canonical Form). The introduction of corresponding "evolution matrices" is a useful formalism for handling these general cases.

In order to properly understand this we'll have to review some basic ideas from Linear Algebra – specifically the idea or a basis, coordinates relative to a basis, changing bases, linear transformations, and the matrix of a linear transformation relative to a basis.

More on this topic in the next lecture, but here's **a preview**:

Solving systems using diagonalization and evolution matrices

Given an $n \times n$ matrix **A**, suppose **S** is a change of basis matrix corresponding to either diagonalization or reduction to Jordan Canonical Form. We will have $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{B}$ in this case, where **B** is diagonal or otherwise in simplest form. We then calculate $\mathbf{A} = \mathbf{SBS}^{-1}$, and substitution gives $\frac{d\mathbf{x}}{dt} = \mathbf{SBS}^{-1}\mathbf{x}$.

Multiplying on the left by $\mathbf{S}^{-1}$ and using the basic calculus fact that $\frac{d}{dt}(\mathbf{Mx}) = \mathbf{M}\frac{d\mathbf{x}}{dt}$ for any (constant) matrix **M**, we have $\mathbf{S}^{-1}\frac{d\mathbf{x}}{dt} = \frac{d(\mathbf{S}^{-1}\mathbf{x})}{dt} = \mathbf{B}(\mathbf{S}^{-1}\mathbf{x})$. If we write $\mathbf{u} = \mathbf{S}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathscr{B}}$, where $\mathscr{B}$ is the new, preferred basis, then in these new coordinates the system becomes $\frac{d\mathbf{u}}{dt} = \mathbf{Bu}$, but now the system will be much more straightforward to solve.

The diagonalizable case
In the case where **B** is a diagonal matrix with the eigenvalues of **A** on the diagonal, the system is just

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \mathbf{u} \text{ or } \left\{ \begin{matrix} \dfrac{du_1}{dt} = \lambda_1 u_1 \\ \vdots \\ \dfrac{du_n}{dt} = \lambda_n u_n \end{matrix} \right\}.$$

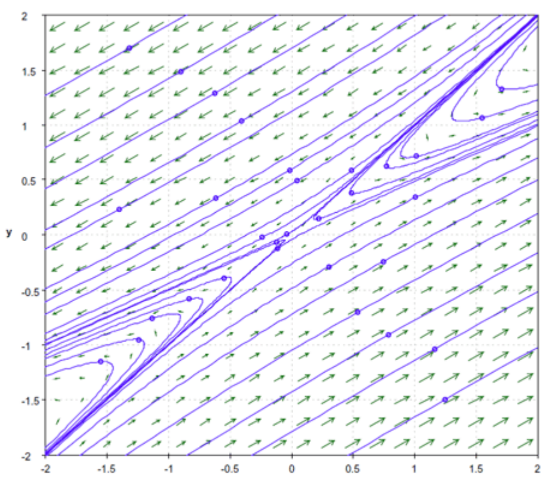
This has the solution $\left\{ \begin{matrix} u_1(t) = e^{\lambda_1 t} u_1(0) \\ \vdots \\ u_n(t) = e^{\lambda_n t} u_n(0) \end{matrix} \right\}$ or $\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} u_1(0) \\ \vdots \\ u_n(0) \end{bmatrix} = [e^{t\mathbf{B}}]\mathbf{u}(0)$.

To revert back to the original coordinates, we write $\mathbf{x} = \mathbf{S}\mathbf{u}$, so $\mathbf{x}(t) = \mathbf{S}\mathbf{u}(t) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{u}(0) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. If we denote the evolution matrix for the system in its original coordinates as $[e^{t\mathbf{A}}]$ where $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$, then the previous calculation gives the simple relation $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$.

In other words, the evolution matrices for the solution are in the same relationship as the matrices **A** and **B**, namely $\mathbf{A} = \mathbf{SBS}^{-1}$. This pattern is very easy to remember, and this same pattern will again be the case where **B** is not diagonal but where the corresponding evolution matrix is still relatively easy to calculate.

$\mathbf{A} = \mathbf{SBS}^{-1} \quad \Rightarrow \quad [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$, and the solution of the original system will be $\boxed{\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)}$.

Problem: Solve the system $\left\{ \begin{matrix} \dfrac{dx}{dt} = 5x - 6y \\ \dfrac{dy}{dt} = 3x - 4y \end{matrix} \right\}$ with initial conditions $x(0) = 3,\ y(0) = 1$.



Solution: In matrix form, we have $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. We start by finding the eigenvalues of the matrix: $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 5 & 6 \\ -3 & \lambda + 4 \end{bmatrix}$, and the characteristic polynomial is $p_{\mathbf{A}}(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$. This gives the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. The first of these gives the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and the second gives the eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So we have $\left\{ \begin{matrix} \mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \end{matrix} \right\}$. The change of basis matrix is $\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and with the new basis (of eigenvectors) $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ we have $[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \mathbf{D}$, a diagonal matrix. [There is no need to carry out the multiplication of the matrices if $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is known to be is a basis of eigenvectors. It will always yield a diagonal matrix with the eigenvalues on the diagonal.]

The evolution matrix for this diagonal matrix is $[e^{t\mathbf{D}}]=\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}$, and the solution of the system is:

$$\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)=\mathbf{S}[e^{t\mathbf{D}}]\mathbf{S}^{-1}\mathbf{x}(0)=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}=\begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$=\begin{bmatrix} 4e^{2t}-e^{-t} \\ 2e^{2t}-e^{-t} \end{bmatrix}=2e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}-e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=2e^{2t}\mathbf{v}_1-e^{-t}\mathbf{v}_2$$

Notes by Robert Winters