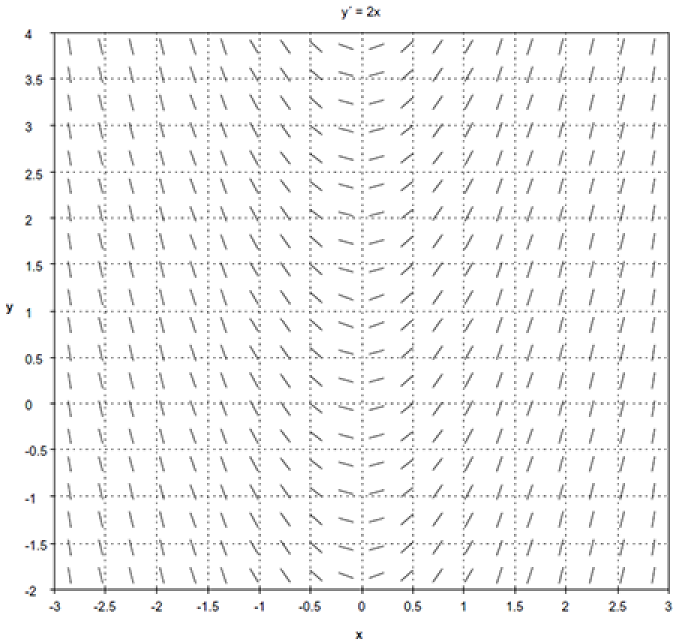
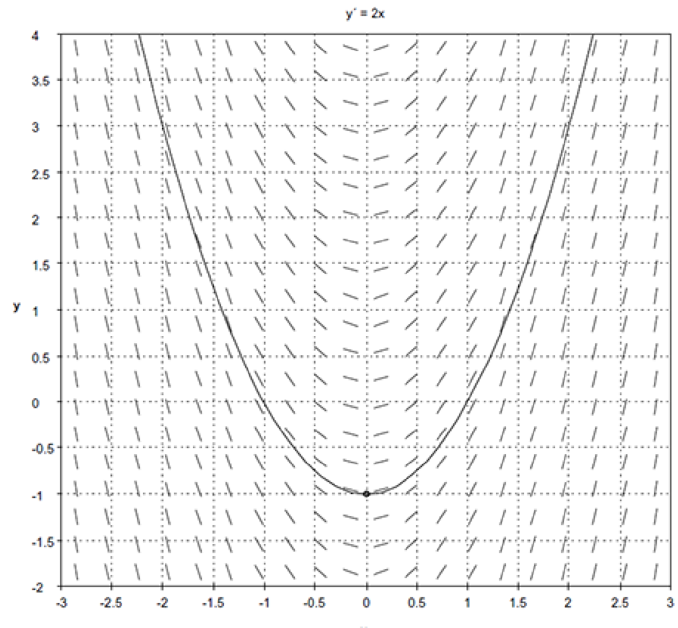
Ordinary Differential Equations – Lecture #1

Basic notions

There are many contexts in which we seek to discover the relationship between two (or more) variables, e.g. $y = y(x)$ or $x = x(t)$. Such relations are often determined by local information – rates that are measurable or which are given by known formulas. This is the essence of an **ordinary differential equation** (ODE).

For example, for the relation $y = x^2 - 1$, it's the case that $\frac{dy}{dx} = 2x$ and with the additional requirement that $y(0) = -1$, this rate information and the initial condition completely determine the relation $y = x^2 - 1$. The derivative statement is a (first order) **differential equation** and the requirement that the graph pass through the point $(0, -1)$ is called an **initial condition**. We can understand this geometrically by looking at the **slope field** – a drawing in the *xy*-plane that indicates the slope at any point as determined by the given differential equation.





The left image shows only the slope field. The right image indicates the unique solution that matches this slope field and the given initial condition. One important theme that we'll explore will be the conditions under which a given differential equation will yield unique solutions for a given initial condition.

Definition: Given a differential equation in the form $\frac{dy}{dx} = F(x, y)$, an **isocline** is a curve along which the slope is constant.

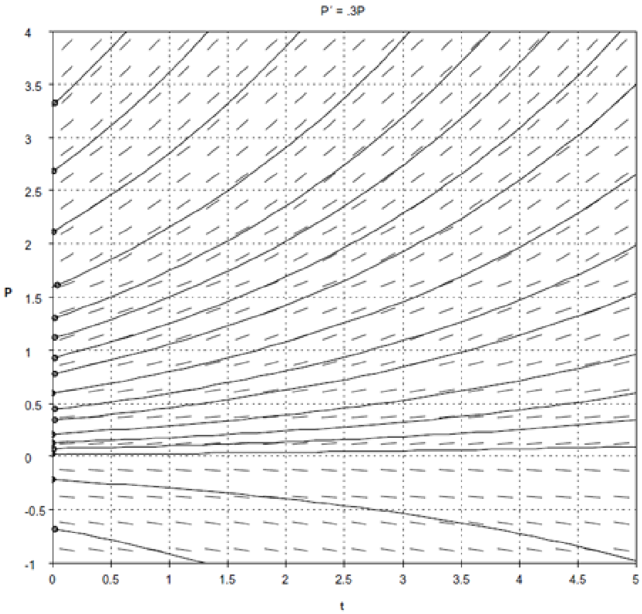
Isoclines are especially useful for drawing slope fields by hand. For any specific slope, we find the corresponding isocline and pencil in little dashes to indicate that slope along the isocline. If we do this for a range of values for the slope, we can usually get a very good indication of the slope field and of the solutions (also called integral curves) it will yield. Note that in the example above, the isoclines were all vertical lines because $\frac{dy}{dx} = 2x$ will be constant where x is constant.

Unrestricted growth

The are many situations from physics to finance in which unrestricted growth at a fixed relative growth rate is the rule. If we express this as a differential equation in how a quantity P grows in time t, the corresponding differential equation may be expressed in terms of a fixed relative growth rate k as $\frac{1}{P}\frac{dP}{dt} = k$ or in terms of absolute growth rate as $\boxed{\frac{dP}{dt} = kP}$. We will presumably also have some initial condition $P(0) = P_0$.

The slope field in this case will have horizontal lines as its isoclines, i.e.

$\frac{dP}{dt} = kP = \text{constant} \quad \Rightarrow \quad P = \text{constant}.$



You have most likely already seen an analytic solution to this differential equation. It is an example of a **separable equation** in which we can algebraically separate the variables and integrate both sides of the equation. That is, we can formally write $\frac{dP}{P} = kdt$ and integrate to get

$$\int \frac{dP}{P} = \int kdt \quad \Rightarrow \quad \ln|P| = kt + c \quad \Rightarrow \quad P = Ae^{kt}$$

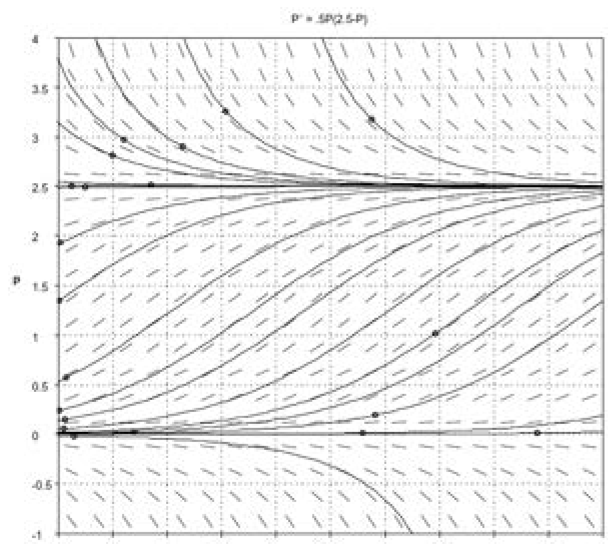
where we've used some basic facts about rules of exponents and absolute value to get the result. Note that this yields exponential growth where the rate *k* appears as the coefficient in the exponent. If we also use the initial condition, we have $P(0) = Ae^0 = A = P_0$, so individual solutions are given by $\boxed{P(t) = P_0 e^{kt}}$. The picture at left indicates the case where $k > 0$ (growth) with several integral curves shown. The $k < 0$ case would give exponential decay.

Differential equations can also involve <u>higher order derivatives</u>. For example, Newton's 2nd Law is usually stated as $F = ma$ where m represents mass, F is the applied force, and a is the acceleration. We know that if x represents position, and v represents velocity, then $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt}$, so Newton's 2nd Law can also be expressed as the **2nd order ordinary differential equation** $\frac{d^2x}{dt^2} = \frac{F}{m} = a$. In the special case of uniform acceleration (or a constant applied force), this is simple to solve. We have $\frac{dv}{dt} = a$ (constant), so $v = at + c_1$. If the initial velocity is $v(0) = v_0$, then $v(0) = c_1 = v_0$, so $\frac{dx}{dt} = v(t) = at + v_0$. One more integration gives $x(t) = \frac{1}{2}at^2 + v_0 t + c_2$, and if the initial position is $x(0) = x_0$ then $x(0) = c_2 = x_0$, so $\boxed{x(t) = \frac{1}{2}at^2 + v_0 t + x_0}$.

Logistic model for growth in a limited environment

In an environment where a population grows with limited resources, it's not realistic to expect unlimited growth. We can model this situation by assuming that the relative growth rate k declines linearly with growing population, at some point (called the carrying capacity) vanishes, and becomes negative when population exceeds this carrying capacity. This is most simply stated as $\frac{1}{P}\frac{dP}{dt} = k(1 - \frac{P}{L})$ or $\frac{dP}{dt} = cP(L - P)$ where $c = \frac{k}{L}$. This is known as the logistic growth model.

The slope field and some trajectories are shown (right) for the differential equation $\frac{dP}{dt} = .5P(2.5 - P)$.



There are two special isoclines for the logistic model, namely the places where $\frac{dP}{dt} = 0$. These occur where $P = 0$ and where $P = L$ and correspond to **equilibria**. If the initial condition lies on either of these lines, the solutions will be constant for all *t*, i.e. $P(t) = 0$ or $P(t) = L$. For initial conditions exceeding *L*, solutions will decay down to the carrying capacity. For any initial condition between 0 and *L*, the solutions will rise and eventually level off at the carrying capacity. Though not meaningful in application, initial values less than 0 will yield solutions that diverge negatively away from 0. We see in this relatively simple model the important distinction between **stable equilibria** (nearby solutions converge toward the equilibrium) and **unstable equilibria** (nearby solutions diverge away from the equilibrium).

Though we can analytically solve the logistic equation (using separation of variables and the method of partial fractions) to give an explicit formula for solutions, the point here is simply that it's often possible to understand qualitatively how the solutions behave just from understanding the slope field – even if we don't produce explicit solutions.

> **Definition**: A differential equation of the form $\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t) = q(t)$, where $p_{n-1}(t),\ldots, p_1(t), p_0(t), q(t)$ are functions of the independent variable *t*, is called an ***n*th order linear ordinary differential equation**. In the case where $q(t) = 0$ for all *t*, we call the equation **homogeneous**. Otherwise we call it **inhomogeneous**.

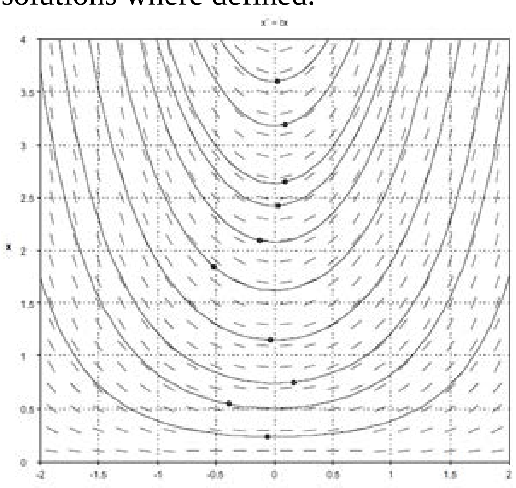
The first three of the previous examples of ODEs (ordinary differential equations) were linear: (a) $\frac{dy}{dx} = 2x$ in inhomogeneous; (b) $\frac{dP}{dt} - kP = 0$ is homogeneous with constant coefficients; and (c) $\frac{d^2x}{dt^2} = a$ (constant) is 2nd order inhomogeneous. The last example given was not linear since it cannot be put into the required form.

In general, any **1st order ODE** can be put in the form $\frac{dy}{dx} = F(x,y)$ for some function $F(x,y)$. If an initial condition $y(a) = b$ is given, we call this an **initial value problem** (IVP). [If we are working with t as the independent variable and x as the dependent variable, we would have $\frac{dx}{dt} = F(t,x)$.]

Question: Under what conditions will this differential equation yield a unique solution for a given initial condition? This is actually <u>two questions</u>: (a) Does a solution exist?; and (b) Is this solution unique?
The answer to these questions is the subject of the following important theorem:

> **Existence and Uniqueness Theorem**: Suppose $F(x,y)$ and the partial derivative F_y are continuous in some rectangle R containing the point $(x_0, y_0) = (a,b)$. Then for some open interval I containing a, the initial value problem $\frac{dy}{dx} = F(x,y)$, $y(a) = b$ has a unique solution defined on the interval I.

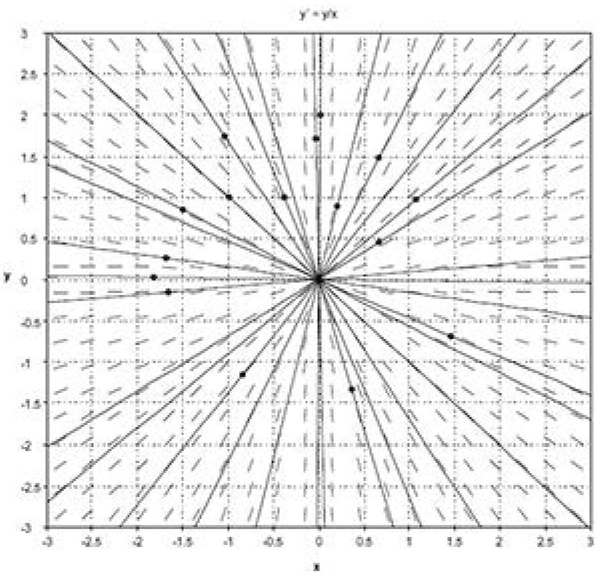
Example: Analyze the ODE $\frac{dx}{dt} = tx$ using its slope field, and solve it analytically to give a formula for all solutions where defined.



Solution: First, note that the right-hand-side is $F(t,x) = tx$ which clearly satisfies the conditions for existence and uniqueness of solutions (for any initial condition).
This is a separable equation. We rewrite $\frac{dx}{dt} = tx$ as $\frac{dx}{x} = tdt$ and integrate $\int \frac{dx}{x} = \int tdt$ to get $\ln|x| = \frac{1}{2}t^2 + C$. If we exponentiate both sides and use some familiar algebra rules we get $x(t) = Ae^{\frac{1}{2}t^2}$. If an initial condition is given as $x(0) = x_0$, this will give us the (unique) solution $x(t) = x_0 e^{\frac{1}{2}t^2}$.

A proof of the Existence and Uniqueness Theorem can be found in the Appendix of the Edwards & Penney text. The condition that F_y be continuous is actually slightly more restrictive than is necessary to prove the theorem, and proofs in other texts use a milder restriction that requires only that this derivative be bounded in a particular way. Another good source for this theorem is the text by Hirsch, Smale, and Devaney. It is quite technical.

Example: Analyze the ODE $\frac{dy}{dx} = \frac{y}{x}$ using its slope field, and solve it analytically to give a formula for all solutions where defined.

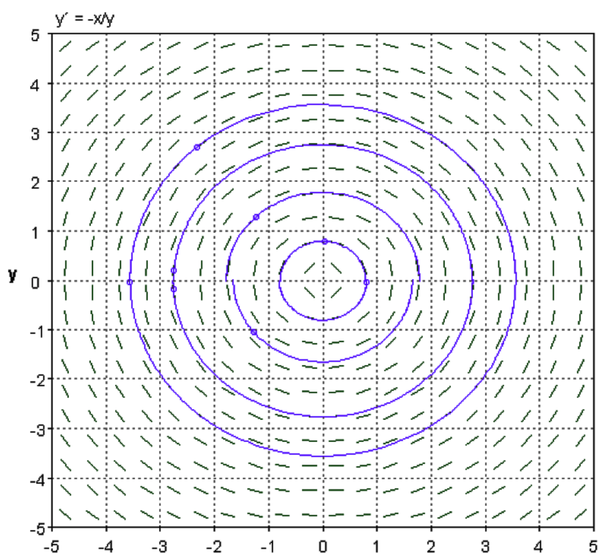


Solution: The slope field for $\frac{dy}{dx} = \frac{y}{x}$ is pretty easy to understand here. Take note, however, that if we have any initial condition where $x = 0$, the only integral curve through that point will be a vertical line, so we'll be unable to solve for $y = y(x)$ near such a point. This coincides with the fact that $F(x, y) = \frac{y}{x}$ is discontinuous at any such point. Further note that though there appear to be local solutions passing through any other point, all such solutions pass through (or at least converge toward) the origin.

This ODE is easy to solve: Rewrite $\frac{dy}{dx} = \frac{y}{x}$ as $\frac{dy}{y} = \frac{dx}{x}$ and integrate $\int \frac{dy}{y} = \int \frac{dx}{x}$ to get that $\ln|y| = \ln|x| + C$ and $y = Ax$ for some constant A. These are just the lines through the origin that we see in the diagram, and there is no unique solution passing through $(0,0)$.

Orthogonal trajectories: Given any ODE of the form $\frac{dy}{dx} = F(x, y)$, since $\frac{dy}{dx}$ represents the slope at any given point, we can rotate all of these to give orthogonal (perpendicular) slopes that are the negative reciprocals of the original slopes. That is, we would look at the new ODE $\frac{dy}{dx} = -\frac{1}{F(x,y)}$.

Example: Using the previous example, the ODE corresponding to its orthogonal trajectories will be $\frac{dy}{dx} = -\frac{x}{y}$. Analyze this using its slope field and solve it analytically to give a formula for all solutions where defined.



Solution: It should be apparent that the integral curves will be circles everywhere perpendicular to the radial lines of the previous example. The Existence and Uniqueness Theorem will fail where $y = 0$, and this corresponds precisely to where these circles would "fold over" between the upper semicircle and lower semicircle. Any solution $y(x)$ will not be extendable beyond such a point.

We can rearrange $\frac{dy}{dx} = -\frac{x}{y}$ as $ydy = -xdx$ and integrate $\int ydy = -\int xdx$ to get $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$ or, more simply, $x^2 + y^2 = C$. These are the circles mentioned above. They yield solutions $y = \sqrt{C - x^2}$ and $y = -\sqrt{C - x^2}$ and we would use initial conditions to determine the value of C and whether we have the upper or lower graph as our solution.

A note on numerical methods

We may go into some detail about this elsewhere. The videos by Prof. Arthur Mattuck (MIT Open Courseware) on this subject are highly recommended. The main thing to keep in mind is that the software used to produce slope fields and graphical solutions (integral curves) does not operate via magic or divine guidance. There are specific algorithms like Euler's method, various improved Runge-Kutta methods, or perhaps the Dormand-Prince method that give solutions using various error correction methods to produce relatively accurate graphical solutions.

Integral Curves, Trajectories, Orbits

A solution to such an initial value problem is called an integral curve. The Existence and Uniqueness Theorem and some parts of its proof can be interpreted in terms of the geometry of integral curves.

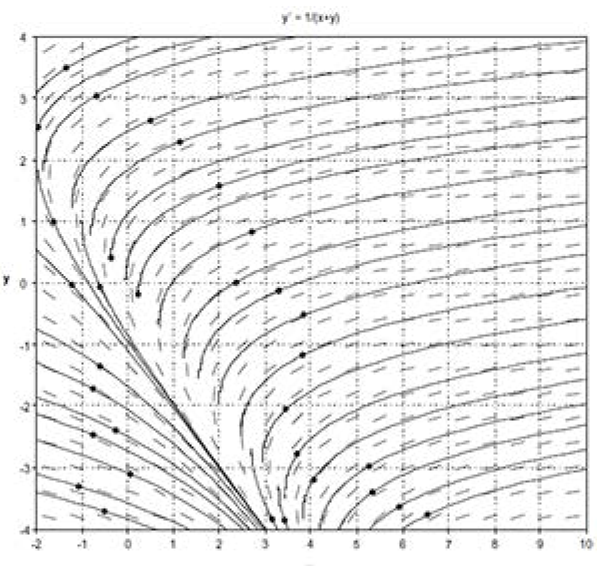
Integral Curve Theorem:

(a) Whenever $F(x,y)$ is defined, integral curves of $\frac{dy}{dx} = F(x,y)$ cannot cross at a positive angle. [This is essentially why the curves have a "parallel" nature.]

(b) If the partial derivative $\frac{\partial F}{\partial y} = F_y$ is continuous in a region, then the integral curves cannot even be tangent to each other at any point in that region.

It sometimes happens when analyzing the slope field of a 1st order ODE that certain integral curves separate other integral curves that are qualitatively fundamentally different. For example, a circular integral curve might separate those curves which spiral inward from those that spiral outward. Such an integral curve is called a **separatrix**. Discovering a separatrix often allows us to separately analyze the behavior of an ODE for initial conditions in different regions.

Example: Analyze the ODE $\dfrac{dy}{dx} = \dfrac{1}{x+y}$ using its slope field, and solve it analytically if possible.



Solution: It's easy to produce isoclines for this example. The slope will be constant wherever $x + y = \text{constant}$, and these are just lines with slope $m = -1$. The line where $x + y = 0$, i.e. the line $y = -x$ is somewhat problematic in that the slope becomes vertical along this line. [The Existence and Uniqueness Theorem will therefore break down everywhere on this line – not surprising as the slope field suggests curves "folding over" at all such points. The isocline where $x + y = -1$ is unusual in that $\frac{dy}{dx} = -1$ everywhere along this line which also has slope -1. This line is, in fact, an integral curve which you can verify by differentiating $y = -x - 1$ and substituting it into the ODE. The integral curves above and below this line are fundamentally different, so this line is a separatrix.

This differential equation does not yield simple analytic solutions of the form $y = y(x)$. However, we can turn things sideways and see if it's possible to solve for $x = x(y)$. Basic calculus permits us to rewrite the differential equation as $\frac{dx}{dy} = x + y$. This can then be written in the form $\frac{dx}{dy} - x = y$, a first order inhomogeneous linear differential equation. We will investigate two approaches to solving such a first order linear ODE.

Definition: A differential equation of the form $\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t) = q(t)$, where $p_{n-1}(t), \ldots, p_1(t), p_0(t), q(t)$ are functions of the independent variable *t*, is called an ***n*th order linear ordinary differential equation**. In the case where $q(t) = 0$ for all *t*, we call the equation **homogeneous**. Otherwise we call it **inhomogeneous**.

We are specifically concerned with 1st order ODEs of the form $\boxed{\frac{dy}{dx} + p(x)y = q(x)}$ (or $\boxed{\frac{dx}{dt} + p(t)x = q(t)}$).

Integrating factors

Definition: An **integrating factor** for a given first order ODE is a function $v(x)$ such that when both sides of the ODE are multiplied by $v(x)$ the resulting differential equation consists of known derivatives on both sides of the equation. The ODE can then be solved by integrating both sides and then solving for the dependent variable in terms of the independent variable.

It's always possible to formally solve $\frac{dy}{dx}+p(x)y=q(x)$ via an integrating factor. We seek $v(x)$ such that we can integrate both sides of the equation $v(x)\left[\frac{dy}{dx}+p(x)y\right]=v(x)q(x)$. The left-hand-side is $v\frac{dy}{dx}+pvy$, and if we note that $\frac{d}{dx}(vy)=v\frac{dy}{dx}+v'(x)y$, we can then look for $v(x)$ such that $\cancel{v\frac{dy}{dx}}+pvy=\cancel{v\frac{dy}{dx}}+v'(x)y$ or simply $pvy=v'y \quad\Rightarrow\quad pv=v'$. This can then be rewritten as $\frac{v'(x)}{v(x)}=p(x) \quad\Rightarrow\quad \int\frac{v'(x)}{v(x)}dx=\int p(x)dx$. This gives $\ln|v(x)|=\int p(x)dx+C \quad\Rightarrow\quad \boxed{v(x)=e^{\int p(x)dx}}$ as an integrating factor. This approach, of course, works best if you can find an antiderivative of the function $p(x)$.

We then have the new ODE $\frac{d}{dx}\big(v(x)y(x)\big)=v(x)q(x)$, so integration gives $v(x)y(x)=\int v(x)q(x)dx+C$. We can then solve for $y(x)=\frac{1}{v(x)}\left[\int v(x)q(x)dx+C\right]$. If we insert the integrating factor $v(x)=e^{\int pdx}$, we can write this solution as $\boxed{y(x)=e^{-\int pdx}\left[\int q(x)e^{\int pdx}dx+C\right]}$. It may not be pretty, but it works if you can actually do the integrals. You may find it simpler to just know how to get the integrating factor and then proceed with the integrations.

If we switch variables in our example (just to be ever so conventional), our equation becomes $\frac{dy}{dx}=y+x$ or $\frac{dy}{dx}-y=x$. In this case, $p(x)=-1$, $\int p(x)dx=-x$ is the simplest antiderivative, and the integrating factor is then $v(x)=e^{-x}$. We then have $e^{-x}\left(\frac{dy}{dx}-y\right)=\frac{d}{dx}\left(e^{-x}y\right)=xe^{-x}$. We can then integrate using integration by parts to get $e^{-x}y=\int xe^{-x}dx=-xe^{-x}-e^{-x}+C$. If we then multiply both sides by e^{x}, we get $\boxed{y(x)=-x-1+Ce^{x}}$. If, for example, we had the initial condition that $y(0)=3$, we would then have $y(0)=-1+C=3$, so $C=4$ and we would get the unique solution $y(x)=-x-1+4e^{x}$ for this initial value problem.

Considering the relatively simple expression for this solution, you might think that there could be a simpler approach. There is, but it requires us to start our way down an important path that will lead to some of the most important methods and perspectives in this entire course. This is the Linearity path.

Linearity

In the context of functions of one variable, linearity is an often abused word. In fact, a function of the form $f(x)=mx+b$ is NOT a linear function. It is more appropriately called a 1st order *affine* function. Linearity is a property most simply characterized by the fact that <u>linear functions preserve scaling and adding</u>. The linear functions of one variable consist only of those of the form $f(x)=mx$. Note that $f(ax)=m(ax)=a(mx)=af(x)$, i.e. it preserves scaling, and $f(x+y)=m(x+y)=mx+my=f(x)+f(y)$, i.e. it preserves addition.

Definition: Formally we say that a function is **linear** if for all inputs x_1, x_2 and constants c_1, c_2 we must have $f(c_1x_1 + c_2x_2) = c_1 f(x_1) + c_2 f(x_2)$.

In the case of functions $T : \mathbf{R}^n \to \mathbf{R}^m$, linearity means that the scaling of vectors and the addition of vectors is preserved via a linear transformation. All such transformations are of the form $T(\mathbf{x}) = \mathbf{Ax}$ where **A** is an $m \times n$ matrix with constant entries. Linearity then translates into the matrix algebra facts that $\mathbf{A}(k\mathbf{x}) = k(\mathbf{Ax})$ and $\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{Ax} + \mathbf{Ay}$, or (combined) $\mathbf{A}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{Ax} + \beta\mathbf{Ay}$ for all scalars α, β and all vectors $\mathbf{x}, \mathbf{y}$.

Our current situation involves working with functions in the same way that we looked at vectors in $\mathbf{R}^n$. Just as we can scale and add vectors, we can also scale and add functions. A transformation that acts on functions in a manner analogous to the way matrices act on vectors is known as a **linear (differential) operator**. The basic examples are differentiation and multiplication by a fixed function. We can then compose these basic operators and add them to form more complicated operators.

There are many spaces of functions in which we can seek solutions to differential equations. Perhaps the most common such space is the space of functions that are differentiable to all orders.

Multiplication by a fixed function is a linear operator

Suppose we have a fixed function $p(x)$ and we define a transformation of functions by $\left[T(f)\right](x) = p(x)f(x)$. We can easily see that for any constant *c*, $\left[T(cf)\right](x) = p(x)cf(x) = cp(x)f(x) = c\left[T(f)\right](x)$, so $T(cf) = cT(f)$, i.e. *T* preserves scaling. Similarly, if f_1 and f_2 are two functions, then $\left[T(f_1 + f_2)\right](x) = p(x)(f_1 + f_2)(x) = p(x)(f_1(x) + f_2(x)) = p(x)f_1(x) + p(x)f_2(x) = \left[T(f_1)\right](x) + \left[T(f_2)\right](x)$.

This is really just the distributive law, but the result is that formally $T(f_1 + f_2) = T(f_1) + T(f_2)$, i.e. *T* preserves addition of functions. Together, this shows that *T* is a linear operator.

Differentiation of functions is a linear operator

Let *D* be the transformation defined by $D(f) = f'$. That is, $\left[D(f)\right](x) = f'(x)$. The old refrains you learned in first semester calculus are precisely what makes this a linear operator: (a) The derivative of a constant times a function is the constant times the derivation of the function; and (b) The derivative of a sum is the sum of the derivatives. In symbolic terms, $D(cf) = cf'$ and $D(f+g) = f' + g'$. We can put these together as a single linearity rule: $D(c_1 f_1 + c_2 f_2) = c_1 D(f_1) + c_2 D(f_2)$.

The composition of linear operators (or any linear function), where defined, is also linear

If *S* and *T* are both linear operators and if the composition $S \circ T$ is defined, then using the linearity properties of both we have that for all scalars c_1, c_2 and functions f_1, f_2,

$$\begin{aligned}\left(S \circ T\right)(c_1 f_1 + c_2 f_2) &= S(T(c_1 f_1 + c_2 f_2)) = S(c_1 T(f_1) + c_2 T(f_2)) \\ &= c_1 S(T(f_1) + c_2 S(T(f_2)) = c_1\left(S \circ T\right)(f_1) + c_2\left(S \circ T\right)(f_2)\end{aligned}$$

For example, since differentiation acts linearly, we can compose this with itself to get the 2nd derivative and this also acts linearly. The same holds for higher order derivatives.

The sum of two linear operators is also a linear operator

The sum of two operators is defined in the same way we add any functions, i.e. $(S+T)(f) = S(f) + T(f)$. If *S* and *T* are both linear operators, then we'll have that for all scalars c_1, c_2 and functions f_1, f_2,

$$\begin{aligned}\left[S+T\right](c_1 f_1 + c_2 f_2) &= S(c_1 f_1 + c_2 f_2) + T(c_1 f_1 + c_2 f_2) = c_1 S(f_1) + c_2 S(f_2) + c_1 T(f_1) + c_2 T(f_2) \\ &= c_1 S(f_1) + c_1 T(f_1) + c_2 S(f_2) + c_2 T(f_2) = c_1\left[S(f_1) + T(f_1)\right] + c_2\left[S(f_2) + T(f_2)\right] = c_1\left[S+T\right](f_1) + c_2\left[S+T\right](f_2)\end{aligned}$$

If we put together the facts that composition of linear operators and the addition of linear operators yields another linear operator, we see that the expression $\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t)$ for functions

$p_{n-1}(t), \ldots, p_1(t), p_0(t)$ represents a linear operator acting on an undetermined function $x(t)$. If we write this operator as $T(x(t)) = \frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x(t)$, we then know by linearity that $T(x_1(t)+x_2(t)) = T(x_1(t)) + T(x_2(t))$ and $T(c\,x(t)) = c\,T(x(t))$ and, more generally, $T(c_1\,x_1(t)+c_2\,x_2(t)) = c_1\,T(x_1(t)) + c_2\,T(x_2(t))$.

Now that we have paved the road to Linearity, we can apply this idea to solving linear differential equations.

Linearity method using homogeneous solutions and particular solutions

Suppose we have an inhomogeneous linear ODE of the form $T(f) = g$ where T is an *n*th order linear differential operator. We can produce ALL solutions to $T(f) = g$ as follows:

(1) First solve the homogeneous equation $T(f) = 0$ to find a general expression for all such solutions. Call this the homogeneous solution f_h. It will generally involve *n* arbitrary constants.

(2) Find a single particular solution to the inhomogeneous equation $T(f) = g$. Call this particular solution f_p.

(3) The general solution to $T(f) = g$ is then $\boxed{f = f_h + f_p}$.

Proof of the method: We know that $T(f_p) = g$, so suppose f is any other solution to $T(f) = g$. Then $T(f - f_p) = T(f) - T(f_p) = g - g = 0$. So $f - f_p$ solves the homogeneous equation and must be included among all homogeneous solution, i.e. $f - f_p = f_h$. Therefore $f = f_h + f_p$.

This fact is really the same thing that we see when solving a consistent, inhomogeneous system of linear algebraic equations. In matrix form, if the system is represented as $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is an $m \times n$ matrix, and if $\mathbf{x}_h$ represents all solutions to the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{x}_p$ is a single solution to $\mathbf{Ax} = \mathbf{b}$, then all solutions to $\mathbf{Ax} = \mathbf{b}$ will be of the form $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$. Typically, these homogeneous solutions are lines, planes or higher-dimensional analogues (subspaces) passing through the origin. This just says that the inhomogeneous solutions are parallel translates of these subspaces.

So, let's solve the problem already:

The ODE $\frac{dy}{dx} - y = x$ is first order, linear, and inhomogeneous.

(1) The homogeneous equation is just $\frac{dy}{dx} - y = 0$ or $\frac{dy}{dx} = y$. We've already solved this to get all solutions in the form $y_h = Ae^x$.

(2) We can find an inhomogeneous solution by educated guessing (formally called the method of undetermined coefficients). Try a solution of the form $y = ax + b$. Calculate $\frac{dy}{dx} = a$ and substitute into the ODE to get $\frac{dy}{dx} - y = a - (ax+b) = (a-b) - bx = x$. We can solve this by choosing $a - b = 0$ and $-b = 1$. So $\boxed{b = -1}$ and $\boxed{a = -1}$, and a particular solution is therefore $y_p = -x - 1$.

(3) By linearity, all solutions are therefore of the form $y = y_h + y_p = Ae^x - x - 1$. This agrees with our previous result.

And **our original problem** $\frac{dx}{dy} - x = y$ gives $\boxed{x = Ae^y - y - 1}$ where the constant *A* is determined by initial conditions. If you compare this with the slope field picture on the first page, you'll see that this accurately describes all of the integral curves including the separatrix which occurs where $A = 0$.

Notes by Robert Winters